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Suzuki type theorems for asymmetric type mappings

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Abstract

We introduce a modified asymmetric $G^{\bigstar}(\psi\varphi)$-contractive mapping with respect to a general family of functions G^* and establish asymmetric type fixed point results for such mappings. As an application of our results, we deduce Suzuki type fixed point results via these mappings. We also derive certain fixed point results for asymmetric type mappings in partial G-metric spaces. Moreover, we discuss an illustrative example to highlight the realized improvements.

Keywords: G-metric space, partial G-metric space, fixed point, Suzuki fixed point theorem.
2010 MSC: 46N40, 47H10, 54H25, 46T99.

1. Introduction

The study of fixed points of given mappings satisfying certain contractive conditions has been at the center of vigorous research activity. There are many concepts of generalized metric spaces. For example, in 2005, Mustafa and Sims introduced a new class of generalized metric spaces (see [4, 5]), which are called G-metric spaces. A G-metric assigns a real number to every triplet of a set. Many fixed point results on such spaces, for mappings satisfying various contractive conditions appeared in [1, 2, 6].

Recently, Samet *et al.* [8] and Vetro and Vetro [9], used a semicontinuous function to establish new fixed point results. As consequences, we deduce some results on fixed point in the setting of partial metric spaces. In this paper, we use the ideas from [8, 9] and the notion of modified asymmetric type mapping to establish existence and uniqueness of fixed points in the setting of G-metric spaces. As consequences, we deduce some results on fixed point in the setting of partial G-metric spaces. An example is furnished to demonstrate the validity of the obtained results.

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2. Preliminaries

In this section, we present necessary definitions and results in G-metric and partial G-metric spaces, which will be useful further on; for more details, we refer to [4, 5, 7]. Denote by $\mathbb{N}$ the set of all positive integers.

Definition 2.1. Let X be a nonempty set. A function $G : X \times X \times X \longrightarrow [0, +\infty)$ is called a G-metric if the following conditions are satisfied:

(G1) If $x = y = z$, then $G(x, y, z) = 0$;

(G2) $0 < G(x, y, y)$, for any $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$ for any points $x, y, z \in X$, with $y \neq z$;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables;

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for any $x, y, z, a \in X$.

Then the pair (X, G) is called a G-metric space.

Definition 2.2. Let (X, G) be a G-metric space and $\{x_n\}$ a sequence in X.

(i) $\{x_n\}$ is a G-Cauchy sequence if for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that for all $n, m, l \geq N$, $G(x_n, x_m, x_l) < \varepsilon$.

(ii) $\{x_n\}$ is G-convergent to $x \in X$ if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $G(x, x_n, x_m) < \varepsilon$.

A G-metric space (X, G) is said to be complete if every G-Cauchy sequence in X is G-convergent in X.

Proposition 2.3. *Let (X, G) be a G-metric space. The followings are equivalent:*

(1) (x_n) is G-convergent to x;

(2) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$;

(3) $G(x_n, x, x) \to 0$ as $n \to +\infty$.

From (G5) and (G3), we obtain the following lemma.

Lemma 2.4 ([3])**.** *Let (X, G) be a G-metric space and $x_1, x_2, y_1, y_2, z_1, z_2, a \in X$ where $x_1 \neq x_2$, $y_1 \neq y_2$ and $z_1 \neq z_2$. Then the following inequality holds.*

$$G(x_1, y_1, z_1) \leq G(x_1, x_2, a) + G(y_1, y_2, a) + G(z_1, z_2, a).$$

Definition 2.5 ([7])**.** Let X be a nonempty set. A function $P : X \times X \times X \longrightarrow [0, +\infty)$ is called a partial G-metric if the following conditions are satisfied:

(P1) If $x = y = z$, then $P(x, y, z) = P(x, x, x) = P(y, y, y) = P(z, z, z)$;

(P2) $P(x, x, x) + P(y, y, y) + P(z, z, z) \leq 3P(x, y, z)$ for all $x, y, z \in X$;

(P3) $\frac{1}{3}P(x, x, x) + \frac{2}{3}P(y, y, y) < P(x, y, y)$ for all $x, y \in X$ with $x \neq y$;

(P4) $P(x, x, y) - \frac{1}{3}P(x, x, x) \leq P(x, y, z) - \frac{1}{3}P(z, z, z)$ for all $x, y, z \in X$, with $y \neq z$;

(P5) $P(x, y, z) = P(x, z, y) = P(y, z, x) = \cdots$, symmetry in all three variables;

(P6) $P(x, y, z) \leq P(x, a, a) + P(a, y, z) - P(a, a, a)$ for any $x, y, z, a \in X$.

Then the pair (X, P) is called a partial G-metric space (in brief PGMS).

The following lemma shows that to every partial G-metric, we can associate one G-metric.

Lemma 2.6 ([7], Lemma 2.2)**.** *Let (X, P) be a PGMS. Define $G_p : X \times X \times X \to [0, +\infty)$ by*

$$G_p(x, y, z) = 3P(x, y, z) - P(x, x, x) - P(y, y, y) - P(z, z, z).$$

Then G_p is a G-metric function on X and the pair (G_p, X) is a G-metric space.

Example 2.7 ([7], Example 2.3)**.** Let $X = [0, +\infty)$ and define $P(x, y, z) = \frac{1}{3}\Big(\max\{x, y\} + \max\{y, z\} + \max\{x, z\}\Big)$, for all $x, y, z \in X$. Then (X, P) is a PGMS.

The following proposition gives some properties of partial G-metric.

Proposition 2.8 ([7], Proposition 2.4)**.** *Let (X, P) be a partial G-metric space; then for any $x, y, z, a \in X$ the following properties hold:*

(i) *If $P(x, y, z) = P(x, x, x) = P(y, y, y) = P(z, z, z)$, then $x = y = z$;*

(ii) *If $P(x, y, z) = 0$, then $x = y = z$;*

(iii) *If $x \neq y$, then $P(x, y, y) > 0$;*

(iv) $P(x, y, z) \leq P(x, x, y) + P(x, x, z) - P(x, x, x)$;

(v) $P(x, y, y) \leq 2P(x, x, y) - P(x, x, x)$;

(vi) $P(x, y, z) \leq P(x, a, a) + P(y, a, a) + P(z, a, a) - 2P(a, a, a)$;

(vii) $P(x, y, z) \leq P(x, a, z) + P(a, y, z) - \frac{2}{3}P(a, a, a) - \frac{1}{3}P(z, z, z)$ *with* $y \neq z$;

(viii) $P(x, y, y) \leq P(x, y, a) + P(a, y, y) - \frac{2}{3}P(a, a, a) - \frac{1}{3}P(y, y, y)$ *with* $x \neq y$.

