Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 940–956
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

# Random coupled and tripled best proximity results with cyclic contraction in metric spaces

Farhana Akbar[a], Marwan Amin Kutbi[b], Masood Hussain Shah[c], Naeem Shafqat[d,*]

[a]Department of Mathematics, GDCW, Bosan Road, Multan, Pakistan.
[b]Department of Mathematics, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia.
[c]Department of Mathematics, SBSSE, Lahore University of Management Sciences, 54792 Lahore, Pakistan.
[d]Centre for Advanced Studies in Pure and Applied Mathematics, Bahauudin Zakariya University, Multan, 60800, Pakistan.

Communicated by N. Hussain

**Abstract**

We consider random best proximity point and cyclic contraction pair problems in uniformly convex Banach spaces. We also prove some tripled best proximity and tripled fixed point theorems in complete metric spaces. Our results present random version of [W. Sintunavarat, P. Kumam, Fixed point Theory Appl., **2012** (2012), 16 pages] and many others. 

*Keywords:* Partially ordered set, coupled best proximity point, tripled best proximity point, random best proximity point.
*2010 MSC:* 47H10, 54H25.

## 1. Introduction and Preliminaries

Random coincidence point theorems are stochastic generalizations of classical coincidence point theorems. Some random fixed point theorems play an important role in the theory of random differential and random integral equations (see [17], [21]). Random fixed point theorems for contractive mappings on separable complete metric spaces have been proved by several authors ([1], [2], [7], [8], [12] and [18]). The stochastic version of the well known Schauder's fixed point theorem was proved by Sehgal and Singh [26]. Ćirić and Lakshmikantham [9], Zhu and Xiao [34], Hussain et al. [16] and Khan et al. [19] proved some coupled

*Corresponding author
*Email addresses:* ridaf75@yahoo.com (Farhana Akbar), mkutbi@ yahoo.com (Marwan Amin Kutbi), mshah@lums.edu.pk (Masood Hussain Shah), naeem781625@yahoo.com (Naeem Shafqat )

*Received 2015-08-01*

random fixed point and coupled random coincidence point results in partially ordered complete metric spaces. Hussain et al. [14] and Kutbi et al. [20] proved coupled and tripled coincidence point results for generalized compatible and hybrid type mappings. In 1969, Fan [11] introduced and established a classical best approximation theorem, that is, if $A$ is a nonempty compact convex subset of a Hausdorff locally convex topological vector space $B$ and $T : A \to B$ is continuous mapping, then there exists an element $x \in A$ such that $d(x, Tx) = d(Tx, A)$. Afterward, many authors have derived extensions of Fan's theorem and best approximation theorem in many directions, such as Prolla [23], Reich [24], Sehgal and Singh [27, 28], Wlodarczyk and Plebaniak [33], Vetrival et al. [31], Eldred and Veeramani [10], Hussain and Hussain et al. [15], [13], Mongkolkeha and Kumam [22] and Sadiq Basha and Veeramani [3], [4], [5], [6].

The purpose of this article is to prove the results for coupled random best proximity points for cyclic contraction for a pair of two binary mappings introduced by W. Sintunavarat and P. Kumam [29]. We prove tripled best proximity and tripled fixed point results in complete metric spaces. Moreover, we apply these results in uniformly convex Banach spaces.

For nonempty subsets $A$ and $B$ of a metric space $(X, d)$, we let

$$d(A, B) := \inf \{d(x, y) : x \in A \text{ and } y \in B\}$$

stand for the distance between $A$ and $B$.

Let

$$B_0 := \{b \in B : d(a, b) = d(A, B) \text{ for an } a \in A\}$$

and

$$A_0 := \{a \in A : d(a, b) = d(A, B) \text{ for a } b \in B\}.$$

**Definition 1.1.** A Banach space $X$ is said to be:

(i) strictly convex if for all $x, y \in X$, $||x|| = ||y|| = 1$ and $x \neq y$ imply that $\left|\left|\frac{x+y}{2}\right|\right| < 1$,

(ii) uniformly convex if for each $\epsilon$ with $0 < \epsilon \leq 2$, there exists a $\delta > 0$ such that

$$||x|| \leq 1,\ ||y|| \leq 1 \text{ and } ||x - y|| \geq \epsilon \Rightarrow \left|\left|\frac{x+y}{2}\right|\right| < 1 - \delta \text{ for all } x, y \in X.$$

It is easy to see that a uniformly convex Banach space is strictly convex, but the converse is not true. Throughout in this article, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

**Definition 1.2** ([30])**.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. The ordered pair $(A, B)$ has the property $UC$ if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ such that $d(x_n, y_n) \to d(A, B)$ and $d(z_n, y_n) \to d(A, B)$, then $d(x_n, z_n) \to 0$.

**Example 1.3** ([30])**.** The following are examples of a pair of nonempty subsets $(A, B)$ having the property $UC$.

(1) Every pair of nonempty subsets $A, B$ of a metric space $(X, d)$ such that $d(A, B) = 0$.
(2) Every pair of nonempty subsets $A, B$ of a uniformly convex Banach space $X$ such that $A$ is convex.
(3) Every pair of nonempty subsets $A, B$ of a strictly convex Banach space where $A$ is convex and relatively compact and the closure of $B$ is weakly compact.

**Definition 1.4.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. The ordered pair $(A, B)$ has the property $UC^*$ if $(A, B)$ has the property $UC$ and the following condition holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ satisfying:

(1) $d(z_n, y_n) \to d(A, B)$;
(2) for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$d(x_m, y_n) \leq d(A, B) + \epsilon$$

for all $m > n \geq N$, then for every $\epsilon > 0$ there exists an $N_1 \in \mathbb{N}$ such that

$$d(x_m, z_n) \leq d(A, B) + \epsilon$$

for all $m > n \geq N_1$.

**Example 1.5.** The following are examples for a pair of nonempty subsets $(A, B)$ having the property $UC^*$:

(1) Every pair of nonempty subsets $A, B$ of a metric space$(X, d)$ such that $d(A, B) = 0$.
(2) Every pair of nonempty closed subsets $A, B$ of a uniformly convex Banach space $X$ such that $A$ is convex [10, Lemma 3.7].

**Definition 1.6.** Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ and $T : A \to B$ a mapping. A point $x \in A$ is said to be a best proximity point of $T$ if

$$d(x, Tx) = d(A, B).$$

**Definition 1.7** ([29])**.** Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $F : A \times A \to B$. A point $(x, x') \in A \times A$ is called a coupled best proximity point of $F$ if

$$d\left(x, F\left(x, x'\right)\right) = d\left(x', F\left(x', x\right)\right) = d(A, B).$$

It is easy to see that if $A = B$ in Definitions 1.6 and 1.7, then a best proximity point (coupled best proximity point) reduces to a fixed point (coupled fixed point).

**Definition 1.8** ([29])**.** Let $A$ and $B$ be nonempty subsets of a metric space $X$, $F : A \times A \to B$ and $G : B \times B \to A$. The ordered pair $(F, G)$ is said to be a cyclic contraction if there exists a nonnegative number $\alpha < 1$ such that

$$d\left(F\left(x, x'\right), G\left(y, y'\right)\right) \leq \frac{\alpha}{2}[d(x, y) + d\left(x', y'\right)] + (1 - \alpha) d(A, B)$$

for all $(x, x') \in A \times A$ and $(y, y') \in B \times B$.

Note that if $(F, G)$ is a cyclic contraction, then $(G, F)$ is also a cyclic contraction.

In [29], Kumam et al. proved the following theorem using cyclic contraction.

**Theorem 1.9.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $(A, B)$ and $(B, A)$ have the property $UC^*$. Let $F : A \times A \to B$, $G : B \times B \to A$ and $(F, G)$ be a cyclic contraction. Let $(x_0, x'_0) \in A \times A$ and define*

$$x_{2n+1} = F(x_{2n}, x'_{2n}),\ x'_{2n+1} = F(x'_{2n}, x_{2n}) \ \text{ and } x_{2n+2} = G\left(x_{2n+1}, x'_{2n+1}\right),\ x'_{2n+1} = G\left(x'_{2n+1}, x_{2n+1}\right)$$

*for all $n \in \mathbb{N} \cup \{0\}$. Then $F$ has a coupled best proximity point $(p, q) \in A \times A$ and $G$ has a coupled best proximity point $(p', q') \in B \times B$ such that $d(p, p') + d(q, q') = 2d(A, B)$. Moreover, we have $x_{2n} \to p$, $x'_{2n} \to q$, $x_{2n+1} \to p'$, $x'_{2n+1} \to q'$.*

## 2. Random best proximity results

Let $(\Omega, \Sigma)$ be a measurable space with $\Sigma$ a sigma algebra of subsets of $\Omega$ and let $(X, d)$ be a metric space. A mapping $T : \Omega \to X$ is called $\Sigma$-measurable if for any open subset $U$ of $X$, one has $T^{-1}(U) = \{\omega : T(\omega) \in U\} \in \Sigma$. In what follows, when we speak of measurability we will mean $\Sigma$-measurability. A mapping $T : \Omega \times X \to X$ is called a random operator if for any $x \in X$, the set $T(., x)$ is measurable. A measurable mapping $\zeta : \Omega \to X$ is called a random fixed point of a random function $T : \Omega \times X \to X$ if $\zeta(\omega) = T(\omega, \zeta(\omega))$ for every $\omega \in \Omega$. A measurable mapping $\zeta : \Omega \to X$ is called a random coincidence of $T : \Omega \times X \to X$ and $g : \Omega \times X \to X$ if $g(\omega, \zeta(\omega)) = T(\omega, \zeta(\omega))$ for every $\omega \in \Omega$.

**Definition 2.1.** Let $A$ and $B$ be nonempty subsets of a separable metric space $(X, d)$ and $(\Omega, \Sigma)$ be a measurable space. If $\zeta, \eta : \Omega \to A$ are measurable mappings, then the random operator $F : \Omega \times (A \times A) \to B$ has a coupled random best proximity point if for each $\omega \in \Omega$, we have

$$d(\zeta(\omega), F(\zeta(\omega), \eta(\omega))) = d(\eta(\omega), F(\eta(\omega), \zeta(\omega)) = d(A, B).$$

**Theorem 2.2.** *Let $(X, d)$ be a complete separable metric space, $(\Omega, \Sigma)$ be a measurable space and $A$ and $B$ be nonempty closed subsets of $X$. Suppose that $F : \Omega \times (A \times A) \to B$ and $G : \Omega \times (B \times B) \to A$ are two random operators. Define*

$$x_{2n+1}(\omega) = F(\omega, (x_{2n}(\omega), y_{2n}(\omega)), \; y_{2n+1}(\omega) = F(\omega, (y_{2n}(\omega), x_{2n}(\omega))) \tag{2.1}$$

*and*

$$x_{2n+2}(\omega) = G(\omega, (x_{2n+1}(\omega), y_{2n+1}\omega)), \; y_{2n+2}(\omega) = G(\omega, (y_{2n+1}(\omega), x_{2n+1}(\omega))) \tag{2.2}$$

*for all $n \in \mathbb{N} \cup \{0\}$ and $\omega \in \Omega$. Let $F$ be continuous and suppose that*

*(i) $F(., v)$ and $G(., u)$ are measurable for all $v \in A \times A$ and $u \in B \times B$ respectively;*

