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Monotone projection algorithms for various nonlinear problems in Hilbert spaces

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Abstract

In this paper, a monotone projection algorithm is investigated for solving common solutions of a fixed point problem of an asymptotically strict pseudocontraction, an equilibrium problem and a zero problem of the sum of two monotone mappings. Strong convergence theorems are established in the framework of real Hilbert spaces.

Keywords: Hilbert space, equilibrium problem, variational inequality, nonexpansive mapping, fixed point.
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1. Introduction and Preliminaries

In this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. P_C denotes the metric projection from H onto C.

Let T be a mapping on C. $F(T)$ stands for the fixed point set of T. Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

T is said to be asymptotically nonexpansive iff there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, n \geq 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [11]. They proved that if C is also bounded, then $F(T)$ is not empty; see [11] for more details.

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T is said to be κ-strictly pseudocontractive iff there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in C.$$

The class of strict pseudocontractions was introduced by Browder and Petryshyn [5]. It is clear that every nonexpansive mapping is a 0-strict pseudocontraction.

T is said to be an asymptotically κ-strict pseudocontraction iff there exist a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and a constant $\kappa \in [0, 1)$ such that

$$\|T^n x - T^n y\|^2 \leq k_n\|x - y\|^2 + \kappa\|(I - T^n)x - (I - T^n)y\|^2, \quad \forall x, y \in C,\, n \geq 1.$$

The class of asymptotically strict pseudocontractions is introduced by Qihou [20]. It is clear that every asymptotically nonexpansive mapping is an asymptotically 0-strict pseudocontraction.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, we also say that A is an α-strongly monotone mapping. A is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, we also say that A is an α-inverse-strongly monotone mapping. It is clear that A is inverse-strongly monotone if and only if A^{-1} is strongly monotone.

Recall that the classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

It is known that $x \in C$ is a solution to problem (1.1) if and only if x is a fixed point of mapping $P_C(I - rA)$, where $r > 0$ is a constant and I is the identity mapping. Recently, iterative methods have been intensively investigated for solving solutions of variational inequality (1.1) by many authors in the framework of Hilbert spaces; see [8], [17], [18], [21], [22], [28], [30], and the references therein.

Let B be a set-valued mapping. In this paper, we use $D(B)$ to denote the domain of B. Recall that B is said to be monotone on H if for all $x, y \in H$, $f \in Bx$ and $g \in By$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping B is maximal on H if the graph $G(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(B)$ implies $f \in Bx$. Let $r > 0$ be a real number. We can define the single-valued resolvent $J_r = (I + rA)^{-1}$. It is known that $J_r : H \to D(B)$ is firmly nonexpansive and $B^{-1}(0) = F(J_r)$. Let A be a monotone mapping of C into H and $N_C v$ the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \quad \forall u \in C\}$$

and define a mapping T on C by

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Av, u - v\rangle \geq 0$ for all $u \in C$; see [24] and the references therein.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Consider the following equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F)$, *i.e.*, $EP(F) = \{x \in C : F(x, y) \geq 0, \forall y \in C\}$.

To study problem (1.2), we may assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Recently, problem (1.2) was studied based on iterative methods by many authors; see [6], [13], [15], [23], [31] and the references therein. The advantage of projection methods is that strong convergence is guaranteed without any compact assumptions. And when C is a line variety, a closed ball, a closed cone or a closed polytope, the computation of P_C is easy to implement. Problem (1.2) is well known to be very useful and efficient tools in mathematics. It provides a unified framework for studying many problems arising in engineering sciences, structural analysis, and other fields; see, e.g., [1], [12], [26], [18], [19], [27]. A closely related subject of current interest is the problem of finding a common solution of nonlinear operator-equations, variational inequality (1.1) and equilibrium problem (1.2). The motivation for this subject is mainly due to its possible applications to mathematical modeling of concrete complex problems. Indeed, a classical strategy to construct such mathematical models consists in introducing constraints which can be expressed as subproblems of a more general problem. In some cases, these constraints can be given by variational inequalities, by fixed point problems, or by problems of different types [2, 3], [6]-[10], [14, 16, 17, 18, 29].

Motivated by the research going on this direction, we study a regularization projection algorithm for solving common solutions of variational inequality (1.1), equilibrium problem (1.2) and fixed points of an asymptotically strict pseudocontraction. Possible computation errors are taken into account. Strong convergence theorems are established in the framework of real Hilbert spaces.