Definition 2.9. Let (X, P) be a PGMS and $\{x_n\}$ a sequence in X.

(1) $\{x_n\}$ is P-G-convergent to $x \in X$ if and only if

$$P(x, x, x) = \lim_{n\to+\infty} P(x, x, x_n) = \lim_{n\to+\infty} P(x, x_n, x_n).$$

(2) $\{x_n\}$ is a $(PG)^*$-Cauchy sequence if

$$\lim_{m,n,l\to+\infty} P(x_n, x_m, x_l) = \lim_{m,n,l\to+\infty} \left[\frac{P(x_n, x_n, x_n) + P(x_m, x_m, x_m) + P(x_l, x_l, x_l)}{3}\right].$$

(3) (X, P) is said to be a $(PG)^*$-complete partial G-metric space if and only if every $(PG)^*$-Cauchy sequence in X is P-G-convergent to a point $x \in X$ such that $\lim_{n\to+\infty} P(x_n, x_n, x_n) = P(x, x, x)$.

Lemma 2.10. *Let (X, P) be a partial G-metric space, $x, y \in X$ and $\{x_n\}$ a sequence in X. Assume that $\lim_{n\to+\infty} P(x, x_n, x_n) = \lim_{n\to+\infty} P(x_n, y, y) = 0$; then $x = y$.*

Proof. By (P6), we have

$$P(x, y, y) \leq P(x, x_n, x_n) + P(x_n, y, y).$$

Letting $n \to +\infty$, we get that $P(x, y, y) = 0$ and so by (ii) of Proposition 2.8 we deduce that $x = y$. □

Lemma 2.11. *Let (X, P) be a partial G-metric space such that $P(x, x, x) \leq P(x, y, y)$ for all $x, y \in X$. Then $P(x, x, x)$ is lower G_p-semicontinuous in (G_p, X).*

Proof. Let $\{x_n\}$ be a sequence in the G-metric space (G_p, X) with $x_n \to x$ as $n \to +\infty$. Then,

$$0 = \lim_{n\to+\infty} G_p(x, x_n, x_n) = \lim_{n\to+\infty} [3P(x, x_n, x_n) - P(x, x, x) - 2P(x_n, x_n, x_n)],$$

which implies

$$3 \lim_{n\to+\infty} P(x, x_n, x_n) = P(x, x, x) + 2 \lim_{n\to+\infty} P(x_n, x_n, x_n).$$

Now since $P(x, x, x) \leq P(x, y, y)$ for all $x, y \in X$, then

$$3P(x, x, x) \leq 3 \lim_{n\to+\infty} P(x, x_n, x_n) = P(x, x, x) + 2 \lim_{n\to+\infty} P(x_n, x_n, x_n),$$

that is

$$P(x, x, x) \leq \lim_{n\to+\infty} P(x_n, x_n, x_n).$$

□

Lemma 2.12. *If (X, P) is a $(PG)^*$-complete partial G-metric space, then (X, G_p) is a complete G-metric space.*

Proof. Let (X, P) be a $(PG)^*$-complete partial G-metric space. Assume that $\{x_n\}$ is a Cauchy sequence in (X, G_p). We have

$$\lim_{m,n,l\to+\infty} G_p(x_n, x_m, x_l) = 0,$$

which implies,

$$\lim_{m,n,l\to+\infty} P(x_n, x_m, x_l) = \lim_{m,n,l\to+\infty} \left[\frac{P(x_n, x_n, x_n) + P(x_m, x_m, x_m) + P(x_l, x_l, x_l)}{3}\right].$$

Thus $\{x_n\}$ is a $(PG)^*$-Cauchy sequence in the partial G-metric space (X, P). Completeness of (X, P) ensures that there exists an $x \in X$ such that

$$P(x, x, x) = \lim_{n\to+\infty} P(x, x, x_n) = \lim_{n\to+\infty} P(x, x_n, x_n),$$

where $\lim_{n\to+\infty} P(x_n, x_n, x_n) = P(x, x, x)$. Hence,

$$\begin{aligned}
\lim_{n\to+\infty} G_p(x, x, x_n) &= 3 \lim_{n\to+\infty} P(x, x, x_n) - P(x, x, x) - P(x, x, x) - \lim_{n\to+\infty} P(x_n, x_n, x_n) \\
&= 3P(x, x, x) - P(x, x, x) - P(x, x, x) - \lim_{n\to+\infty} P(x_n, x_n, x_n) \\
&= P(x, x, x) - \lim_{n\to+\infty} P(x_n, x_n, x_n) = 0,
\end{aligned}$$

that is, $\{x_n\}$ converges to x with respect to G-metric G_p. □

3. Main result

We denote by $\mathcal{G}^\star$ the set of continuous functions $G^\star : [0, +\infty)^4 \to [0, +\infty)$ satisfying the following conditions:

(G^*1) $\max\{a, b\} \leq G^\star(a, b, c, d)$ for all $a, b, c, d \in [0, +\infty)$;

(G^*2) if $G^\star(a, b, c, d) = 0$, then $a = 0$;

(G^*3) $G^\star$ is nondecreasing in fourth variable.

As examples, the following functions belong to $\mathcal{G}^\star$:

- $G^\bigstar(a,b,c,d) = (a+b+c)(d+1)$;
- $G^\bigstar(a,b,c,d) = a+b+c+d$;
- $G^\bigstar(a,b,c,d) = \max\{a,b,c,d\}$;
- $G^\bigstar(a,b,c,d) = a + \max\{a,b,c,d\}$.

We denote by Ψ the set of nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ.