*(ii) $(A, B)$ and $(B, A)$ have the property $UC^*$;*

*(iii) $(F, G)$ is a cyclic contraction.*

*Then $F$ and $G$ have a coupled random best proximity point.*

*Proof.* Let $\Theta = \{\zeta : \Omega \to X\}$ be a family of measurable mappings. Define a function $h = \Omega \times X \to \mathbb{R}^+$ by

$$h(\omega, x) = d(x, F(\omega, x)).$$

Since $x \to F(\omega, x)$ is continuous for all $\omega \in \Omega$, we conclude that $h(\omega, .)$ is continuous for all $\omega \in \Omega$. Also, since $x \to F(\omega, x)$ is measurable for all $x \in X$, we conclude that $h(., x)$ is measurable for all $\omega \in \Omega$ (see [32, p. 868]). Thus $h(\omega, x)$ is the Caratheodory function. Therefore, if $\zeta : \Omega \to X$ is a measurable mapping, then $\omega \to h(\omega, \zeta(\omega))$ is also measurable (see [25]). Also, for each $\zeta \in \Theta$, the function $\eta : \Omega \to X$ defined by $\eta(\omega) = F(\omega, \zeta(\omega))$ is also measurable, that is, $\eta \in \Theta$. Now, we shall construct two sequences $\{\zeta_n(\omega)\}$ and $\{\eta_n(\omega)\}$ of measurable mappings in $\Omega$ and will prove the theorem in three steps:

**Step I**: For each $n \in \mathbb{N} \cup \{0\}$, we have from (2.1) and (2.2)

$$\begin{aligned}
&d(\zeta_{2n}(\omega), \zeta_{2n+1}(\omega)) = d(\zeta_{2n}(\omega), F(\omega, (\zeta_{2n}(\omega), \eta_{2n}(\omega))))\\
&= d(G(\omega, (\zeta_{2n-1}(\omega), \eta_{2n-1}(\omega))), F(\omega, (G(\omega, (\zeta_{2n-1}(\omega), \eta_{2n-1}(\omega))), G(\omega, (\eta_{2n-1}(\omega), \zeta_{2n-1})(\omega)))))\\
&\le \tfrac{\alpha}{2}[d(\zeta_{2n-1}(\omega), G(\omega, (\zeta_{2n-1}(\omega), \eta_{2n-1}(\omega)))) + d(\eta_{2n-1}(\omega), G(\omega, (\eta_{2n-1}(\omega), \zeta_{2n-1}(\omega))))]\\
&\quad + (1-\alpha) d(A, B)\\
&= \tfrac{\alpha}{2}[d(F(\omega, (\zeta_{2n-2}(\omega), \eta_{2n-2}(\omega))), G(\omega, F(\omega, (\zeta_{2n-2}(\omega), \eta_{2n-2}(\omega))), F(\omega, (\eta_{2n-2}(\omega), \zeta_{2n-2}(\omega)))))\\
&\quad + d(F(\omega, (\eta_{2n-2}(\omega), \zeta_{2n-2}(\omega))), G(\omega, F(\omega, (\eta_{2n-2}(\omega), \zeta_{2n-2}(\omega))), F(\omega, (\zeta_{2n-2}(\omega), \eta_{2n-2}(\omega)))))]\\
&\quad + (1-\alpha) d(A, B)\\
&\le \tfrac{\alpha}{2}[\tfrac{\alpha}{2}[d(\zeta_{2n-2}(\omega), F(\omega, (\zeta_{2n-2}(\omega), \eta_{2n-2}(\omega)))) + d(\eta_{2n-2}(\omega), F(\omega, (\eta_{2n-2}(\omega), \zeta_{2n-2}(\omega))))\\
&\quad + (1-\alpha) d(A, B)] + \tfrac{\alpha}{2}[d(\eta_{2n-2}(\omega), F(\omega, (\eta_{2n-2}(\omega), \zeta_{2n-2}(\omega))))\\
&\quad + d(\zeta_{2n-2}(\omega), F(\omega, (\zeta_{2n-2}(\omega), \eta_{2n-2}(\omega)))) + (1-\alpha) d(A, B)]] + (1-\alpha) d(A, B)\\
&= \tfrac{\alpha^2}{2}[d(\zeta_{2n-2}(\omega), F(\omega, (\zeta_{2n-2}(\omega), \eta_{2n-2}(\omega)))) d(\eta_{2n-2}(\omega), F(\omega, (\eta_{2n-2}(\omega), \zeta_{2n-2}(\omega))))]\\
&\quad + \left(1-\alpha^2\right) d(A, B).
\end{aligned}$$

By induction, we can see that

$$\begin{aligned}
d(\zeta_{2n}(\omega), \zeta_{2n+1}(\omega)) \;\le\; & \frac{\alpha^{2n}}{2}[d(\zeta_0(\omega), F(\omega, (\zeta_0(\omega), \eta_0(\omega)))) + d(\eta_0(\omega), F(\omega, (\eta_0(\omega), \zeta_0(\omega))))]\\
& + \left(1-\alpha^{2n}\right) d(A, B).
\end{aligned}$$

Taking $n \to \infty$, we obtain

$$d(\zeta_{2n}(\omega), \zeta_{2n+1}(\omega)) \to d(A, B). \tag{2.3}$$

By similar arguments, we can prove that

$$d\left(\zeta_{2n+1}(\omega),\zeta_{2n+2}\left(\omega\right)\right)\to d\left(A,B\right), \tag{2.4}$$

$$d\left(\eta_{2n}(\omega),\eta_{2n+1}(\omega)\right)\to d\left(A,B\right), \tag{2.5}$$

$$d\left(\eta_{2n+1}(\omega),\eta_{2n+2}(\omega)\right)\to d\left(A,B\right). \tag{2.6}$$

Now, we have to show that for every $\epsilon>0$, there exists a positive integer $N_0$ such that for all $m>n>N_0$,

$$\tfrac{1}{2}[d\left(\eta_{2m}(\omega),\eta_{2n+1}(\omega)\right)+d\left(\zeta_{2m}\left(\omega\right),\zeta_{2n+1}(\omega)\right)]<d\left(A,B\right)+\epsilon. \tag{2.7}$$

Since the pairs $(A,B)$ and $(B,A)$ have the property $UC$, therefore from (2.3), (2.4), (2.5), and (2.6) we get $d\left(\zeta_{2n},\zeta_{2n+2}\right)\to 0$, $d\left(\eta_{2n},\eta_{2n+2}\right)\to 0$, $d\left(\zeta_{2n+1},\zeta_{2n+3}\right)\to 0$ and $d\left(\eta_{2n+1},\eta_{2n+3}\right)\to 0$. Assume contrary that (2.7) does not hold. Then there would exists an $\epsilon'>0$ such that for all $k\in\mathbb{N}$, there would be an $m_k>n_k\geq k$ satisfying

$$\frac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k}(\omega),\zeta_{2n_k+1}(\omega)\right)]\geq d\left(A,B\right)+\epsilon'$$

and

$$\frac{1}{2}[d\left(\eta_{2m_k-2}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k-2}(\omega),\zeta_{2n_k+1}(\omega)\right)]<d\left(A,B\right)+\epsilon'.$$

That is, we would have

$$\begin{aligned} d\left(A,B\right)+\epsilon' &\leq \frac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k}(\omega),\zeta_{2n_k+1(\omega)}\right)]\\ &\leq \frac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2m_k-2}(\omega)\right)+d\left(\eta_{2m_k-2}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k}(\omega),\zeta_{2m_k-2}(\omega)\right)\\ &\quad +d\left(\zeta_{2m_k-2}(\omega),\zeta_{2n_k+1}(\omega)\right)]\\ &< \frac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2m_k-2}(\omega)\right)+d\left(\zeta_{2m_k}(\omega),\zeta_{2m_k-2}(\omega)\right)]+d\left(A,B\right)+\epsilon'. \end{aligned}$$

Letting $k\to\infty$, we would have

$$\frac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k}(\omega),\zeta_{2n_k+1}(\omega)\right)]\to d\left(A,B\right)+\epsilon'. \tag{2.8}$$

By using the triangle inequality, we would get

$$\begin{aligned} &\tfrac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k}(\omega),\zeta_{2n_k+1}(\omega)\right)]\\ &\leq \tfrac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2m_k+2}(\omega)\right)+d\left(\eta_{2m_k+2}(\omega),\eta_{2n_k+3}(\omega)\right)+d\left(\eta_{2n_k+3}(\omega),\eta_{2n_k+1}(\omega)\right)\\ &\quad +d\left(\zeta_{2m_k-2}(\omega),\zeta_{2m_k+2}(\omega)\right)+d\left(\zeta_{2m_k+2}(\omega),\zeta_{2n_k+3}(\omega)\right)+d\left(\zeta_{2n_k+3}(\omega),\zeta_{2n_k+1}(\omega)\right)]\\ &= \tfrac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2m_k+2}(\omega)\right)+d\left(G\left(\omega,\left(\eta_{2m_k+1}(\omega),\zeta_{2m_k+1}\left(\omega\right)\right)\right),F\left(\omega,\left(\eta_{2n_k+2}(\omega),\zeta_{2n_k+2}(\omega)\right)\right)\right)\\ &\quad +d\left(\eta_{2n_k+3}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k}(\omega),\zeta_{2m_k+2}(\omega)\right)\\ &\quad +d\left(G\left(\omega,\left(\zeta_{2m_k+1}(\omega),\zeta_{2m_k+1}(\omega)\right)\right),F\left(\omega,\left(\zeta_{2n_k+2}(\omega),\eta_{2m_k+2}(\omega)\right)\right)\right)+d\left(\zeta_{2n_k+3}(\omega),\zeta_{2n_k+1}(\omega)\right)]\\ &\leq \tfrac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2m_k+2}(\omega)\right)+\tfrac{\alpha}{2}[d\left(\eta_{2m_k+1}(\omega),\eta_{2n_k+2}(\omega)\right)+d\left(\zeta_{2m_k+1}(\omega),\zeta_{2n_k+2}(\omega)\right)]\\ &\quad +(1-\alpha)\,d\left(A,B\right)+d\left(\eta_{2n_k+3}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k-2}(\omega),\zeta_{2m_k+2}(\omega)\right)+\tfrac{\alpha}{2}[d\left(\zeta_{2m_k+1}(\omega),\zeta_{2n_k+2}(\omega)\right)\\ &\quad +d\left(\eta_{2m_k+1}(\omega),\eta_{2n_k+2}(\omega)\right)]+(1-\alpha)\,d\left(A,B\right)+d\left(\zeta_{2n_k+3}(\omega),\zeta_{2n_k+1}\left(\omega\right)\right)]\\ &= \tfrac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2m_k+2}(\omega)\right)+d\left(\eta_{2n_k+3}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k}(\omega),\zeta_{2m_k+2}(\omega)\right)\\ &\quad +d\left(\zeta_{2n_k+3}(\omega),\zeta_{2n_k+1}(\omega)\right)]+\tfrac{\alpha}{2}[d\left(\zeta_{2m_k+1}(\omega),\zeta_{2n_k+2}(\omega)\right)+d\left(\eta_{2m_k+1}(\omega),\eta_{2n_k+2}(\omega)\right)]+(1-\alpha)\,d\left(A,B\right)\\ &= \tfrac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2m_k+2}(\omega)\right)+d\left(\eta_{2n_k+3}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k}(\omega),\zeta_{2m_k+2}(\omega)\right)\\ &\quad +d\left(\zeta_{2n_k+3}(\omega),\zeta_{2n_k+1}(\omega)\right)]+\tfrac{\alpha}{2}[d\left(F\left(\omega,\left(\zeta_{2m_k}(\omega),\eta_{2m_k}(\omega)\right)\right),G\left(\omega,\left(\zeta_{2n_k+1}\left(\omega\right),\eta_{2n_k+1}(\omega)\right)\right)\right)\\ &\quad +d\left(F\left(\omega,\left(\eta_{2m_k}(\omega),\zeta_{2m_k}(\omega)\right)\right),G\left(\omega,\left(\eta_{2n_k+1}(\omega),\zeta_{2n_k+1}(\omega)\right)\right)\right)]+(1-\alpha)\,d\left(A,B\right)\\ &\leq \tfrac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2m_k+2}(\omega)\right)+d\left(\eta_{2n_k+3}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k}(\omega),\zeta_{2m_k+2}(\omega)\right)\\ &\quad +d\left(\zeta_{2n_k+3}(\omega),\zeta_{2n_k+1}(\omega)\right)]+\tfrac{\alpha}{2}[\tfrac{\alpha}{2}[d\left(\zeta_{2m_k}\left(\omega\right),\zeta_{2n_k+1}(\omega)\right),d\left(\eta_{2m_k+1}(\omega),\eta_{2n_k+1}(\omega)\right)+(1-\alpha)\,d\left(A,B\right)]\\ &\quad +\tfrac{\alpha}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2n_k+1}(\omega)\right),d\left(\zeta_{2m_k+1}(\omega),\zeta_{2n_k+1}(\omega)\right)+(1-\alpha)\,d\left(A,B\right)]]+(1-\alpha)\,d\left(A,B\right)\\ &= \tfrac{1}{2}[d\left(\eta_{2m_k}(\omega),\eta_{2m_k+2}(\omega)\right)+d\left(\eta_{2n_k+3}(\omega),\eta_{2n_k+1}(\omega)\right)+d\left(\zeta_{2m_k}(\omega),\zeta_{2m_k+2}(\omega)\right)\\ &\quad +d\left(\zeta_{2n_k+3}(\omega),\zeta_{2n_k+1}(\omega)\right)]+\tfrac{\alpha^2}{2}[d\left(\zeta_{2m_k}(\omega),\zeta_{2n_k+1}(\omega)\right)+d\left(\eta_{2m_k}(\omega),\eta_{2n_k+1}(\omega)\right)]+\left(1-\alpha^2\right)d\left(A,B\right). \end{aligned}$$