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 ([4]). *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(a) *T_r is single-valued;*

(b) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(c) *$F(T_r) = EP(F)$;*

(d) *$EP(F)$ is closed and convex.*

Lemma 1.2 ([25]). *Let C be a nonempty closed convex subset of H and let $T : C \to C$ an asymptotically strict pseudocontraction. Then $I - T$ is demi-closed, this is, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in F(T)$.*

2. Main results

Theorem 2.1. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : H \rightrightarrows H$ be a maximal monotone mapping such that $D(B) \subset C$. Let $T : C \to C$ be an asymptotically κ-strict pseudocontraction. Assume that $\Omega = F(T) \cap (A + B)^{-1}(0) \cap EP(F)$ is not empty and bounded. Let $\{r_n\}$ and $\{s_n\}$ be two positive real number sequences. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ F(z_n, z) + \frac{1}{s_n}\langle z - z_n, z_n - J_{r_n}(x_n - r_n A x_n + e_n)\rangle \geq 0, \quad \forall z \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)\beta_n z_n + (1 - \beta_n)(1 - \alpha_n)T^n z_n, \\ C_{n+1} = \{\lambda \in C_n : \|y_n - \lambda\| \leq \|x_n - \lambda\| + (\sqrt{k_n} - 1)\Theta_n + \sqrt{k_n}\|e_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1, \end{cases}$$

where $J_{r_n} = (I + r_n A)^{-1}$, $\{e_n\}$ is a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\Theta_n = \sup\{\|x_n - q\| : q \in \Omega\}$. Assume that the control sequences satisfy the following restrictions: $0 \leq \alpha_n \leq a < 1$, $\kappa \leq \beta_n \leq b < 1$, $\liminf_{n\to\infty} s_n > 0$ and $0 < r \leq r_n \leq r' < 2\alpha$, where a, b, r, r' are real constants. Then $\{x_n\}$ converges strongly to $P_\Omega x_1$.

Proof. From the construction of C_n, we see that C_n is convex and closed so that the metric projection onto C_n is well defined. For any $x, y \in C$, we see that

$$\begin{aligned} &\|(I - r_n A)x - (I - r_n A)y\|^2 \\ &= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2. \end{aligned}$$

Using the restrictions imposed on $\{r_n\}$, we see that $\|(I - r_n A)x - (I - r_n A)y\| \leq \|x - y\|$. This proves that $I - r_n A$ is nonexpansive.

Next, we show that $\Omega \subset C_n$. It is clear that $\Omega \subset C_1 = C$. Suppose that $\Omega \subset C_h$ for some $h \geq 1$. Next, we show that $\Omega \subset C_{h+1}$ for the same h. Let $p \in \Omega$ be fixed arbitrarily. By use of Lemma 1.1, we find that $z_h = T_{s_h} w_h$, where $w_h = J_{r_h}\big(x_h - r_h A x_h + e_h\big)$. It follows from the firm nonexpansivity of the resolvent that

$$\begin{aligned} \|z_h - p\| &\leq \|w_h - p\| \\ &\leq \|\big(x_h - r_h A x_h + e_h\big) - \big(p - r_h A p\big)\| \\ &\leq \|x_h - p\| + \|e_h\|. \end{aligned} \tag{2.1}$$

Since T is an asymptotically κ-strict pseudocontraction, we have

$$\begin{aligned} &\|\beta_h z_h + (1 - \beta_h)T^h z_h - p\|^2 \\ &\leq \beta_h\|y_h - p\|^2 + (1 - \beta_h)(k_h\|y_h - p\|^2 + \kappa\|(z_h - p) - (T^h z_h - T^h p)\|^2) \\ &\quad - \beta_h(1 - \beta_h)\|(y_h - p) - (T^h z_h - T^h p)\|^2 \\ &\leq k_h\|z_h - p\|^2 - (1 - \beta_h)(\beta_h - \kappa)\|(z_h - p) - (T^h z_h - T^h p)\|^2. \end{aligned}$$

Using the restrictions imposed on sequence $\{\beta_n\}$, we find that

$$\|\beta_h z_h + (1 - \beta_h)T^h z_h - p\| \leq \sqrt{k_h}\|z_h - p\|. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned}\|y_h - p\| &\leq \alpha_h \|x_h - p\| + (1-\alpha_h)\|\beta_h z_h + (1-\beta_h)T^h z_h - p\| \\ &\leq \alpha_h \|x_h - p\| + (1-\alpha_h)\sqrt{k_h}\|z_h - p\| \\ &\leq \|x_h - p\| + (\sqrt{k_h} - 1)\|x_h - p\| + \sqrt{k_h}\|e_h\|.\end{aligned}$$

This proves that $\Omega \subset C_n$.