It is not difficult to show that if $\psi \in \Psi$, then $\psi(t) < t$ for every $t > 0$.

Definition 3.1. Let (X,G) be a G-metric space, $T : X \to X$ and $\alpha, \eta : X^2 \to [0,+\infty)$. We say that T is an α-admissible mapping with respect to η if $\alpha(Tx,Ty) \geq \eta(Tx,Ty)$ whenever $\alpha(x,y) \geq \eta(x,y)$.

Definition 3.2. Let (X,G) be a G-metric space, $T : X \to X$, $\alpha,\eta : X^2 \to [0,+\infty)$ and $\varphi : [0,+\infty) \to [0,+\infty)$. We say that T is a modified asymmetric $G^\bigstar(\psi\varphi)$-contractive mapping if there exist $\psi \in \Psi$ and $G^\bigstar \in \mathcal{G}^\bigstar$ such that for all $x,y \in X$ with $\alpha(x,y) \geq \eta(x,Tx)$ we have

$$G^\bigstar\left(G(Tx,T^2x,Ty),\varphi(Tx),\varphi(T^2x),\varphi(Ty)\right) \leq \psi\left(G^\bigstar(G(x,Tx,y),\varphi(x),\varphi(Tx),\varphi(y))\right).$$

Our main result is given by the following theorem.

Theorem 3.3. *Let (X,G) be a G-complete G-metric space and $\varphi : X \to [0,+\infty)$ a lower G-semicontinuous mapping. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) T is a G-continuous modified asymmetric $G^\bigstar(\psi\varphi)$-contractive mapping;

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq \eta(x_0,Tx_0)$.

Then T has a fixed point $z \in X$ such that $\varphi(z) = 0$. Moreover, T has a unique fixed point whenever $\alpha(x,y) \geq \eta(x,x)$ for all $x,y \in Fix(T)$.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0,Tx_0) \geq \eta(x_0,Tx_0)$ and let $\{x_n\}$ be the sequence of Picard starting at x_0, that is, $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Now, since T is an α-admissible mapping with respect to η, we deduce that

$$\alpha(x_1,x_2) = \alpha(Tx_0,T^2x_0) \geq \eta(Tx_0,T^2x_0) = \eta(x_1,Tx_1).$$

By continuing this process, we get

$$\alpha(x_{n-1},x_n) \geq \eta(x_{n-1},Tx_{n-1}) \quad \text{for all } n \in \mathbb{N}.$$

Now if, for some $n_0 \in \mathbb{N}$, we have $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and we have nothing to prove. Hence, we assume $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. Therefore, using condition (ii) with $x = x_{n-1}$ and $y = x_n$, we obtain that

$$\begin{aligned} &G^\bigstar\left(G(Tx_{n-1},T^2x_{n-1},Tx_n),\varphi(Tx_{n-1}),\varphi(T^2x_{n-1}),\varphi(Tx_n)\right) \\ &\quad \leq \psi\left(G^\bigstar(G(x_{n-1},Tx_{n-1},x_n),\varphi(x_{n-1}),\varphi(Tx_{n-1}),\varphi(x_n))\right), \end{aligned}$$

which implies

$$\begin{aligned} &G^\bigstar\left(G(x_n,x_{n+1},x_{n+1}),\varphi(x_n),\varphi(x_{n+1}),\varphi(x_{n+1})\right) \\ &\quad \leq \psi\left(G^\bigstar(G(x_{n-1},x_n,x_n),\varphi(x_{n-1}),\varphi(x_n),\varphi(x_n))\right). \end{aligned}$$

By using monotony of the function ψ, we obtain that

$$\begin{aligned}&G^{\bigstar}\left(G(x_n,x_{n+1},x_{n+1}),\varphi(x_n),\varphi(x_{n+1}),\varphi(x_{n+1})\right)\\ &\quad\leq \psi^n\left(G^{\bigstar}(G(x_0,x_1,x_1),\varphi(x_0),\varphi(x_1),\varphi(x_1))\right)\end{aligned}$$

for every $n\in\mathbb{N}$. Finally, by (G^*1), we get

$$\begin{aligned}\max\{G(x_n,x_{n+1},x_{n+1}),\varphi(x_n)\}&\leq G^{\bigstar}\left(G(x_n,x_{n+1},x_{n+1}),\varphi(x_n),\varphi(x_{n+1}),\varphi(x_{n+1})\right)\\ &\leq \psi^n\left(G^{\bigstar}(G(x_0,x_1,x_1),\varphi(x_0),\varphi(x_1),\varphi(x_1))\right)\end{aligned}\tag{3.1}$$

for every $n\in\mathbb{N}$.

Note that, if $G^{\bigstar}(G(x_0,x_1,x_1),\varphi(x_0),\varphi(x_1),\varphi(x_1))=0$, by (G^*2), $G(x_0,x_1,x_1)=0$ and by (G2), we have $x_0=x_1=Tx_0$. Then x_0 is a fixed point of T and we have nothing to prove. Therefore, we assume $G^{\bigstar}(G(x_0,x_1,x_1),\varphi(x_0),\varphi(x_1),\varphi(x_1))>0$.

Fix $\varepsilon>0$ and let $h=h(\varepsilon)$ be a positive integer such that

$$\sum_{n=h}^{+\infty}\psi^n\left(G^{\bigstar}(G(x_0,x_1,x_1),\varphi(x_0),\varphi(x_1),\varphi(x_1))\right)<\varepsilon.$$

Let $m>n\geq h$. Using condition (G5) and (3.1), we obtain

$$\begin{aligned}G(x_n,x_m,x_m)&\leq\sum_{k=n}^{m-1}G(x_k,x_{k+1},x_{k+1})\\ &\leq\sum_{n=h}^{+\infty}\psi^n\left(G^{\bigstar}(G(x_0,x_1,x_1),\varphi(x_0),\varphi(x_1),\varphi(x_1))\right)\\ &<\varepsilon.\end{aligned}$$

Consequently, $\{x_n\}$ is a G-Cauchy sequence. Since (X,G) is G-complete, there exists an $z\in X$ such that $x_n\to z$ as $n\to+\infty$. Finally, the G-continuity of T implies that $z=Tz$. The lower G-semicontinuity of φ and (3.1) give $\varphi(z)=0$.