Taking $k \to \infty$, we would get

$$\begin{aligned} d(A,B)+\epsilon' &\le \alpha^2[d(A,B)+\epsilon'] + \left(1-\alpha^2\right) d(A,B) \\ &= d(A,B)+\alpha^2\epsilon', \end{aligned}$$

which is a contradiction. Therefore, we can conclude that (2.7) holds.

**Step II**: Now, we will show that $\{\zeta_{2n}(\omega)\}, \{\eta_{2n}(\omega)\}, \{\zeta_{2n+1}(\omega)\}$, and $\{\eta_{2n+1}(\omega)\}$ are Cauchy sequences. Since from (2.3) and (2.4), we have $d(\zeta_{2n}, \zeta_{2n+1}) \to d(A,B)$ and $d(\zeta_{2n+1}, \zeta_{2n+2}) \to d(A,B)$ and $(A,B)$ has the property $UC^*$, we get $d(\zeta_{2n}, \zeta_{2n+2}) \to 0$. As $(B,A)$ has the same property, we have $d(\zeta_{2n+1}, \zeta_{2n+3}) \to 0$. Here, we show that for every $\epsilon > 0$ there exists an $N$ such that

$$d(\zeta_{2m}(\omega), \zeta_{2n+1}(\omega)) \le d(A,B)+\epsilon \tag{2.9}$$

for all $m > n \ge N$. Assume contrary, that there exists an $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there exists an $m_k > n_k \ge k$ such that

$$d(\zeta_{2m_k}(\omega), \zeta_{2n_k+1}(\omega)) > d(A,B)+\epsilon.$$

Now, we would have

$$\begin{aligned} d(A,B)+\epsilon &< d(\zeta_{2m_k}(\omega), \zeta_{2n_k+1}(\omega)) \\ &\le d(\zeta_{2m_k}(\omega), \zeta_{2n_k-1}(\omega)) + d(\zeta_{2n_k-1}(\omega), \zeta_{2n_k+1}(\omega)) \\ &\le d(A,B)+\epsilon + d(\zeta_{2n_k-1}(\omega), \zeta_{2n_k+1}(\omega)). \end{aligned}$$

Taking $k \to \infty$, we would get $d(\omega, (\zeta_{2m_k}(\omega), \zeta_{2n_k+1}(\omega))) \to d(A,B)+\epsilon$. By using the triangle inequality and (2.7) we would have,

$$\begin{aligned} & d(\zeta_{2m_k}(\omega), \zeta_{2n_k+1}(\omega)) \\ & \le d(\zeta_{2m_k}(\omega), \zeta_{2m_k+2}(\omega)) + d(\zeta_{2m_k+2}(\omega), \zeta_{2n_k+3}(\omega)) + d(\zeta_{2n_k+3}(\omega), \zeta_{2n_k+1}(\omega)) \\ & = d(\zeta_{2m_k}(\omega), \zeta_{2m_k+2}(\omega)) + d(G(\omega, (\zeta_{2m_k+1}(\omega), \eta_{2m_k+1}(\omega))), F(\omega, (\zeta_{2n_k+2}(\omega), \eta_{2n_k+2}(\omega)))) \\ & \quad + d(\zeta_{2n_k+3}(\omega), \zeta_{2n_k+1}(\omega)) \\ & \le d(\zeta_{2m_k}(\omega), \zeta_{2m_k+2}(\omega)) + \tfrac{\alpha}{2}[d(\zeta_{2m_k+1}(\omega), \zeta_{2n_k+2}(\omega)) + d(\eta_{2m_k+1}(\omega), \eta_{2n_k+2}(\omega))] + (1-\alpha)\, d(A,B)) \\ & \quad + d(\zeta_{2n_k+3}(\omega), \zeta_{2n_k+1}(\omega)). \\ & = \tfrac{\alpha}{2}[d(F(\omega, (\zeta_{2m_k}(\omega), \eta_{2m_k}(\omega))), G(\omega, (\zeta_{2n_k+1}(\omega), \eta_{2n_k+1}(\omega)))) \\ & \quad + d(F(\omega, (\eta_{2m_k}(\omega), \zeta_{2m_k}(\omega))), G(\omega, (\eta_{2n_k+1}(\omega), \zeta_{2n_k+1}(\omega))))] + (1-\alpha)\, d(A,B) \\ & \quad + d(\zeta_{2m_k}(\omega), \zeta_{2m_k+2}(\omega)) + d(\zeta_{2n_k+3}(\omega), \zeta_{2n_k+1}(\omega)) \\ & \le \tfrac{\alpha}{2}[\tfrac{\alpha}{2}[d(\zeta_{2m_k}(\omega), \zeta_{2n_k+1}(\omega)) + d(\eta_{2m_k}(\omega), \eta_{2n_k+1}(\omega)) + (1-\alpha)\, d(A,B)] \\ & \tfrac{\alpha}{2}[d(\eta_{2m_k}(\omega), \eta_{2n_k+1}(\omega)) + d(\zeta_{2m_k}(\omega), \zeta_{2n_k+1}(\omega)) + (1-\alpha)\, d(A,B)]] + (1-\alpha)\, d(A,B) \\ & \quad + d(\zeta_{2m_k}(\omega), \zeta_{2m_k+2}(\omega)) + d(\zeta_{2n_k+3}(\omega), \zeta_{2n_k+1}(\omega)). \\ & = \alpha^2 \tfrac{1}{2}[d(\zeta_{2m_k}(\omega), \zeta_{2n_k+1}(\omega)) + d(\eta_{2m_k}(\omega), \eta_{2n_k+1}(\omega))] + \left(1-\alpha^2\right) d(A,B) + d(\zeta_{2m_k}(\omega), \zeta_{2m_k+2}(\omega)) \\ & \quad + d(\zeta_{2n_k+3}(\omega), \zeta_{2n_k+1}(\omega)) \\ & < \alpha^2 (d(A,B)+\epsilon) + \left(1-\alpha^2\right) d(A,B) + d(\zeta_{2m_k}(\omega), \zeta_{2m_k+2}(\omega)) + d(\zeta_{2n_k+3}(\omega), \zeta_{2n_k+1}(\omega)) \\ & = d(A,B) + \alpha^2\epsilon + d(\zeta_{2m_k}(\omega), \zeta_{2m_k+2}(\omega)) + d(\zeta_{2n_k+3}(\omega), \zeta_{2n_k+1}(\omega)). \end{aligned}$$

Taking $k \to \infty$, we would get

$$d(A,B)+\epsilon \le d(A,B)+\alpha^2\epsilon,$$

which contradicts the assumption. Therefore, condition (2.9) holds. By (2.9) and $d((\zeta_{2n}(\omega), \zeta_{2n+1}(\omega))) \to d((A,B))$ and using the property $UC^*$ of $(A,B)$, we have that $\{\zeta_{2n}(\omega)\}$ is a Cauchy sequence. In a similar way, we can prove that $\{\eta_{2n}(\omega)\}, \{\zeta_{2n+1}(\omega)\}$ and $\{\eta_{2n+1}(\omega)\}$ are Cauchy sequences.

**Step III:** Since $A$ and $B$ are subsets of a complete metric space $X$, therefore there exist $\zeta(\omega)$ and $\eta(\omega)$ such that $\zeta_{2n}(\omega) \to \zeta(\omega)$ and $\eta_{2n}(\omega) \to \eta(\omega)$. We have

$$d(A,B) \le d(\zeta(\omega), \zeta_{2n-1}(\omega)) \le d(\zeta(\omega), \zeta_{2n}(\omega)) + d(\zeta_{2n}(\omega), \zeta_{2n-1}(\omega)).$$

Letting $n \to \infty$, we get $d(\zeta(\omega), \zeta_{2n-1}(\omega)) \to d(A,B)$. By a similar argument, we can also get $d(\eta(\omega), \eta_{2n-1}(\omega)) \to d(A,B)$. It follows that

$$\begin{aligned}
&d(\zeta_{2n}(\omega), F(\omega, (\zeta(\omega), \eta(\omega))))\\
&= d(G(\omega, (\zeta_{2n-1}(\omega), \eta_{2n-1}(\omega))), F(\omega, (\zeta(\omega), \eta(\omega))))\\
&\le \tfrac{\alpha}{2}[d(\zeta_{2n-1}(\omega), \zeta(\omega)) + d(\eta_{2n-1}(\omega), \eta(\omega))] + (1-\alpha)\, d(A,B).
\end{aligned}$$

Taking $n \to \infty$, we get

$$d(\zeta(\omega), F(\omega, (\zeta(\omega), \eta(\omega)))) = d(A,B).$$

Similarly, we can prove that $d(\eta(\omega), F(\omega, (\eta(\omega), \zeta(\omega)))) = d(A,B)$. Therefore, we have that $(\zeta(\omega), \eta(\omega))$ is a coupled random best proximity point of $F$. By the same argument, we can prove that there exist $\zeta'(\omega)$, $\eta'(\omega) \in B$ such that $\zeta_{2n+1}(\omega) \to \zeta'(\omega)$ and $\eta'_{2n+1}(\omega) \to \eta'(\omega)$.

Moreover, we also have $d(\zeta'(\omega), G(\omega, (\zeta'(\omega), \eta'(\omega)))) = d(A,B)$ and $d(\eta'(\omega), G(\omega, (\eta'(\omega), \zeta'(\omega)))) = d(A,B)$ and so $(\zeta'(\omega), \eta'(\omega))$ is a coupled random best proximity point of $G$. □

Here, we note that if $(A,B)$ is a pair of nonempty closed subsets of a uniformly convex Banach space $X$ such that $A$ is convex, then $(A,B)$ has the property $UC^*$. Then, we have the following corollary.