Now, we are in a position to show that $\{x_n\}$ is bounded. Note that $x_n = P_{C_n}x_1$. For any $p \in \Omega \subset C_n$, we have $\|x_1 - x_n\| \leq \|x_1 - p\|$. In particular, we have

$$\|x_1 - x_n\| \leq \|x_1 - P_\Omega x_1\|.$$

This implies that $\{x_n\}$ is bounded. Since $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to x. Since $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we have that

$$\begin{aligned}0 &\leq \langle x_1 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\ &\leq -\|x_1 - x_n\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|.\end{aligned}$$

Hence, we have

$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\|.$$

It follows that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. Since

$$\begin{aligned}&\|x_n - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2,\end{aligned}$$

we find that

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{2.3}$$

Since $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1}$, we see that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + (\sqrt{k_n} - 1)\Theta_n + \sqrt{k_n}\|e_n\|$. It follows that

$$\|y_n - x_n\| \leq 2\|x_{n+1} - x_n\| + (\sqrt{k_n} - 1)\Theta_n + \sqrt{k_n}\|e_n\|.$$

Since $(\sqrt{k_n} - 1)\Theta_n + \sqrt{k_n}\|e_n\| \to 0$ as $n \to \infty$, we obtain from (2.3) that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0.$$

On the other hand, we have

$$\|x_n - y_n\| = (1-\alpha_n)\|x_n - \big(\beta_n z_n + (1-\beta_n)T^n z_n\big)\|.$$

Using the restriction imposed on $\{\alpha_n\}$, we find that

$$\lim_{n\to\infty}\|x_n - \big(\beta_n z_n + (1-\beta_n)T^n z_n\big)\| = 0. \tag{2.4}$$

Since T_{s_n} is firmly nonexpansive, we find that

$$\begin{aligned}\|z_n - p\|^2 &\leq \langle w_n - p, z_n - p\rangle \\ &= \frac{1}{2}\big(\|w_n - p\|^2 + \|z_n - p\|^2 - \|z_n - w_n\|^2\big).\end{aligned}$$

That is,

$$\begin{aligned}\|z_n - p\|^2 &\le \|w_n - p\|^2 - \|w_n - z_n\|^2 \\ &\le \|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) - \|w_n - z_n\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|y_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n\|z_n - p\|^2 \\ &\le k_n\|x_n - p\|^2 + k_n\|e_n\|(\|e_n\| + 2\|x_n - p\|) - (1-\alpha_n)k_n\|w_n - z_n\|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}(1-\alpha_n)k_n\|w_n - z_n\|^2 \le{}& (k_n - 1)\|x_n - p\|^2 + k_n\|e_n\|(\|e_n\| + 2\|x_n - p\|) \\ &+ \|x_n - p\|^2 - \|x_{n+1} - p\|^2.\end{aligned}$$

It follows from the restriction imposed on $\{\alpha_n\}$ that

$$\lim_{n\to\infty} \|w_n - z_n\| = 0. \tag{2.5}$$

Since A is inverse-strongly monotone, we find that

$$\begin{aligned}\|w_n - p\|^2 &\le \|\big(x_n - r_nAx_n + e_n\big) - \big(p - r_nAp\big)\|^2 \\ &\le \|\big(x_n - r_nAx_n\big) - \big(p - r_nAp\big)\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) \\ &\le \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|).\end{aligned}$$

It follows that

$$\begin{aligned}\|y_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n\|z_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n\|w_n - p\|^2 \\ &\le k_n\|x_n - p\|^2 - r_n(2\alpha - r_n)(1-\alpha_n)k_n\|Ax_n - Ap\|^2 \\ &\quad + \|e_n\|k_n(\|e_n\| + 2\|x_n - p\|).\end{aligned}$$

This implies that

$$\begin{aligned}&r_n(2\alpha - r_n)(1-\alpha_n)k_n\|Ax_n - Ap\|^2 \\ &\le (k_n - 1)\|x_n - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \|e_n\|k_n(\|e_n\| + 2\|x_n - p\|).\end{aligned}$$

Using the restrictions imposed on $\{\alpha_n\}$ and $\{r_n\}$, we find that

$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0. \tag{2.6}$$

Since J_{r_n} is firmly nonexpansive, we find that

$$\begin{aligned}\|w_n - p\|^2 &\le \langle\big(x_n - r_nAx_n + e_n\big) - \big(p - r_nAp\big), w_n - p\rangle \\ &= \frac{1}{2}\{\|\big(x_n - r_nAx_n + e_n\big) - \big(p - r_nAp\big)\|^2 + \|w_n - p\|^2 \\ &\quad - \|\big(x_n - r_nAx_n + e_n\big) - \big(p - r_nAp\big) - (w_n - p)\|^2\} \\ &\le \frac{1}{2}\{\|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) + \|w_n - p\|^2 \\ &\quad - \|x_n - w_n - \big(r_n(Ax_n - Ap) - e_n\big)\|^2\} \\ &\le \frac{1}{2}\{\|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) + \|w_n - p\|^2 - \|x_n - w_n\|^2 \\ &\quad + 2\|x_n - w_n\|\|r_n(Ax_n - Ap) - e_n\| - \|r_n(Ax_n - Ap) - e_n\|^2\}.\end{aligned}$$