The uniqueness of the fixed point follows from the hypothesis that T is a modified asymmetric $G^{\bigstar}(\psi\varphi)$-contractive mapping. In fact, if $y\in Fix(T)$ with $y\neq z$ and $G(z,z,y)>0$, then

$$G^{\bigstar}\left(G(z,z,y),\varphi(z),\varphi(z),\varphi(y)\right)>0.$$

Since $\alpha(z,y)\geq\eta(z,z)$, we deduce that

$$\begin{aligned}G^{\bigstar}\left(G(z,z,y),\varphi(z),\varphi(z),\varphi(y)\right)&\leq\psi\left(G^{\bigstar}(G(z,z,y),\varphi(z),\varphi(z),\varphi(y))\right)\\ &<G^{\bigstar}\left(G(z,z,y),\varphi(z),\varphi(z),\varphi(y)\right).\end{aligned}$$

This is a contradiction and hence $G(z,z,y)=0$, which implies $z=y$ by condition (G2). □

In the following theorem, we omit the G-continuity hypothesis of T.

Theorem 3.4. *Let (X,G) be a G-complete G-metric space and $\varphi:X\to[0,+\infty)$ be a lower G-semicontinuous mapping. Let $T:X\to X$ be a self-mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) T is a modified asymmetric $G^{\bigstar}(\psi\varphi)$-contractive mapping with respect to a continuous $\psi\in\Psi$;

(iii) there exists an $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq\eta(x_0,Tx_0)$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to z$ as $n \to +\infty$, then either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, z) \quad \text{or} \quad \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, z)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point $z \in X$ such that $\varphi(z) = 0$. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. As in the proof of Theorem 3.3, we can conclude that the Picard sequence $\{x_n\}$, starting at x_0, satisfies the following conditions:

$$\alpha(x_{n-1}, x_n) \geq \eta(x_{n-1}, Tx_{n-1}), \quad \lim_{n\to+\infty} x_n = z \in X \quad \text{and} \quad \varphi(z) = \lim_{n\to+\infty} \varphi(x_n) = 0. \tag{3.2}$$

So, from condition (iv), either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, z) \quad \text{or} \quad \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, z)$$

holds for all $n \in \mathbb{N}$. This implies that either

$$\eta(x_{n+1}, x_{n+2}) \leq \alpha(x_{n+1}, z) \quad \text{or} \quad \eta(x_{n+2}, x_{n+3}) \leq \alpha(x_{n+2}, z)$$

holds for all $n \in \mathbb{N}$. Equivalently, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Tx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leq \alpha(x_{n_k}, z) \quad \text{for all } k \in \mathbb{N}.$$

Now, from (ii), we obtain

$$\begin{aligned} &G^\bigstar\left(G(Tx_{n_k}, T^2x_{n_k}, Tz), \varphi(Tx_{n_k}), \varphi(T^2x_{n_k}), \varphi(Tz)\right) \\ &\quad \leq \psi\left(G^\bigstar(G(x_{n_k}, Tx_{n_k}, z), \varphi(x_{n_k}), \varphi(Tx_{n_k}), \varphi(z))\right), \end{aligned}$$

that is,

$$\begin{aligned} &G^\bigstar\left(G(x_{n_k+1}, x_{n_k+2}, Tz), \varphi(x_{n_k+1}), \varphi(x_{n_k+2}), \varphi(Tz)\right) \\ &\quad \leq \psi\left(G^\bigstar(G(x_{n_k}, x_{n_k+1}, z), \varphi(x_{n_k}), \varphi(x_{n_k+1}), 0)\right). \end{aligned}$$

Now, we claim that z is a fixed point of T. Assume $G(z, z, Tz) > 0$; this implies that $G^\bigstar(G(z, z, Tz), 0, 0, 0) > 0$. Letting $k \to +\infty$ in the above inequality, by the continuity of the functions $G^\bigstar$, G and ψ and (3.2), we get

$$\begin{aligned} G^\bigstar(G(z, z, Tz), 0, 0, \varphi(Tz)) &\leq \psi\left(G^\bigstar(G(z, z, Tz), 0, 0, 0)\right) \\ &< G^\bigstar(G(z, z, Tz), 0, 0, 0). \end{aligned}$$

This is a contradiction since $G^\bigstar$ is nondecreasing and so $G(z, z, Tz) = 0$. Consequently, $z = Tz$, that is, z is a fixed point of T. □

Example 3.5. Let $X = [0, +\infty)$. Define, $G : X^3 \to [0, +\infty)$ by

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y\} + \max\{y, z\} + \max\{x, z\}, & \text{otherwise.} \end{cases}$$

Clearly, (X, G) is a G-complete G-metric space. Define $T : X \to X$, $\alpha, \eta : X^2 \to [0, +\infty)$, $\psi : [0, +\infty) \to [0, +\infty)$, $G^\bigstar : [0, +\infty)^4 \to [0, +\infty)$ and $\varphi : X \to [0, +\infty)$ by

$$Tx = \begin{cases} \frac{1}{8}x, & \text{if } x \in [0, 1], \\ \frac{x^{\sin x} + 2|x-2||x-3|\ln(x+1)}{2}, & \text{if } x \in (1, +\infty), \end{cases}$$

$$\alpha(x,y)=\begin{cases}\frac{t}{2}, & \text{if } x,y\in[0,1],\\ 0, & \text{otherwise,}\end{cases}\qquad \eta(x,y)=\tfrac{1}{4},$$

$$\psi(t)=\frac{t}{2},\quad G^\bigstar(a,b,c,d)=a+b+c+d \quad\text{and}\quad \varphi(t)=t.$$