**Corollary 2.3.** *Let $(X,d)$ be a complete separable metric space, $(\Omega, \Sigma)$ a measurable space and let $A$ and $B$ be nonempty closed subsets of a uniformly convex separable Banach space $X$. Suppose that $F : \Omega \times (A \times A) \to B$ and $G : \Omega \times (B \times B) \to A$ are two random operators. Define*

$$x_{2n+1}(\omega) = F(\omega, (x_{2n}(\omega), y_{2n}(\omega))),\ y_{2n+1}(\omega) = F(\omega, (y_{2n}(\omega), x_{2n}(\omega)))$$

*and*

$$x_{2n+2}(\omega) = G(\omega, (x_{2n+1}(\omega), y_{2n+1}\omega)),\ y_{2n+2}(\omega) = G(\omega, (y_{2n+1}(\omega), x_{2n+1}(\omega)))$$

*for all $n \in \mathbb{N} \cup \{0\}$ and $\omega \in \Omega$. Let $F$ be continuous and suppose that*

*(i) $F(., v)$ and $G(., u)$ are measurable for all $v \in A \times A$ and $u \in B \times B$ respectively;*

*(ii) $(F,G)$ is a cyclic contraction.*

*Then $F$ and $G$ have a coupled random best proximity point.*

**Theorem 2.4.** *Let $(X,d)$ be a separable metric space, $(\Omega, \Sigma)$ a measurable space and let $A$ and $B$ be nonempty compact subsets of $X$. Suppose that $F : \Omega \times (A \times A) \to B$ and $G : \Omega \times (B \times B) \to A$ are two random operators. Define*

$$x_{2n+1}(\omega) = F(\omega, (x_{2n}(\omega), y_{2n}(\omega))),\ y_{2n+1}(\omega) = F(y_{2n}(\omega), x_{2n}(\omega))$$

*and*

$$x_{2n+2}(\omega) = G(x_{2n+1}(\omega), y_{2n+1}(\omega)),\ y_{2n+2}(\omega) = G(y_{2n+1}(\omega), x_{2n+1}(\omega))$$

*for all $n \in \mathbb{N} \cup \{0\}$ and $\omega \in \Omega$. Let $F$ be continuous and suppose that*

*(i) $F(., v)$ and $G(., u)$ are measurable for all $v \in A \times A$ and $u \in B \times B$ respectively;*

*(ii) $(F,G)$ is a cyclic contraction.*

*Then $F$ and $G$ have a coupled random best proximity point.*

*Proof.* As in Theorem 2.2, we have that $\zeta, \eta : \Omega \to X$ are measurable mappings and

$$\zeta_{2n+1}(\omega) = F(\omega, (\zeta_{2n}(\omega), \eta_{2n}(\omega))), \eta_{2n+1}(\omega) = F(\eta_{2n}(\omega), \zeta_{2n}(\omega))$$

and

$$\zeta_{2n+2}(\omega) = G(\zeta_{2n+1}(\omega), \eta_{2n+1}(\omega)),\ \eta_{2n+2}(\omega) = G(\eta_{2n+1}(\omega), \zeta_{2n+1}(\omega))$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $\zeta_{2n}(\omega), \eta_{2n}(\omega) \in A$ and $\zeta_{2n+1}(\omega), \eta_{2n+1}(\omega) \in B$ for all $n \in \mathbb{N} \cup \{0\}$. Since $A$ is compact, the sequences $\{\zeta_{2n}(\omega)\}$ and $\{\eta_{2n}(\omega)\}$ have convergent subsequences $\{\zeta_{2n_k}(\omega)\}$ and $\{\eta_{2n_k}(\omega)\}$ respectively, that is, $\zeta_{2n_k}(\omega) \to \zeta(\omega)$ and $\eta_{2n_k}(\omega) \to \eta(\omega)$. Now, we have

$$d(A,B) \le d(\zeta(\omega), \zeta_{2n_k-1}(\omega)) \le d(\zeta(\omega), \zeta_{2n_k}(\omega)) + d(\zeta_{2n_k}(\omega), \zeta_{2n_k-1}(\omega)). \tag{2.10}$$

By (2.3), we have $d(\zeta_{2n_k}(\omega), \zeta_{2n_k-1}(\omega)) \to d(A,B)$. Taking $k \to \infty$ in (2.10), we get $d(\zeta(\omega), \zeta_{2n_k-1}(\omega)) \to d(A,B)$. Similarly, we can prove that $d(\eta(\omega), \zeta_{2n_k-1}(\omega)) \to d(A,B)$. Note that

$$\begin{aligned}
&d(A,B) \le d(\zeta_{2n_k}(\omega), F(\omega, (\zeta(\omega), \eta(\omega))))\\
&= d(G(\omega, (\zeta_{2n_k-1}(\omega), \eta_{2n_k-1}(\omega))), F(\omega, (\zeta(\omega), \eta(\omega))))\\
&\le \tfrac{\alpha}{2}[d(\zeta_{2n_k-1}(\omega), \zeta(\omega)) + d(\eta_{2n_k-1}(\omega), \eta(\omega))] + (1-\alpha)\, d(A,B).
\end{aligned}$$

Taking $k \to \infty$ in the above inequality, we get $d(\zeta(\omega), F(\zeta(\omega), \eta(\omega))) = d(A,B)$. Using the same argument, we can prove that $d(\eta(\omega), F(\eta(\omega), \zeta(\omega))) = d(A,B)$. Thus $F$ has a coupled random best proximity point $(\zeta(\omega), \eta(\omega))$. In a similar way, since $B$ is compact, we can prove that $G$ has a coupled random best proximity point. □

## 3. Tripled best proximity point results

In this section, we study the existence and convergence of tripled best proximity points for cyclic contraction pairs. We begin by the notion of tripled best proximity point.

**Definition 3.1.** Let $A$ and $B$ be nonempty subsets of a metric space $X$ and $F : A \times A \times A \to B$. A point $(x, y, z) \in A \times A \times A$ is called a tripled best proximity point of $F$ if

$$d(x, F(x,y,z)) = d(y, F(y,x,z)) = d(z, F(z,y,x)) = d(A,B).$$

It is easy to see that if $A = B$ in the definition above, then a tripled best proximity point reduces to a tripled fixed point.

Next, we introduce the notion of a cyclic contraction for a pair of two binary mappings.

**Definition 3.2.** Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$ and $F : A \times A \times A \to B$, and $G : B \times B \times B \to A$ two mappings. The ordered pair $(F,G)$ is said to be cyclic $\alpha$-contractive if there exists a scalar $\alpha$ with $0 \le \alpha < 1$ such that

$$d(F(x,y,z), G(u,v,w)) \le \frac{\alpha}{3}[d(x,u) + d(y,v) + d(z,w)] + (1-\alpha)d(A,B)$$

for all $(x,y,z) \in A \times A \times A$ and $(u,v,w) \in B \times B \times B$.

Observe that if $(F,G)$ is cyclic $\alpha$-contractive, then so is $(G,F)$.

**Example 3.3.** Let $X = \mathbb{R}$ with the usual metric $d(x,y) = |x-y|$ and let $A = [2,4]$ and $B = [-4,-2]$. Then $d(A,B) = 4$. Define $F : A \times A \times A \to B$ and $G : B \times B \times B \to A$ by

$$F(x,y,z) = \frac{-(x+y+z+6)}{6} \text{ and } G(x,y,z) = \frac{-(x+y+z-6)}{6}.$$

Then for $(x,y,z) \in A \times A \times A$, $(u,v,w) \in B \times B \times B$ and for $\alpha = \frac{1}{2}$, we have

$$\begin{aligned}
d(F(x,y,z), G(u,v,w)) &= \left| \frac{-(x+y+z+6)}{6} - \frac{-(u+v+w-6)}{6} \right|\\
&\le \frac{|x-u| + |y-v| + |z-w|}{6} + \frac{1}{2}(4)\\
&= \frac{\alpha}{3}[d(x,u) + d(y,v) + d(z,w)] + (1-\alpha)d(A,B).
\end{aligned}$$

Thus $(F,G)$ is cyclic $\frac{1}{2}$-contractive.

**Example 3.4.** Let $X=\mathbb{R}^3$ with the metric $d(F(x,y,z),G(u,v,w))=\max\{|x-u|,|y-v|,|z-w|\}$ and let $A=\{(x,0,0):0\le x\le 1\}$ and $B=\{(x,0,1):0\le x\le 1\}$. It is easy to prove that $d(A,B)=1$. Define the mappings $F:A\times A\times A\to B$ and $G:B\times B\times B\to A$ by

$$F((x,0,0),(y,0,0),(z,0,0))=\left(\frac{x+y+z}{3},0,1\right) \text{ and } G((x,0,1),(y,0,1),(z,0,1))=\left(\frac{x+y+z}{3},0,0\right).$$

Then

$$\begin{aligned}&d(F((x,0,0),(y,0,0),(z,0,0)),G((u,0,1),(v,0,1),(w,0,1)))\\&=d\left(\left(\frac{x+y+z}{3},0,1\right),\left(\frac{u+v+w}{3},0,0\right)\right)=1,\end{aligned}$$

and for all $\alpha>0$, we get

$$\begin{aligned}&\frac{\alpha}{3}[d((x,0,0),(u,0,1))+d((y,0,0),(v,0,1)+d((z,0,0),(w,0,1))]+(1-\alpha)d(A,B)\\&\quad=\frac{\alpha}{3}[\max\{|x-u|,0,1\}+\max\{|y-v|,0,1\}+\max\{|z-w|,0,1\}]+(1-\alpha)d(A,B)\\&\quad=\frac{\alpha}{3}\times 3+(1-\alpha)=1.\end{aligned}$$

This implies that $(F,G)$ is cyclic $\alpha$-contractive.

The following lemma is very useful to prove our main results.

**Lemma 3.5.** *Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$, $F:A\times A\times A\to B$, $G:B\times B\times B\to A$ two mappings and $(F,G)$ cyclic $\alpha$-contractive. Let $(x_0,y_0,z_0)\in A\times A\times A$. If for each $n\ge 0$, we define*

$$\begin{aligned}&x_{2n+1}=F(x_{2n},y_{2n},z_{2n}),\quad y_{2n+1}=F(y_{2n},x_{2n},z_{2n}),\quad z_{2n+1}=F(z_{2n},y_{2n},x_{2n}) \text{ and}\\&x_{2n+2}=G(x_{2n+1},y_{2n+1},z_{2n+1}),\quad y_{2n+2}=G(y_{2n+1},x_{2n+1},z_{2n+1}),\quad z_{2n+2}=G(z_{2n+1},y_{2n+1},x_{2n+1}),\end{aligned}$$

*then*

$$\begin{aligned}\lim_{n\to\infty}d(x_{2n},x_{2n+1})&=d(A,B)=\lim_{n\to\infty}d(x_{2n+1},x_{2n+2});\\\lim_{n\to\infty}d(y_{2n},y_{2n+1})&=d(A,B)=\lim_{n\to\infty}d(y_{2n+1},y_{2n+2});\\\lim_{n\to\infty}d(z_{2n},z_{2n+1})&=d(A,B)=\lim_{n\to\infty}d(z_{2n+1},z_{2n+2}).\end{aligned}$$