Let $\alpha(x,y)\geq\eta(x,Tx)$; then $x,y\in[0,1]$. At first, we assume that $x\leq y$. Then

$$\begin{aligned}&G^\bigstar(G(x,Tx,y),\varphi(x),\varphi(Tx),\varphi(y))\\&=\max\{x,Tx\}+\max\{Tx,y\}+\max\{x,y\}+x+\frac{1}{8}x+y\\&=\frac{17}{8}x+3y,\end{aligned}$$

and

$$\begin{aligned}&G^\bigstar(G(Tx,T^2x,Ty),\varphi(Tx),\varphi(T^2x),\varphi(Ty))\\&=\max\{Tx,T^2x\}+\max\{T^2x,Ty\}+\max\{Tx,Ty\}+\varphi(Tx)+\varphi(T^2x)+\varphi(Ty)\\&=\frac{17}{64}x+\frac{3}{8}y.\end{aligned}$$

Next, assume that $y<x$. Then

$$\begin{aligned}&G^\bigstar(G(x,Tx,y),\varphi(x),\varphi(Tx),\varphi(y))\\&=\max\{x,Tx\}+\max\{Tx,y\}+\max\{x,y\}+x+\frac{1}{8}x+y\\&=\frac{25}{8}x+y+\max\{\frac{1}{8}x,y\},\end{aligned}$$

and

$$\begin{aligned}&G^\bigstar(G(Tx,T^2x,Ty),\varphi(Tx),\varphi(T^2x),\varphi(Ty))\\&=\max\{Tx,T^2x\}+\max\{T^2x,Ty\}+\max\{Tx,Ty\}+Tx+T^2x+Ty\\&=\frac{25}{64}x+\frac{1}{8}y+\frac{1}{8}\max\{\frac{1}{8}x,y\}.\end{aligned}$$

Therefore,

$$G^\bigstar(G(Tx,T^2x,Ty),\varphi(Tx),\varphi(T^2x),\varphi(Ty))\leq\psi(G^\bigstar(G(x,Tx,y),\varphi(x),\varphi(Tx),\varphi(y))).$$

Now, if $\alpha(x,y)\geq\eta(x,y)$, then $x,y\in[0,1]$. On the other hand, for all $w\in[0,1]$, we have $Tw\in[0,1]$. Hence $\alpha(Tx,Ty)\geq\eta(Tx,Ty)$. This implies that T is an α-admissible mapping with respect to η. Clearly, $\alpha(0,T0)\geq\eta(0,T0)$. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ with $x_n\to x$ as $n\to+\infty$, then $Tx_n,T^2x_n,T^3x_n\in[0,1]$ for all $n\in\mathbb{N}$ and hence

$$\eta(Tx_n,T^2x_n)\leq\alpha(Tx_n,x)\text{ and }\eta(T^2x_n,T^3x_n)\leq\alpha(T^2x_n,x)$$

hold for all $n\in\mathbb{N}$. Thus all of the hypotheses of Theorem 3.4 hold and T has a fixed point.

4. Consequences

In this section, if we choose opportunely the function $G^\bigstar$, then we obtain different kinds of contractive conditions. By taking $G^\bigstar(a,b,c,d)=(a+b+c)(d+1)$ in Theorem 3.3, we deduce the following result.

Corollary 4.1. *Let (X,G) be a G-complete G-metric space, $\varphi:X\to[0,+\infty)$ be a lower G-semicontinuous mapping and $\alpha,\eta:X^2\to[0,+\infty)$. Let $T:X\to X$ be a self-mapping satisfying the following assertions:*

(i) T is a G-continuous α-admissible mapping with respect to η;

(ii) there exist xz[a] $\psi \in \Psi$ such that for all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, Tx)$, we have

$$G(Tx, T^2x, Ty) \leq \frac{\psi\left((G(x, Tx, y) + \varphi(x) + \varphi(Tx))(\varphi(y) + 1)\right)}{\varphi(Ty) + 1} - \varphi(Tx) - \varphi(T^2x);$$

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$.

Then T has a fixed point $z \in X$ such that $\varphi(z) = 0$. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

By taking $G^\bigstar(a, b, c, d) = (a + b + c)(d + 1)$ in Theorem 3.4, we deduce the following result.

Corollary 4.2. *Let (X, G) be a G-complete G-metric space, $\varphi : X \to [0, +\infty)$ be a lower G-semicontinuous mapping and $\alpha, \eta : X^2 \to [0, +\infty)$. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) there exists a continuous $\psi \in \Psi$ such that for all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, Tx)$, we have

$$G(Tx, T^2x, Ty) \leq \frac{\psi\left((G(x, Tx, y) + \varphi(x) + \varphi(Tx))(\varphi(y) + 1)\right)}{\varphi(Ty) + 1} - \varphi(Tx) - \varphi(T^2x);$$

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to +\infty$, then either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \quad \text{or} \quad \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point $z \in X$ such that $\varphi(z) = 0$. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

By taking $G^\bigstar(a, b, c, d) = a + b + c + d$ in Theorem 3.3, we deduce the following result.

Corollary 4.3. *Let (X, G) be a G-complete G-metric space, $\varphi : X \to [0, +\infty)$ be a lower G-semicontinuous mapping and $\alpha, \eta : X^2 \to [0, +\infty)$. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is a G-continuous α-admissible mapping with respect to η;

(ii) there exists a $\psi \in \Psi$ such that for all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, Tx)$, we have

$$G(Tx, T^2x, Ty) \leq \psi\left(G(x, Tx, y) + \varphi(x) + \varphi(Tx) + \varphi(y)\right) - \varphi(Tx) - \varphi(T^2x) - \varphi(Ty);$$

(iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$.

Then T has a fixed point $z \in X$ such that $\varphi(z) = 0$. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

By taking $G^\bigstar(a, b, c, d) = a + b + c + d$ in Theorem 3.4, we deduce the following result.

Corollary 4.4. *Let (X, G) be a G-complete G-metric space, $\varphi : X \to [0, +\infty)$ be a lower G-semicontinuous mapping and $\alpha, \eta : X^2 \to [0, +\infty)$. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) there exists a continuous $\psi \in \Psi$ such that for all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, Tx)$, we have

$$G(Tx, T^2x, Ty) \leq \psi\left(G(x, Tx, y) + \varphi(x) + \varphi(Tx) + \varphi(y)\right) - \varphi(Tx) - \varphi(T^2x) - \varphi(Ty);$$

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to +\infty$, then either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \quad or \quad \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point $z \in X$ such that $\varphi(z) = 0$. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

From the previous corollary, if we choose $\psi(t) = rt$ for all $t \geq 0$ where $r \in (0, 1)$, we get the following corollary.

Corollary 4.5. *Let (X, G) be a G-complete G-metric space, $\varphi : X \to [0, \infty)$ be a lower G-semicontinuous mapping and $\alpha, \eta : X^2 \to [0, +\infty)$. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) there exists an $r \in (0, 1)$ such that for all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, Tx)$, we have

$$G(Tx, T^2x, Ty) \leq r\,[(x, Tx, y) + \varphi(x) + \varphi(Tx) + \varphi(y)] - \varphi(Tx) - \varphi(T^2x) - \varphi(Ty);$$

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to +\infty$, then either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \quad or \quad \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point $z \in X$ such that $\varphi(z) = 0$. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

5. Suzuky type results

In this section, we give some results on fixed point for self-mappings that satisfy a Suzuki type condition.

Theorem 5.1. *Let (X, G) be a G-complete G-metric space, $\varphi : X \to [0, +\infty)$ be a lower G-semicontinuous mapping. Let $T : X \to X$ be a G-continuous self-mapping such that for all $x, y \in X$ with*

$$G(x, Tx, Tx) + \varphi(Tx) \leq G(x, Tx, y) + \varphi(y),$$

we have

$$G^\star\left(G(Tx, T^2x, Ty), \varphi(Tx), \varphi(T^2x), \varphi(Ty)\right) \leq \psi\left(G^\star(G(x, Tx, y), \varphi(x), \varphi(Tx), \varphi(y))\right).$$

Then T has a unique fixed point $z \in X$ such that $\varphi(z) = 0$.

Proof. First, we note that for each $x \in Fix(T)$, we have $G^\star(G(x, x, x), \varphi(x), \varphi(x), \varphi(x)) = 0$ and hence $\varphi(x) = 0$. Define $\alpha, \eta : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \eta(x, y) = G(x, Tx, y) + \varphi(y)$$

for all $x, y \in X$. Clearly, $\eta(x, y) \leq \alpha(x, y)$ for all $x, y \in X$. This ensures that T is an α-admissible mapping with respect to $\eta(x, y)$ and that condition (iii) of Theorem 3.3 holds. Let $\eta(x, Tx) \leq \alpha(x, y)$; then

$$G(x, Tx, Tx) + \varphi(Tx) \leq G(x, Tx, y) + \varphi(y)$$

and so by assumption, we get

$$G^\star\left(G(Tx, T^2x, Ty), \varphi(Tx), \varphi(T^2x), \varphi(Ty)\right) \leq \psi\left(G^\star(G(x, Tx, y), \varphi(x), \varphi(Tx), \varphi(y))\right).$$

Hence, all the conditions of Theorem 3.3 hold and T has a fixed point. The uniqueness of the fixed point follows since $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$. □

By taking $G^{\bigstar}(a,b,c,d) = (a+b+c)(d+1)$ in Theorem 5.1, we deduce the following result.

Corollary 5.2. *Let (X,G) be a G-complete G-metric space, $\varphi : X \to [0,+\infty)$ be a lower G-semicontinuous mapping. Let $T : X \to X$ be a G-continuous self-mapping such that for all $x, y \in X$ with*

$$G(x,Tx,Tx) + \varphi(Tx) \leq G(x,Tx,y) + \varphi(y),$$

we have

$$G(Tx,T^2x,Ty) \leq \frac{\psi\left(G(x,Tx,y) + \varphi(x) + \varphi(Tx))(\varphi(y)+1)\right)}{\varphi(Ty)+1} - \varphi(Tx) - \varphi(T^2x).$$

Then T has a unique fixed point $z \in X$ such that $\varphi(z) = 0$.

By taking $G^{\bigstar}(a,b,c,d) = a+b+c+d$ in Theorem 5.1, we deduce the following result.

Corollary 5.3. *Let (X,G) be a G-complete G-metric space, $\varphi : X \to [0,+\infty)$ be a lower G-semicontinuous mapping. Let $T : X \to X$ be a G-continuous self-mapping such that for all $x, y \in X$ with*

$$G(x,Tx,Tx) + \varphi(Tx) \leq G(x,Tx,y) + \varphi(y),$$

we have

$$G(Tx,T^2x,Ty) \leq \psi\left(G(x,Tx,y) + \varphi(x) + \varphi(Tx) + \varphi(y)\right) - \varphi(Tx) - \varphi(T^2x) - \varphi(Ty).$$

Then T has a unique fixed point $z \in X$ such that $\varphi(z) = 0$.

Theorem 5.4. *Let (X,G) be a G-complete G-metric space, $\varphi : X \to [0,+\infty)$ be a lower G-semicontinuous mapping. Let $T : X \to X$ be a self-mapping and let there exist an $r \in (0,1)$ such that for all $x, y \in X$ with*

$$\frac{1}{1+2r}[G(x,Tx,Tx) + \varphi(x) + \varphi(Tx) + \varphi(Tx))] \leq G(x,Tx,y) + \varphi(x) + \varphi(Tx) + \varphi(y),$$

we have

$$G(Tx,T^2x,Ty) \leq r\left[G(x,Tx,y) + \varphi(x) + \varphi(Tx) + \varphi(y)\right] - \varphi(Tx) - \varphi(T^2x) - \varphi(Ty).$$

Then T has a fixed point $z \in X$ such that $\varphi(z) = 0$.