*Proof.* For each $n\ge 0$, we have

$$\begin{aligned}&d(x_{2n},x_{2n+1})=d(x_{2n},F(x_{2n},y_{2n},z_{2n}))\\&=d(G(x_{2n-1},y_{2n-1},z_{2n-1}),F(G(x_{2n-1},y_{2n-1},z_{2n-1}),G(y_{2n-1},x_{2n-1},z_{2n-1}),G(z_{2n-1},y_{2n-1},x_{2n-1})))\\&\le\tfrac{\alpha}{3}[d(x_{2n-1},G(x_{2n-1},y_{2n-1},z_{2n-1}))+d(y_{2n-1},G(y_{2n-1},x_{2n-1},z_{2n-1})\\&\quad+d(z_{2n-1},G(z_{2n-1},y_{2n-1},x_{2n-1}))]+(1-\alpha)d(A,B)\\&=\tfrac{\alpha}{3}[d(F(x_{2n-2},y_{2n-2},z_{2n-2}),G(F(x_{2n-2},y_{2n-2},z_{2n-2}),F(y_{2n-2},x_{2n-2},z_{2n-2}),F(z_{2n-2},y_{2n-2},x_{2n-2})))\\&\quad+d(F(y_{2n-2},x_{2n-2},z_{2n-2}),G(F(y_{2n-2},x_{2n-2},z_{2n-2}),F(x_{2n-2},y_{2n-2},z_{2n-2}),F(z_{2n-2},y_{2n-2},x_{2n-2})))\\&\quad+d(F(z_{2n-2},y_{2n-2},z_{2n-2}),G(F(z_{2n-2},y_{2n-2},x_{2n-2}),F(y_{2n-2},x_{2n-2},z_{2n-2}),F(x_{2n-2},y_{2n-2},z_{2n-2})))]\\&\quad+(1-\alpha)d(A,B)\\&\le\tfrac{\alpha}{3}[\tfrac{\alpha}{3}[dx_{2n-2},F(x_{2n-2},y_{2n-2},z_{2n-2})+d(y_{2n-2},F(y_{2n-2},x_{2n-2},z_{2n-2})\\&\quad+d(z_{2n-2},Fz_{2n-2},y_{2n-2},x_{2n-2})+(1-\alpha)d(A,B)+\tfrac{\alpha}{3}[d(y_{2n-2},F(y_{2n-2},x_{2n-2},z_{2n-2})\\&\quad+d(x_{2n-2},F(x_{2n-2},y_{2n-2},z_{2n-2}))+d(z_{2n-2},F(z_{2n-2},y_{2n-2},x_{2n-2}))]+(1-\alpha)d(A,B)\\&\quad+\tfrac{\alpha}{3}[d(z_{2n-2},,F(z_{2n-2},y_{2n-2},x_{2n-2}))+d(y_{2n-2},F(y_{2n-2},x_{2n-2},z_{2n-2}))\\&\quad+d(x_{2n-2},Fx_{2n-2},y_{2n-2},z_{2n-2})]+(1-\alpha)d(A,B)]+(1-\alpha)d(A,B)\\&=\tfrac{\alpha^2}{3}[d(x_{2n-2},F(x_{2n-2},y_{2n-2},z_{2n-2})+d(y_{2n-2},F(y_{2n-2},x_{2n-2},z_{2n-2})\\&\quad+d(z_{2n-2},F(z_{2n-2},y_{2n-2},x_{2n-2}))]+\left(1-\alpha^2\right)d(A,B).\end{aligned}$$

Using induction on $n$, we get

$$d\left(x_{2n}, x_{2n+1}\right) \leq \frac{\alpha^{2n}}{3}\left[d\left(x_0, F\left(x_0, y_0, z_0\right)\right) + d\left(y_0, F\left(y_0, x_0, z_0\right)\right) + d(z_0, F(z_0, y_0, x_0)\right] + \left(1 - \alpha^{2n}\right) d\left(A, B\right).$$

Since $\alpha < 1$ and taking $n \to \infty$, we have

$$\lim_{n\to\infty} d\left(x_{2n}, x_{2n+1}\right) = d\left(A, B\right).$$

By the same arguments, we can prove that

$$\begin{aligned}\lim_{n\to\infty} d\left(x_{2n+1}, x_{2n+2}\right) &= \lim_{n\to\infty} d\left(y_{2n}, y_{2n+1}\right) = \lim_{n\to\infty} d\left(y_{2n+1}, y_{2n+2}\right) \\ &= \lim_{n\to\infty} d\left(z_{2n}, z_{2n+1}\right) = \lim_{n\to\infty} d\left(z_{2n+1}, z_{2n+2}\right) = d\left(A, B\right).\end{aligned}$$ □

**Lemma 3.6.** *Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ such that $(A, B)$ and $(B, A)$ have the property UC. Let $F : A \times A \times A \to B$ and $G : B \times B \times B \to A$ be mappings such that the ordered pair $(F, G)$ is cyclic $\alpha$-contractive. Let $(x_0, y_0, z_0) \in A \times A \times A$. If for each $n \geq 0$, we define*

$$\begin{aligned}&x_{2n+1} = F\left(x_{2n}, y_{2n}, z_{2n}\right), \quad y_{2n+1} = F\left(y_{2n}, x_{2n}, z_{2n}\right), \quad z_{2n+1} = F\left(z_{2n}, y_{2n}, x_{2n}\right) \\ &x_{2n+2} = G\left(x_{2n+1}, y_{2n+1}, z_{2n+1}\right), \quad y_{2n+2} = G\left(y_{2n+1}, x_{2n+1}, z_{2n+1}\right), \quad z_{2n+2} = G\left(z_{2n+1}, y_{2n+1}, x_{2n+1}\right),\end{aligned}$$

*then for $\epsilon > 0$, there exists a positive integer $n_0$ such that for all $m > n \geq n_0$,*

$$\frac{1}{3}[d\left(x_{2m}, x_{2n+1}\right) + d\left(y_{2m}, y_{2n+1}\right) + d(z_{2m}, z_{2n+1})] < d\left(A, B\right) + \epsilon. \tag{3.1}$$

*Proof.* By Lemma 3.5, we have $\lim_{n\to\infty} d\left(x_{2n}, x_{2n+1}\right) = d\left(A, B\right) = \lim_{n\to\infty} d\left(x_{2n+1}, x_{2n+2}\right)$. Since $(A, B)$ has the property $UC$, we get $d\left(x_{2n}, x_{2n+2}\right) \to 0$. A similar argument shows that $d\left(y_{2n}, y_{2n+2}\right) \to 0$ and $d\left(z_{2n}, z_{2n+2}\right) \to 0$. As $(B, A)$ has the property $UC$, we also have $d\left(x_{2n+1}, x_{2n+3}\right) \to 0$, $d\left(y_{2n+1}, y_{2n+3}\right) \to 0$ and $d\left(z_{2n1}, z_{2n+3}\right) \to 0$. Suppose that (3.1) does not hold. Then there would exists an $\epsilon' > 0$ such that for all $k \in \mathbb{N}$, there would be an $m_k > n_k \geq k$ satisfying

$$\frac{1}{3}[d\left(x_{2m_k}, x_{2n_k+1}\right) + d\left(y_{2m_k}, y_{2n_k+1}\right) + d(z_{2m_k}, z_{2n_k+1})] \geq d\left(A, B\right) + \epsilon'$$

and

$$\frac{1}{3}[d\left(x_{2m_k-2}, x_{2n_k+1}\right) + d\left(y_{2m_k-2}, y_{2n_k+1}\right) + d(z_{2m_k-2}, z_{2n_k+1})] < d\left(A, B\right) + \epsilon'.$$

Therefore, we would get

$$\begin{aligned}d\left(A, B\right) + \epsilon' &\leq \frac{1}{3}[d\left(x_{2m_k}, x_{2n_k+1}\right) + d\left(y_{2m_k}, y_{2n_k+1}\right) + d(z_{2m_k}, z_{2n_k+1})] \\ &\leq \frac{1}{3}[d\left(x_{2m_k}, x_{2m_k-2}\right) + d\left(x_{2m_k-2}, x_{2n_k+1}\right) + d\left(y_{2m_k}, y_{2m_k-2}\right) + d\left(y_{2m_k-2}, y_{2n_k+1}\right) \\ &\quad + d\left(z_{2m_k}, z_{2m_k-2}\right) + d\left(z_{2m_k-2}, z_{2n_k+1}\right)] \\ &< \frac{1}{3}[d\left(x_{2m_k}, x_{2m_k-2}\right) + d\left(y_{2m_k}, y_{2m_k-2}\right) + d\left(z_{2m_k}, z_{2m_k-2}\right)] + d\left(A, B\right) + \epsilon'.\end{aligned}$$

Applying limit as $k \to \infty$, we would get

$$\begin{aligned}d\left(A, B\right) + \epsilon' &\leq \frac{1}{3}\lim_{k\to\infty}[d\left(x_{2m_k}, x_{2n_k+1}\right) + d\left(y_{2m_k}, y_{2n_k+1}\right) + d(z_{2m_k}, z_{2n_k+1})] \\ &\leq \frac{1}{3}\lim_{k\to\infty}[d\left(x_{2m_k}, x_{2m_k-2}\right) + d\left(y_{2m_k}, y_{2m_k-2}\right) + d\left(z_{2m_k}, z_{2m_k-2}\right)] + d\left(A, B\right) + \epsilon' \\ &= d\left(A, B\right) + \epsilon',\end{aligned}$$

that is,

$$\frac{1}{3}\lim_{k\to\infty}[d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(z_{2m_k},z_{2n_k+1})]=d(A,B)+\epsilon'. \tag{3.2}$$

Applying the triangle inequality to each term on the left of (3.2), we would get

$$\begin{aligned}
&\frac{1}{3}[d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(z_{2m_k},z_{2n_k+1})]\\
&\le \frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(x_{2m_k+2},x_{2n_k+3})+d(x_{2n_k+3},x_{2n_k+1})\\
&\quad +d(y_{2m_k},y_{2m_k+2})+d(y_{2m_k+2},y_{2n_k+3})+d(y_{2n_k+3},y_{2n_k+1})\\
&\quad +d(z_{2m_k},z_{2m_k+2})+d(z_{2m_k+2},z_{2n_k+3})+d(z_{2n_k+3},z_{2n_k+1})].\\
&= \frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(G(x_{2m_k+1},y_{2m_k+1},z_{2m_k+1}),F(x_{2n_k+2},y_{2n_k+2},z_{2n_k+2}))+d(x_{2n_k+3},x_{2n_k+1})\\
&\quad +d(y_{2m_k},y_{2m_k+2})+d(G(y_{2m_k+1},x_{2m_k+1},z_{2m_k+1}),F(y_{2n_k+2},x_{2n_k+2},z_{n_k+2}))+d(y_{2n_k+3},y_{2n_k+1})\\
&\quad +d(z_{2m_k},z_{2m_k+2})+d(G(z_{2m_k+1},y_{2m_k+1},x_{2m_k+1}),F(z_{2n_k+2},y_{2n_k+2},x_{2n_k+2}))+d(x_{2n_k+3},x_{2n_k+1})]\\
&\le \frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+\frac{\alpha}{3}[d(x_{2m_k+1},x_{2n_k+2})+d(y_{2m_k+1},y_{2n_k+2})+d(z_{2m_k+1},z_{2n_k+2})]\\
&\quad +(1-\alpha)d(A,B)+d(x_{2n_k+3},x_{2n_k+1})]\\
&+\frac{1}{3}[d(y_{2m_k},y_{2m_k+2})+\frac{\alpha}{3}[d(y_{2m_k+1},y_{2n_k+2})+d(x_{2m_k+1},x_{2n_k+2})+d(z_{2m_k+1},z_{n_k+2})]\\
&\quad +(1-\alpha)d(A,B)+d(y_{2n_k+3},y_{2n_k+1})]\\
&\quad +\frac{1}{3}[d(z_{2m_k},z_{2m_k+2})+\frac{\alpha}{3}[d(z_{2m_k+1},z_{2n_k+2})+d(y_{2m_k+1},y_{2n_k+2})+d(x_{2m_k+1},x_{2n_k+2})]\\
&\quad +(1-\alpha)d(A,B)+d(z_{2n_k+3},z_{2n_k+1})].\\
&= \frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})+d(y_{2m_k},y_{2m_k+2})+d(y_{2n_k+3},y_{2n_k+1})+d(z_{2m_k},z_{2m_k+2})\\
&\quad +d(z_{2n_k+3},z_{2n_k+1})]+\frac{\alpha}{3}[d(x_{2m_k+1},x_{2n_k+2})+d(y_{2m_k+1},y_{2n_k+2})+d(z_{2m_k+1},z_{2n_k+2})]\\
&\quad +(1-\alpha)d(A,B).\\
&= \frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})+d(y_{2m_k},y_{2m_k+2})+d(y_{2n_k+3},y_{2n_k+1})+d(z_{2m_k},z_{2m_k+2})\\
&\quad +d(z_{2n_k+3},z_{2n_k+1})]+\frac{\alpha}{3}[d(F(x_{2m_k},y_{2m_k},z_{2m_k}),G(x_{2n_k+1},y_{2n_k+1},z_{2n_k+1}))\\
&\quad +d(F(y_{2m_k},x_{2m_k},z_{2m_k}),G(y_{2n_k+1},x_{2n_k+1},z_{2n_k+1}))\\
&\quad +d(F(z_{2m_k},y_{2m_k},x_{2m_k}),G(z_{2n_k+1},y_{2n_k+1},x_{2n_k+1}))]+(1-\alpha)d(A,B)\\
&\le \frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})+d(y_{2m_k},y_{2m_k+2})+d(y_{2n_k+3},y_{2n_k+1})+d(z_{2m_k},z_{2m_k+2})\\
&\quad +d(z_{2n_k+3},z_{2n_k+1})]+\frac{\alpha^2}{9}[d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(z_{2m_k},z_{2n_k+1})+\frac{\alpha}{3}(1-\alpha)d(A,B)\\
&\quad +\frac{\alpha^2}{9}[d(y_{2m_k},y_{2n_k+1})+d(x_{2m_k},x_{2n_k+1})+d(z_{2m_k},z_{2n_k+1})]+\frac{\alpha}{3}(1-\alpha)d(A,B)\\
&\quad +\frac{\alpha^2}{9}[d(z_{2m_k},z_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(x_{2m_k},x_{2n_k+1})]+\frac{\alpha}{3}(1-\alpha)d(A,B)+(1-\alpha)d(A,B).\\
&= \frac{1}{3}[d(x_{2m_k},x_{2m_k+2})+d(x_{2n_k+3},x_{2n_k+1})+d(y_{2m_k},y_{2m_k+2})+d(y_{2n_k+3},y_{2n_k+1})\\
&\quad +d(z_{2m_k},z_{2m_k+2})+d(z_{2n_k+3},z_{2n_k+1})]\\
&\quad +\frac{\alpha^2}{3}[d(x_{2m_k},x_{2n_k+1})+d(y_{2m_k},y_{2n_k+1})+d(z_{2m_k},z_{2n_k+1})]+\left(1-\alpha^2\right)d(A,B).
\end{aligned}$$

Let $k \to \infty$. Then by (3.2), and since $(A, B)$ and $(B, A)$ have the $UC$, we would get

$$d(A,B) + \epsilon' \leq \alpha^2[d(A,B) + \epsilon'] + \left(1-\alpha^2\right) d(A,B) = d(A,B) + \alpha^2\epsilon',$$

which would imply that $\alpha \geq 1$, a contradiction. Thus (3.1) must hold. □

**Lemma 3.7.** *Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$ such that $(A, B)$ and $(B, A)$ have the property $UC^*$. Let $F : A \times A \times A \to B$ and $G : B \times B \times B \to A$ be mappings with $(F, G)$ cyclic $\alpha$-contractive. If $(x_0, y_0, z_0) \in A \times A \times A$ and if for $n \geq 0$, we define*

$$\begin{aligned} x_{2n+1} =&F(x_{2n}, y_{2n}, z_{2n}), \quad y_{2n+1} = F(y_{2n}, x_{2n}, z_{2n}), \quad z_{2n+1} = F(z_{2n}, y_{2n}, x_{2n});\\ x_{2n+2} =&G(x_{2n+1}, y_{2n+1}, z_{2n+1}), \quad y_{2n+2} = G(y_{2n+1}, x_{2n+1}, z_{2n+1}), \quad z_{2n+2} = G(z_{2n+1}, y_{2n+1}, x_{2n+1}), \end{aligned}$$

*then $\{x_{2n}\}, \{y_{2n}\}, \{z_{2n}\}$, and $\{x_{2n+1}\}, \{y_{2n+1}\}, \{z_{2n+1}\}$ are Cauchy sequences.*

*Proof.* By Lemma 3.5, we have $d(x_{2n}, x_{2n+1}) \to d(A, B)$ and $d(x_{2n+1}, x_{2n+2}) \to d(A, B)$. Since $(A, B)$ and $(B, A)$ have the property $UC^*$, we get $d(x_{2n}, x_{2n+2}) \to 0$ and $d(x_{2n+1}, x_{2n+3}) \to 0$. We only show that $\{x_{2n}\}$ is a Cauchy sequence. That other sequences are Cauchy ones, can be proved in a similar way. We first show that for every $\epsilon > 0$ there exist an $N$ such that

$$d(x_{2m}, x_{2n+1}) \leq d(A,B) + \epsilon \tag{3.3}$$

for all $m > n \geq N$. Suppose (3.3) does not hold. Then there would exist an $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there would exist an $m_k > n_k \geq k$ such that

$$d(x_{2m_k}, x_{2n_k+1}) > d(A,B) + \epsilon. \tag{3.4}$$

Now

$$d(A,B) + \epsilon < d(x_{2m_k}, x_{2n_k+1}) \leq d(x_{2m_k}, x_{2n_k-1}) + d(x_{2n_k-1}, x_{2n_k+1}) \leq d(A,B) + \epsilon + d(x_{2n_k-1}, x_{2n_k+1})$$

would imply that $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = d(A, B) + \epsilon$. By Lemma 3.6, there would exist an $n_0 \in \mathbb{N}$ such that

$$\frac{1}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1})] < d(A,B) + \epsilon \tag{3.5}$$

for all $m_k > n_k \geq n_0$. By using the triangle inequality we would get

$$\begin{aligned} d(x_{2m_k}, x_{2n_k+1}) \leq& d(x_{2m_k}, x_{2m_k+2}) + d(x_{2m_k+2}, x_{2n_k+3}) + d(x_{2n_k+3}, x_{2n_k+1})\\ =& d(x_{2m_k}, x_{2m_k+2}) + d(G(x_{2m_k+1}, y_{2m_k+1}, z_{2m_k+1}), F(x_{2n_k+2}, y_{2n_k+2}, z_{2m_k+2}))\\ &+ d(x_{2n_k+3}, x_{2n_k+1})\\ \leq& d(x_{2m_k}, x_{2m_k+2}) + \frac{\alpha}{3}[d(x_{2m_k+1}, x_{2n_k+2}) + d(y_{2m_k+1}, y_{2n_k+2}) + d(z_{2m_k+1}, z_{2n_k+2})]\\ &+ (1-\alpha) d(A,B) + d(x_{2n_k+3}, x_{2n_k+1})\\ =& \frac{\alpha}{3}[d(F(x_{2m_k}, y_{2m_k}, z_{2m_k}), G(x_{2n_k+1}, y_{2n_k+1}, z_{2n_k+1}))\\ &+ d(F(y_{2m_k}, x_{2m_k}, z_{2m_k}), G(y_{2n_k+1}, x_{2n_k+1}, z_{2n_k+1}))]\\ &+ d(F(z_{2m_k}, y_{2m_k}, x_{2m_k}), G(z_{2n_k+1}, y_{2n_k+1}, x_{2n_k+1}))\\ &+ (1-\alpha) d(A,B) + d(x_{2m_k}, x_{2m_k+2}) + d(x_{2n_k+3}, x_{2n_k+1})\\ \leq& \frac{\alpha}{3}[\frac{\alpha}{3}[d(x_{2m_k}, x_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1}) + (1-\alpha) d(A,B)]\\ &+ \frac{\alpha}{3}[d(y_{2m_k}, y_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1}) + d(z_{2m_k}, z_{2n_k+1}) + (1-\alpha) d(A,B)]\\ &+ \frac{\alpha}{3}[d(z_{2m_k}, z_{2n_k+1}) + d(y_{2m_k}, y_{2n_k+1}) + d(x_{2m_k}, x_{2n_k+1}) + (1-\alpha) d(A,B)]] \end{aligned}$$

$$
\begin{aligned}
&+ (1-\alpha)\, d\,(A,B) + d\,(x_{2m_k}, x_{2m_k+2}) + d\,(x_{2n_k+3}, x_{2n_k+1}) \\
=&\frac{\alpha^2}{3}[d\,(x_{2m_k}, x_{2n_k+1}) + d\,(y_{2m_k}, y_{2n_k+1}) + d\,(z_{2m_k}, z_{2n_k+1})] + \left(1-\alpha^2\right) d\,(A,B) \\
&+ d\,(x_{2m_k}, x_{2m_k+2}) + d\,(x_{2n_k+3}, x_{2n_k+1}) \\
<&\alpha^2\,(d\,(A,B)+\epsilon) + \left(1-\alpha^2\right) d\,(A,B) + d\,(x_{2m_k}, x_{2m_k+2}) + d\,(x_{2n_k+3}, x_{2n_k+1}) \\
=&d\,(A,B) + \alpha^2\epsilon + d\,(x_{2m_k}, x_{2m_k+2}) + d\,(x_{2n_k+3}, x_{2n_k+1})\,.
\end{aligned}
$$

Letting $k \to \infty$, we would get

$$d\,(A,B) + \epsilon \le d\,(A,B) + \alpha^2\epsilon,$$

which would imply that $\alpha \ge 1$, a contradiction. Thus, (3.3) must hold. Now by (3.3) and since $d\,(x_{2n}, x_{2n+1}) \to d\,(A,B)$, and $(A,B)$ has the property $UC^*$ , we can conclude that $\{x_{2n}\}$ is a Cauchy sequence. This completes the proof of the lemma. □

The following two theorems are the main results of this section which ensure the existence and convergence of tripled best proximity points for cyclic $\alpha$-contractive pairs on nonempty subsets of the metric spaces having the property $UC^*$.

**Theorem 3.8.** *Let $A$ and $B$ be nonempty closed subsets of a complete metric space $X$ such that $(A,B)$ and $(B,A)$ have the property $UC^*$. Let $F : A\times A \to B, G : B \times B \to A$ and $(F,G)$ be cyclic $\alpha$-contractive. Let $(x_0, y_0, z_0) \in A\times A\times A$ and define*

$$
\begin{aligned}
x_{2n+1} =&F\,(x_{2n}, y_{2n}, z_{2n})\,,\ \ y_{2n+1} = F\,(y_{2n}, x_{2n}, z_{2n})\,,\ \ z_{2n+1} = F\,(z_{2n}, y_{2n}, x_{2n})\,; \\
x_{2n+2} =&G\,(x_{2n+1}, y_{2n+1}, z_{2n+1})\,,\ \ y_{2n+2} = G\,(y_{2n+1}, x_{2n+1}, z_{2n+1})\,,\ \ z_{2n+2} = G\,(z_{2n+1}, y_{2n+1}, x_{2n+1})\,,
\end{aligned}
$$

*for all $n \ge 0$. Then $F$ has a tripled best proximity point $(a_1, a_2, a_3) \in A\times A\times A$ and $G$ has a tripled best proximity point $(b_1, b_2, b_3) \in B\times B\times B$ such that*

$$\frac{1}{3}[d\,(a_1, b_1) + d(a_2, b_2) + d(a_3, b_3)] = d\,(A,B)\,.$$

*Furthermore, $(x_{2n}, y_{2n}, z_{2n}) \to (a_1,a_2, a_3)$, and $(x_{2n+1}, y_{2n+1}, z_{2n+1}) \to (b_1,b_2, b_3)$ as $n \to \infty$.*

*Proof.* By Lemma 3.5, $d\,(x_{2n}, x_{2n+1}) \to d(A,B)$, and by Lemma 3.7, the sequences $\{x_{2n}\}$, $\{y_{2n}\}$, and $\{z_{2n}\}$ are Cauchy ones. Thus, there exists an $(a_1, a_2, a_3) \in A\times A\times A$ such that $(x_{2n}, y_{2n}, z_{2n}) \to (a_1,a_2, a_3)$. Hence we have

$$d\,(A,B) \le d\,(a_1, x_{2n-1}) \le d\,(a_1, x_{2n}) + d\,(x_{2n}, x_{2n-1}) \to d(A,B). \tag{3.6}$$