Proof. First, we note that for each $x \in Fix(T)$, we have $\varphi(x) = 0$. Define $\alpha, \eta : X \times X \to [0,+\infty)$ by

$$\alpha(x,y) = G(x,Tx,y) + \varphi(x) + \varphi(Tx) + \varphi(y)$$

and

$$\eta(x,y) = \frac{1}{1+2r}(G(x,Tx,y) + \varphi(x) + \varphi(Tx) + \varphi(y))$$

for all $x, y \in X$. Clearly, $\eta(x,y) \leq \alpha(x,y)$ for all $x, y \in X$, that is, T is an α-admissible mapping with respect to $\eta(x,y)$. Let $\{x_n\}$ be a sequence with $x_n \to x$ as $n \to +\infty$. Since

$$\frac{1}{1+2r}[G(Tx_n,T^2x_n,T^2x_n) + \varphi(Tx_n) + 2\varphi(T^2x_n)] \leq G(Tx_n,T^2x_n,T^2x_n) + \varphi(Tx_n) + 2\varphi(T^2x_n)$$

for all $n \in \mathbb{N}$, then by assumption, we get

$$\begin{aligned} &G(T^2x_n,T^3x_n,T^3x_n) + \varphi(T^2x_n) + \varphi(T^3x_n) + \varphi(T^3x_n) \\ &\quad \leq r\left[G(Tx_n,T^2x_n,T^2x_n) + \varphi(Tx_n) + \varphi(T^2x_n) + \varphi(T^2x_n)\right] \end{aligned} \tag{5.1}$$

for all $n \in \mathbb{N}$. Assume there exists $n_0 \in \mathbb{N}$ such that

$$\eta(Tx_{n_0}, T^2x_{n_0}) > \alpha(Tx_{n_0}, x) \text{ and } \eta(T^2x_{n_0}, T^3x_{n_0}) > \alpha(T^2x_{n_0}, x),$$

then

$$\begin{aligned} &\frac{1}{1+2r}\left(G(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}) + \varphi(Tx_{n_0}) + \varphi(T^2x_{n_0}) + \varphi(T^2x_{n_0})\right) \\ &\quad > G(Tx_{n_0}, T^2x_{n_0}, x) + \varphi(Tx_{n_0}) + \varphi(T^2x_{n_0}) + \varphi(x), \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} &\frac{1}{1+2r}\left(G(T^2x_{n_0}, T^3x_{n_0}, T^3x_{n_0}) + \varphi(T^2x_{n_0}) + \varphi(T^3x_{n_0}) + \varphi(T^3x_{n_0})\right) \\ &\quad > G(T^2x_{n_0}, T^3x_{n_0}, x) + \varphi(T^2x_{n_0}) + \varphi(T^3x_{n_0}) + \varphi(x). \end{aligned} \tag{5.3}$$

Since $x_1 = Tx_{n_0} \neq T^2x_{n_0} = x_2$ and $y_1 = z_1 = T^2x_{n_0} \neq T^3x_{n_0} = y_2 = z_2$, by Lemma 2.4, we deduce

$$G(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}) \leq G(Tx_{n_0}, T^2x_{n_0}, x) + 2G(T^2x_{n_0}, T^3x_{n_0}, x). \tag{5.4}$$

Therefore, by (5.1), (5.2), (5.3), and (5.4), we get

$$\begin{aligned} &G(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}) + \varphi(Tx_{n_0}) + \varphi(T^2x_{n_0}) + \varphi(T^2x_{n_0}) \\ &\quad \leq G(Tx_{n_0}, T^2x_{n_0}, x) + 2G(T^2x_{n_0}, T^3x_{n_0}, x) + \varphi(Tx_{n_0}) + \varphi(T^2x_{n_0}) + \varphi(T^2x_{n_0}) \\ &\quad \leq [G(Tx_{n_0}, T^2x_{n_0}, x) + \varphi(Tx_{n_0}) + \varphi(T^2x_{n_0}) + \varphi(x)] \\ &\qquad + 2[G(T^2x_{n_0}, T^3x_{n_0}, x) + \varphi(T^2x_{n_0}) + \varphi(T^3x_{n_0}) + \varphi(x)] \\ &\quad < \frac{1}{1+2r}(G(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}) + \varphi(Tx_{n_0}) + \varphi(T^2x_{n_0}) + \varphi(T^2x_{n_0})) \\ &\qquad + \frac{2}{1+2r}(G(T^2x_{n_0}, T^3x_{n_0}, T^3x_{n_0}) + \varphi(T^2x_{n_0}) + \varphi(T^3x_{n_0}) + \varphi(T^3x_{n_0})) \\ &\quad \leq \frac{1}{1+2r}(G(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}) + \varphi(Tx_{n_0}) + \varphi(T^2x_{n_0}) + \varphi(T^2x_{n_0})) \\ &\qquad + \frac{2r}{1+2r}(G(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}) + \varphi(Tx_{n_0}) + \varphi(T^2x_{n_0}) + \varphi(T^2x_{n_0})) \\ &\quad = G(Tx_{n_0}, T^2x_{n_0}, T^2x_{n_0}) + \varphi(Tx_{n_0}) + \varphi(T^2x_{n_0}) + \varphi(T^2x_{n_0}), \end{aligned}$$

which is a contradictions. Hence, either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \quad \text{or} \quad \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$. Let $\eta(x, Tx) \leq \alpha(x, y)$. Thus

$$\frac{1}{1+2r}[G(x, Tx, Tx) + \varphi(x) +, \varphi(Tx) + \varphi(Tx))] \leq G(x, Tx, y) + \varphi(x) +, \varphi(Tx) + \varphi(y).$$

Then from assumption, we get

$$G(Tx, T^2x, Ty) \leq r\,[G(x, Tx, y) + \varphi(x) + \varphi(Tx) + \varphi(y)] - \varphi(Tx) - \varphi(T^2x) - \varphi(Ty).$$

Hence, all the conditions of Corollary 4.5 hold and T has a fixed point. The uniqueness of the fixed point follows since $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$. □

If in Theorem 5.4 we choose $\varphi(x) = 0$ for all $x \in X$, then we get the following corollary.

Corollary 5.5. *Let (X, G) be a G-complete G-metric space. Let $T : X \to X$ be a self-mapping and let there exist an $r \in (0, 1)$ such that for all $x, y \in X$ with*

$$\frac{1}{1+2r} G(x, Tx, Tx) \leq G(x, Tx, y),$$

we have

$$G(Tx, T^2x, Ty) \leq rG(x, Tx, y).$$

Then T has a unique fixed point.

6. Some results in partial G-metric spaces

In this section, using the previous results, we give some results on fixed points in the setting of partial G-metric spaces.

Theorem 6.1. *Let (X, P) be a $(PG)^*$-complete partial G-metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) there exists a $\psi \in \Psi$ such that for all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, Tx)$, we have

$$P(Tx, T^2x, Ty) \leq \psi(P(x, Tx, y));$$

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) $P(x, x, x) \leq P(x, y, y)$ for all $x, y \in X$;

(v) T is a G_p-continuous mapping in (X, G_p).

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

Proof. Let $\overline{G}(x, y, z) := \frac{G_p(x,y,z)}{3}$ for all $x, y \in X$, $\varphi(x) = \frac{P(x,x,x)}{3}$ for all $x \in X$ and $G^\star(a, b, c, d) = a+b+c+d$. From Lemma 2.6 we get

$$\begin{aligned} P(x, y, z) &= \frac{G_p(x, y, z)}{3} + \frac{P(x, x, x)}{3} + \frac{P(y, y, y)}{3} + \frac{P(z, z, z)}{3} \\ &= \overline{G}(x, y, z) + \varphi(x) + \varphi(y) + \varphi(z). \end{aligned} \tag{6.1}$$

Now, if $\alpha(x, y) \geq \eta(x, Tx)$, then from (ii), we have

$$P(Tx, T^2x, Ty) \leq \psi(P(x, Tx, y))$$

and so from (6.1), we get

$$G^\star\left(\overline{G}(Tx, T^2x, Ty), \varphi(Tx), \varphi(T^2x), \varphi(Ty)\right) \leq \psi\left(G^\star(\overline{G}(x, Tx, y), \varphi(x), \varphi(Tx), \varphi(y))\right).$$

Note that by Lemma 2.11 and (iv), the function φ is lower $\overline{G}$-semicontinuous. Therefore, all the conditions of Theorem 3.3 hold true and T has a fixed point. □

In the following theorem, we omit the G_p-continuity hypothesis of T.

Theorem 6.2. *Let (X, P) be a $(PG)^*$-complete partial G-metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) there exists a $\psi \in \Psi$ such that for all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, Tx)$, we have

$$P(Tx, T^2x, Ty) \leq \psi(P(x, Tx, y));$$

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) $P(x, x, x) \leq P(x, y, y)$ for all $x, y \in X$;

(v) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to +\infty$, then either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \text{ or } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

If in the previous theorem we choose $\psi(t) = rt$ for some $r \in (0, 1)$, then we deduce the following corollary.

Corollary 6.3. *Let (X, P) be a $(PG)^*$-complete partial G-metric space. Let $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) T is an α-admissible mapping with respect to η;

(ii) there exists an $r \in (0, 1)$ such that for all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, Tx)$, we have

$$P(Tx, T^2x, Ty) \leq rP(x, Tx, y);$$

(iii) there exists an $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;

(iv) $P(x, x, x) \leq P(x, y, y)$ for all $x, y \in X$;

(v) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \to x$ as $n \to +\infty$, then either

$$\eta(Tx_n, T^2x_n) \leq \alpha(Tx_n, x) \text{ or } \eta(T^2x_n, T^3x_n) \leq \alpha(T^2x_n, x)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in Fix(T)$.

7. Suzuky type results in partial G-metric spaces

By using a similar proof as in Theorem 6.1 (and applying Theorem 5.1) we can deduce the following Suzuki type result.

Theorem 7.1. *Let (X, P) be a $(PG)^*$-complete partial G-metric space such that $P(x, x, x) \leq P(x, y, y)$ for all $x, y \in X$. Let $T : X \to X$ be a G-continuous self-mapping and let there exist an $r \in (0, 1)$ such that for all $x, y \in X$ with $P(x, Tx, Tx) \leq P(x, Tx, y)$, we have*

$$P(Tx, T^2x, Ty) \leq \psi(P(x, Tx, y)).$$

Then T has a fixed point.

By using a similar proof as that in Theorem 6.1 (and applying Theorem 5.4) we can deduce the following Suzuki type result.

Theorem 7.2. *Let (X, P) be a $(PG)^*$-complete partial G-metric space such that $P(x, x, x) \leq P(x, y, y)$ for all $x, y \in X$. Let $T : X \to X$ be a self-mapping and let there exist an $r \in (0, 1)$ such that for all $x, y \in X$ with $\frac{1}{1+2r}P(x, Tx, Tx) \leq P(x, Tx, y)$, we have*

$$P(Tx, T^2x, Ty) \leq rP(x, Tx, y).$$

Then T has a fixed point.

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References

[1] M. Abbas, T. Nazir, P. Vetro, *Common fixed point results for three maps in G-metric spaces*, Filomat, **25** (2011), 1–17. 1

[2] H. Aydi, W. Shatanawi, C. Vetro, *On generalized weak G-contraction mapping in G-metric spaces*, Comput. Math. Appl., **62** (2011), 4223–4229. 1

[3] F. Moradlou, P. Salimi, P.Vetro, *Some new extensions of Edelstein-Suzuki-type fixed point theorem to G-metric and G-cone metric spaces*, Acta Math. Sci. Ser. B Engl. Ed., **33** (2013), 1049–1058. 2.4

[4] Z. Mustafa, *A new structure for generalized metric spaces with applications to fixed point theory*, PhD Thesis, the University of Newcastle, Australia, (2005). 1, 2

[5] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), 289–297. 1, 2

[6] R. Saadati, S. M. Vaezpour, P. Vetro, B. E. Rhoades, *Fixed point theorems in generalized partially ordered G-metric spaces*, Math. Comput. Modelling, **52** (2010), 797–801. 1

[7] P. Salimi, P. Vetro, *A result of Suzuki type in partial G-metric spaces*, Acta Math. Sci. Ser. B Engl. Ed., **34** (2014), 274–284. 2, 2.5, 2.6, 2.7, 2.8

[8] B. Samet, C. Vetro, F. Vetro, *From metric spaces to partial metric spaces*, Fixed Point Theory Appl., **2013** (2013), 11 pages. 1

[9] C. Vetro, F. Vetro, *Metric or partial metric spaces endowed with a finite number of graphs: a tool to obtain fixed point results*, Topology Appl., **164** (2014), 125–137. 1