This implies that $\lim_{n\to\infty} d\,(a_1, x_{2n-1}) = d\,(A,B)$. Similarly, we get $\lim_{n\to\infty} d\,(a_2, y_{2n-1}) = d\,(A,B) = \lim_{n\to\infty} d\,(a_3, z_{2n-1})$. From this it follows that

$$
\begin{aligned}
d\,(x_{2n}, F(a_1, a_2, a_3)) &= d\,(G\,(x_{2n-1}, y_{2n-1}, z_{2n-1})\,, F(a_1, a_2, a_3)) \\
&\le \frac{\alpha}{3}[d\,(x_{2n-1}, a_1) + d\,(y_{2n-1}, a_2) + d\,(z_{2n-1}, a_3)] + (1-\alpha)\, d\,(A,B)\,.
\end{aligned}
$$

Letting $n \to \infty$, we get $d\,(a_1, F(a_1, a_2, a_3)) = d\,(A,B)$. Similarly, we get $d\,(a_2, F(a_2, a_1, a_3)) = d\,(A,B) = d\,(a_3, F(a_3, a_2, a_1))$. Thus, $(a_1, a_2, a_3) \in A\times A\times A$ is a tripled best proximity point of $F$. By similar arguments, we can prove that there exists a $(b_1, b_2, b_3) \in B\times B\times B$ such that $(x_{2n+1}, y_{2n+1}, z_{2n+1}) \to (b_1, b_2, b_3)$ and hence $d\,(b_1, F(b_1, b_2, b_3)) = d\,(b_2, F(b_2, b_1, b_3)) = d\,(b_3, F(b_3, b_2, b_1)) = d\,(A,B)$. Thus $(b_1, b_2, b_3) \in B\times B\times B$ is a tripled best proximity point of $G$.

To show that $\frac{1}{3}[d\,(a_1, b_1) + d(a_2, b_2) + d(a_3, b_3)] = d\,(A,B)$, for $n \ge 0$, we use

$$
\begin{aligned}
d\,(x_{2n}, x_{2n+1}) &= d\,(G\,(x_{2n-1}, y_{2n-1}, z_{2n-1})\,, F\,(x_{2n}, y_{2n}, z_{2n})) \\
&\le \frac{\alpha}{3}[d\,(x_{2n-1}, x_{2n}) + d\,(y_{2n-1}, y_{2n}) + d\,(z_{2n-1}, z_{2n})] + (1-\alpha)\, d\,(A,B)\,,
\end{aligned}
$$

$$\begin{aligned} d\left(y_{2n}, y_{2n+1}\right) &= d(G\left(y_{2n-1}, x_{2n-1}, z_{2n-1}\right), F\left(y_{2n}, x_{2n}, z_{2n}\right)) \\ &\leq \frac{\alpha}{3}[d\left(y_{2n-1}, y_{2n}\right) + d\left(x_{2n-1}, x_{2n}\right) + d(z_{2n-1}, z_{2n})] + (1-\alpha)\, d\left(A, B\right), \end{aligned}$$

$$\begin{aligned} d\left(z_{2n}, z_{2n+1}\right) &= d(G\left(z_{2n-1}, y_{2n-1}, x_{2n-1}\right), F\left(z_{2n}, y_{2n}, x_{2n}\right)) \\ &\leq \frac{\alpha}{3}[d\left(z_{2n-1}, z_{2n}\right) + d\left(y_{2n-1}, y_{2n}\right) + d(x_{2n-1}, x_{2n})] + (1-\alpha)\, d\left(A, B\right). \end{aligned}$$

Taking the limit as $n \to \infty$, we get

$$d\left(a_1, b_1\right) \leq \frac{\alpha}{3}[d(a_1, b_1) + d\left(a_2, b_2\right) + d(a_3, b_3)] + (1-\alpha)\, d\left(A, B\right), \tag{3.7}$$

$$d\left(a_2, b_2\right) \leq \frac{\alpha}{3}[d\left(a_2, b_2\right) + d\left(a_1, b_1\right) + d(a_3, b_3)] + (1-\alpha)\, d\left(A, B\right) \tag{3.8}$$

and

$$d\left(a_3, b_3\right) \leq \frac{\alpha}{3}[d\left(a_3, b_3\right) + d\left(a_2, b_2\right) + d(a_1, b_1)] + (1-\alpha)\, d\left(A, B\right). \tag{3.9}$$

It follows from (3.7), (3.8), and (3.9) that

$$d(a_1, b_1) + d\left(a_2, b_2\right) + d(a_3, b_3) \leq \alpha[d(a_1, b_1) + d\left(a_2, b_2\right) + d(a_3, b_3)] + 3\left(1-\alpha\right) d\left(A, B\right),$$

which implies that

$$d(a_1, b_1) + d\left(a_2, b_2\right) + d(a_3, b_3) \leq 3d\left(A, B\right). \tag{3.10}$$

Since $d\left(A, B\right) \leq d(a_1, b_1)$, $d\left(a_2, b_2\right)$, $d(a_3, b_3)$, we get

$$3d(A, B) \leq d(a_1, b_1) + d\left(a_2, b_2\right) + d(a_3, b_3).$$

Thus $d(a_1, b_1) + d\left(a_2, b_2\right) + d(a_3, b_3) = 3d(A, B)$ and the thorem is proved. □

*Remark* 3.9. Since every pair of nonempty closed subsets $A, B$ of a uniformly convex Banach space $X$ with $A$ convex have the property $UC^*$, we have Corollary 3.10.

**Corollary 3.10.** *Let $A$ and $B$ be nonempty closed convex subsets of a uniformly convex Banach space $X$. Let $F : A \times A \times A \to B$, and $G : B \times B \times B \to A$ be mappings such that $(F, G)$ is cyclic $\alpha$-contractive. Let $(x_0, y_0, z_0) \in A \times A \times A$ and define*

$$\begin{aligned} &x_{2n+1} = F\left(x_{2n}, y_{2n}, z_{2n}\right), \quad y_{2n+1} = F\left(y_{2n}, x_{2n}, z_{2n}\right), \quad z_{2n+1} = F\left(z_{2n}, y_{2n}, x_{2n}\right); \\ &x_{2n+2} = G\left(x_{2n+1}, y_{2n+1}, z_{2n+1}\right), \quad y_{2n+2} = G\left(y_{2n+1}, x_{2n+1}, z_{2n+1}\right), \quad z_{2n+2} = G\left(z_{2n+1}, y_{2n+1}, x_{2n+1}\right), \end{aligned}$$

*for $n \geq 0$. Then $F$ has a tripled best proximity point $(a_1, a_2, a_3) \in A \times A \times A$ and $G$ has a tripled best proximity point $(b_1, b_2, b_3) \in B \times B \times B$ such that*

$$d(a_1, b_1) + d\left(a_2, b_2\right) + d(a_3, b_3) = 3d(A, B).$$

*Furthermore, $(x_{2n}, y_{2n}, z_{2n}) \to (a_1, a_2, a_3)$, and $(x_{2n+1}, y_{2n+1}, z_{2n+1}) \to (b_1, b_2, b_3)$ as $n \to \infty$.*

**Example 3.11.** The space $X = \mathbb{R}$ with the usual norm is a uniformly convex Banach space. Let $A = [1, 2]$ and $B = [-2, -1]$. Then $d\left(A, B\right) = 2$. Define $F : A \times A \times A \to B$ and $G : B \times B \times B \to A$ by

$$F\left(x, y, z\right) = \frac{-(x+y+z+3)}{6} \text{ and } G(x, y, z) = \frac{-(x+y+z-3)}{6}.$$

Then for $(x, y, z) \in A \times A \times A$, $(u, v, w) \in B \times B \times B$ and for $\alpha = \frac{1}{2}$, we have

$$d\left(F\left(x, y, z\right), G\left(u, v, w\right)\right) = \left|\frac{-(x+y+z+3)}{6} - \frac{-(u+v+w-3)}{6}\right|$$

$$\begin{aligned} &\leq \frac{|x-u|+|y-v|+|z-w|}{6}+\frac{1}{2}(2) \\ &= \frac{\alpha}{3}[d\,(x,u)+d\,(y,v)+d(z,w)]+(1-\alpha)d\,(A,B)\,. \end{aligned}$$

Thus $(F,G)$ is cyclic $\frac{1}{2}$-contractive. Since $A$ and $B$ are convex, we have that $(A,B)$ and $(B,A)$ have the property $UC^*$. Therefore, all the hypothesis of Corollary 3.10 hold. So each of $F$ and $G$ has a tripled best proximity point. Observe that $(1,1,1)\in A\times A\times A$ is a unique tripled best proximity point of $F$ and $(-1,-1,-1)\in B\times B\times B$ is a unique tripled best proximity point of $G$. Furthermore, we get

$$d\,(1,-1,)+d(1,-1)+d(1,-1)=6=3d\,(A,B)\,.$$

**Theorem 3.12.** *Let $A$ and $B$ be nonempty compact subsets of a metric space $(X,d)$. Let $F:A\times A\times A\to B$, and $G:B\times B\times B\to A$ be mappings such that $(F,G)$ is cyclic $\alpha$-contractive and $(x_0,y_0,z_0)\in A\times A\times A$. For $n\geq 0$, define*

$$x_{2n+1}=F\,(x_{2n},y_{2n},z_{2n})\,,\ \ y_{2n+1}=F\,(y_{2n},x_{2n},z_{2n})\,,\ \ z_{2n+1}=F\,(z_{2n},y_{2n},x_{2n})\,,$$

$$x_{2n+2}=G\,(x_{2n+1},y_{2n+1},z_{2n+1})\,,\ \ y_{2n+2}=G\,(y_{2n+1},x_{2n+1},z_{2n+1})\,,\ \ z_{2n+2}=G\,(z_{2n+1},y_{2n+1},x_{2n+1})\,.$$

*Then $F$ has a trippled best proximity point $(a_1,a_2,a_3)\in A\times A\times A$ and $G$ has a tripled best proximity point $(b_1,b_2,b_3)\in B\times B\times B$ such that*

$$\frac{1}{3}[d\,(a_1,b_1)+d(a_2,b_2)+d(a_3,b_3)]=d\,(A,B)\,.$$

*Furthermore, $(x_{2n},y_{2n},z_{2n})\to(a_1,a_2,a_3)$, and $(x_{2n+1},y_{2n+1},z_{2n+1})\to(b_1,b_2,b_3)$ as $n\to\infty$.*

*Proof.* By definition, for each $n\geq 0$, we have $x_{2n},\ y_{2n},\ z_{2n}\in A$ and $x_{2n+1},\ y_{2n+1},\ z_{2n+1}\in B$. Since $A$ is compact, the sequences $\{x_{2n}\}$, $\{y_{2n}\}$ and $\{z_{2n}\}$ have convergent subsequences $\{x_{2n_k}\}$, $\{y_{2n_k}\}$ and $\{z_{2n_k}\}$, respectively, such that

$$(x_{2n_k},\ y_{2n_k},\ z_{2n_k})\to(a_1,a_2,a_3)\in A\times A\times A.$$

Now, we have

$$d\,(A,B)\leq d\,(a_1,x_{2n_k-1})\leq d\,(a_1,x_{2n_k})+d\,(x_{2n_k},x_{2n_k-1})\,. \tag{3.11}$$

By Lemma 3.5, $\lim_{k\to\infty}d\,(x_{2n_k},x_{2n_k-1})=d\,(A,B)$. Therefore, letting $k\to\infty$ in (3.11), we get $d\,(a_1,x_{2n_k-1})\to d\,(A,B)$. By a similar argument we can get $d\,(a_2,x_{2n_k-1})\to d\,(A,B)$ and $d\,(a_3,x_{2n_k-1})\to d\,(A,B)$. Also, observe that

$$\begin{aligned} d\,(A,B) &\leq d\,(x_{2n_k},F\,(a_1,a_2,a_3)) \\ &= d\,(G\,(x_{2n_k-1},y_{2n_k-1},z_{2n_k-1})\,,F\,(a_1,a_2,a_3)) \\ &\leq \frac{\alpha}{3}[d(x_{2n_k-1},a_1)+d\,(y_{2n_k-1},a_2)+d\,(z_{2n_k-1},a_3)]+(1-\alpha)\,d\,(A,B)\,. \end{aligned}$$

So letting $k\to\infty$, we get $d\,(a_1,F\,(a_1,a_2,a_3))=d\,(A,B)$. Similarly, we can prove that $d\,(a_2,F\,(a_2,a_1,a_3))=d(A,B)=d\,(a_3,F\,(a_3,a_2,a_1))$. Thus $F$ has a tripled best proximity point $(a_1,a_2,a_3)\in A\times A\times A$. In a similar way, since $B$ is compact, we can also prove that $G$ has a tripled best proximity point $(b_1,b_2,b_3)\in B\times B\times B$. The proof that $d\,(a_1,b_1)+d(a_2,b_2)+d(a_3,b_3)=3d\,(A,B)$ is similar to the last part of the proof of Theorem 3.8. Hence the theorem follows. □

As a consequence of the above best proximity results, we get the following tripled fixed point theorem.

**Theorem 3.13.** *. Let $A$ and $B$ be nonempty closed subsets of a complete metric space $(X,d)$. Let $F:A\times A\times A\to B$, and $G:B\times B\times B\to A$ be mappings such that $(F,G)$ is cyclic $\alpha$-contractive. Let $(x_0,y_0,z_0)\in A\times A\times A$. For $n\geq 0$, define*

$$x_{2n+1}=F\,(x_{2n},y_{2n},z_{2n})\,,\ \ y_{2n+1}=F\,(y_{2n},x_{2n},z_{2n})\,,\ \ z_{2n+1}=F\,(z_{2n},y_{2n},x_{2n})\,,$$

*and*

$$x_{2n+2} = G\left(x_{2n+1}, y_{2n+1}, z_{2n+1}\right), \quad y_{2n+2} = G\left(y_{2n+1}, x_{2n+1}, z_{2n+1}\right), \quad z_{2n+2} = G\left(z_{2n+1}, y_{2n+1}, x_{2n+1}\right).$$

*If $d(A,B) = 0$, then $F$ and $G$ have a unique common tripled fixed point $(a,b,c) \in A \cap B \times A \cap B \times A \cap B$. Moreover, we have $x_{2n}$, $x_{2n+1} \to a$, $y_{2n}$, $y_{2n+1} \to b$ and $z_{2n}$, $z_{2n+1} \to c$.*

**Example 3.14.** Consider $X = \mathbb{R}$ with the usual metric, $A = [-1, 0]$ and $B = [0, 1]$. Define $F : A \times A \times A \to B$ by $F(x, y, z) = -\frac{x+y+z}{6}$ and $G : B \times B \times B \to A$ by $G(x, y) = -\frac{x+y+z}{6}$. Then $d(A,B) = 0$, and $(F, G)$ is a cyclic $\alpha$-contraction with $\alpha = \frac{1}{2}$. Indeed, for arbitrary $(x, y, z) \in A \times A \times A$ and $(u, v, w) \in B \times B \times B$, we have

$$\begin{aligned} d\left(F(x,y,z), G(u,v,w)\right) &= \left| -\frac{x+y+z}{6} + \frac{u+v+w}{6} \right| \\ &\leq \frac{1}{6}\left(|x-u| + |y-v| + |z-w|\right) \\ &= \frac{\alpha}{3}[d(x,u) + d(y,v) + d(z,w)] + (1-\alpha)\, d(A,B). \end{aligned}$$

Therefore, all the hypotheses of Theorem 3.13 holds and so, by Theorem 3.13, $F$ and $G$ have a unique common tripled fixed point which is $(0,0,0)$.

If we take $A = B$ in Theorem 3.13, then we get the following results.

**Corollary 3.15.** *Let $A$ be a nonempty closed subset of a complete metric space $(X, d)$. Let $F, G : A \times A \times A \to A$ be mappings such that $(F, G)$ is cyclic $\alpha$-contractive, and let $(x_0, y_0, z_0) \in A \times A \times A$. For $n \geq 0$, define*

$$x_{2n+1} = F\left(x_{2n}, y_{2n}, z_{2n}\right), \quad y_{2n+1} = F\left(y_{2n}, x_{2n}, z_{2n}\right), \quad z_{2n+1} = F\left(z_{2n}, y_{2n}, x_{2n}\right),$$

*and*

$$x_{2n+2} = G\left(x_{2n+1}, y_{2n+1}, z_{2n+1}\right), \quad y_{2n+2} = G\left(y_{2n+1}, x_{2n+1}, z_{2n+1}\right), \quad z_{2n+2} = G\left(z_{2n+1}, y_{2n+1}, x_{2n+1}\right).$$

*Then $F$ and $G$ have a unique common tripled fixed point $(a, b, c) \in A \times A \times A$. Moreover, we have $x_{2n}$, $x_{2n+1} \to a$, $y_{2n}$, $y_{2n+1} \to b$, and $z_{2n}$, $z_{2n+1} \to c$.*

## Acknowledgement

This article was supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. Therefore, the second author acknowledges with thanks DSR, KAU for financial support.

## References


[1] M. Abbas, N. Hussain, B. E. Rhoades, *Coincidence point theorems for multivalued $f$-weak contraction mappings and applications*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM, **105** (2011), 261–272. 1

[2] R. P. Agarwal, D. O. Regan, M. Sambandham, *Random and deterministic fixed point theory for generalized contractive maps*, Appl. Anal., **83** (2004), 711–725. 1

[3] S. S. Basha, *Best proximity point theorems generalizing the contraction principal*, Nonlinear Anal., **74** (2011), 5844–5850. 1

[4] S. S. Basha, P. Veeramani, *Best proximity pairs and best approximations*, Acta Sci. Math. (Szeged), **63** (1997), 289–300. 1

[5] S. S. Basha, P. Veeramani, *Best proximity pair theorems for multifunctions with open fibers*, J. Approx. Theory, **103** (2000), 119–129. 1

[6] S. S. Basha, P. Veeramani, D. V. Pai, *Best proximity pair theorems*, Indian J. Pure Appl. Math., **32** (2001), 1237–1246. 1

[7] I. Beg, A. R. Khan, N. Hussain, *Approximation of $\star$-nonexpansive random multivalued operators on Banach spaces*, J. Australian . Math. Soc., **76** (2004), 51–66. 1

[8] L. Ćirić, J. S. Ume, S. N. Ješić, *On random coincidence for a pair of measurable mappings*, Ital. J. Pure Appl. Math., **23** (2008), 37–44. 1
[9] L. Ćirić, V. Lakshmikantham, *Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Stoch. Anal. Appl., **27** (2009), 1246–1259. 1
[10] A. A. Eldred, P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl., **323** (2006), 1001–1006 1, 1.5
[11] K. Fan, *Extensions of two fixed point theorems of F. E. Browder*, Math. Z., **112** (1969), 234–240. 1
[12] N. J. Huang, *Aprinciple of randomization of coincidence points with appliactions*, Appl. Math. Lett., **12** (1999), 107–113. 1
[13] N. Hussain, *Asymptotically pseudo-contractions, Banach operator pairs and best simultaneous approximations*, Fixed Point Theory Appl., **2011** (2011), 11 pages. 1
[14] N. Hussain, M. abbas, A. Azam, J. Ahmad, *Coupled coincidence point results for a generalized compatible pair with applications*, Fixed Point Theory Appl., **2014** (2014), 20 pages. 1
[15] N. Hussain, V. Berinde, N. Shafqat, *Common fixed point and approximation results for generalized $\phi$-contractions*, Fixed Point Theory, **10** (2009), 111–124. 1
[16] N. Hussain, A. Latif, N. Shafqat, *Weak contractive inequalities and compatible mixed monotone random operators in ordered metric spaces*, J. Inequal. Appl., **2012** (2012), 20 pages. 1
[17] S. Ioth, *A random fixed point theorem for a multi-valued contraction mapping*, Pac. J. Math., **68** (1977), 85–90. 1
[18] A. R. Khan, N. Hussain, *Random coincidence point theoremin Frechet spaces with applications*, Stochastic Anal. Appl., **22** (2004), 155–167. 1
[19] A. R. Khan, N. Hussain, N. Yasmin, N. Shafqat, *Random coincidence point results for weakly increasing functions in partially ordered metric spaces*, Bull. Iranian Math. Soc., **41** (2015), 407–422. 1
[20] M. A. Kutbi, J. Ahmad, M. Abbas, M. Arshad, *Trippled coincidence and common fixed results for two pairs of hybrid mappings*, Abstract Appl. Anal., **2014** (2014), 11 pages. 1
[21] T. C. Lin, *Random approximations and random fixed point theorems for non-self-maps*, Proc. Amer. Math. Soc., **103** (1988), 1129–1135. 1
[22] C. Mongkolkeha, P. Kumam, *Best proximity point theorems for generalized cyclic contractions in ordered metric spaces*, J. Optim. Theory Appl., **155** (2012), 215–226. 1
[23] J. B. Prolla, *Fixed point theorems for set valued mappings and existence of best approximations*, Numer. Funct. Anal. Optim., **5** (1982/83), 449–455. 1
[24] S. Reich, *Approximate selections, best approximations, fixed points and invarient sets*, J. Math. Anal. Appl., **62** (1978), 104–113. 1
[25] R. T. Rockafellar, *Measurable dependence of convex sets and functions in parameters*, J. Math. Anal. Appl., **28** (1969), 4–25. 2
[26] V. M. Sehgal, S. P. Singh, *On random approximations and a random fixed point theorem for set valued mappings*, Proc. Amer. Math. Soc., **95** (1985), 91–94. 1
[27] V. M. Sehgal, S. P. Singh, *A generalization to multifunctions of Fan's best approximation theorem*, Proc. Amer. Math. Soc., **102** (1988), 534–537. 1
[28] V. M. Sehgal, S. P. Singh, *A theorem on best approximations*, Numer. Funct. Anal. Optim., **10** (1989), 181–184. 1
[29] W. Sintunavarat, P. Kumam, *Coupled best proximity point theorem in metric spaces*, Fixed Point Theory Appl., **2012** (2012), 16 pages. 1, 1.7, 1.8, 1
[30] T. Suzuki, M. Kikkawa, C. Vetro, *The existence of best proximity points in metric spaces with the property UC*, Nonlinear Anal., **71** (2009), 2918–2926. 1.2, 1.3
[31] V. Vetrival, P. Veeramani, P. Bhattachayya, *Some extensions of Fan's best approximation theorem*, Numer. Funct. Anal. Optim., **13** (1992), 397–402. 1
[32] D. H. Wagner, *Survey of measurable selection theorems*, SIAM J. Control Optim., **15** (1977), 859–903. 2
[33] K. Wlodarczyk, R. Plebaniak, A. Banach, *Best proximity points for cyclic and noncyclic set-valued relatively quas-asymptotic contractions in uniform spaces*, Nonlinear Anal., **70** (2009), 3332–3342. 1
[34] X. H. Zhu, J. Z. Xiao, *Random periodic point and fixed point results for random monotone mappings in ordered Polish spaces*, Fixed Point Theory Appl., **2010** (2010), 13 pages. 1