It follows that

$$\begin{aligned}\|w_n - p\|^2 &\le \|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) - \|x_n - w_n\|^2\\ &\quad + 2r_n\|x_n - w_n\|\|Ax_n - Ap\| + 2\|x_n - w_n\|\|e_n\|.\end{aligned}$$

This implies that

$$\begin{aligned}\|y_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n\|z_n - p\|^2\\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n\|w_n - p\|^2\\ &\le k_n\|x_n - p\|^2 + k_n\|e_n\|(\|e_n\| + 2\|x_n - p\|) - (1-\alpha_n)k_n\|x_n - w_n\|^2\\ &\quad + 2r_nk_n\|x_n - w_n\|\|Ax_n - Ap\| + 2k_n\|x_n - w_n\|\|e_n\|.\end{aligned}$$

Hence, we have

$$\begin{aligned}&(1-\alpha_n)k_n\|x_n - w_n\|^2\\ &\le (k_n - 1)\|x_n - p\|^2 + k_n\|e_n\|(\|e_n\| + 2\|x_n - p\|) + \|x_n - p\|^2 - \|x_{n+1} - p\|^2\\ &\quad + 2r_nk_n\|x_n - w_n\|\|Ax_n - Ap\| + 2k_n\|x_n - w_n\|\|e_n\|.\end{aligned}$$

By use of (2.6), we find from the restriction imposed on $\{\alpha_n\}$ that

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{2.7}$$

Since $\|x_n - z_n\| \le \|x_n - w_n\| + \|w_n - z_n\|$, we find from (2.5) and (2.7) that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{2.8}$$

Next, we show $x \in F(T)$. Note that

$$\begin{aligned}&\|\big(\beta_nx_n + (1-\beta_n)T^nx_n\big) - x_n\|\\ &\le \|\big(\beta_nx_n + (1-\beta_n)T^nx_n\big) - \big(\beta_nz_n + (1-\beta_n)T^nz_n\big)\|\\ &\quad + \|\big(\beta_nz_n + (1-\beta_n)T^nz_n\big) - x_n\|\\ &\le \beta_n\|x_n - z_n\| + (1-\beta_n)\|T^nx_n - T^nz_n\| + \|\big(\beta_nz_n + (1-\beta_n)T^nz_n\big) - x_n\|\\ &\le \beta_n\|x_n - z_n\| + (1-\beta_n)L\|x_n - z_n\| + \|\big(\beta_nz_n + (1-\beta_n)T^nz_n\big) - x_n\|,\end{aligned}$$

where L stands for the Lipschitz constant of T. By use of (2.4) and (2.8), we find that

$$\lim_{n\to\infty}\|\big(\beta_nx_n + (1-\beta_n)T^nx_n\big) - x_n\| = 0. \tag{2.9}$$

Note that

$$\begin{aligned}\|T^nx_n - x_n\| &\le \|T^nx_n - \big(\beta_nx_n + (1-\beta_n)T^nx_n\big)\|\\ &\quad + \|\big(\beta_nx_n + (1-\beta_n)T^nx_n\big) - x_n\|\\ &\le \beta_n\|T^nx_n - x_n\| + \|\big(\beta_nx_n + (1-\beta_n)T^nx_n\big) - x_n\|,\end{aligned}$$

which yields that

$$(1-\beta_n)\|T^nx_n - x_n\| \le \|\big(\beta_nx_n + (1-\beta_n)T^nx_n\big) - x_n\|.$$

Using the restriction imposed on $\{\beta_n\}$, we find from (2.9) that $\lim_{n\to\infty}\|T^nx_n - x_n\| = 0$. Since T is uniformly L-Lipschitz continuous, we can obtain that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. By use of Lemma 1.2, we find that $x \in F(T)$.

Next, we prove $x \in (A+B)^{-1}(0)$. Since $w_n = J_{r_n}(x_n - r_nAx_n + e_n)$, we find that

$$\frac{x_n - w_n + e_n}{r_n} - Ax_n \in Bw_n.$$

Let $\mu \in B\nu$. Since B is monotone, we find that

$$\left\langle \frac{x_n - w_n + e_n}{r_n} - Ax_n - \mu, w_n - \nu \right\rangle \geq 0.$$

It follows that $\langle -Ax - \mu, x - \nu\rangle \geq 0$. This implies that $-Ax \in Bx$, that is, $x \in (A + B)^{-1}(0)$.

Finally, we show that $x \in EP(F)$. Note that

$$F(z_n, z) + \frac{1}{s_n}\langle z - z_n, z_n - w_n\rangle \geq 0, \quad \forall z \in C.$$

Since F is monotone, we see that

$$\frac{1}{s_{n_i}}\langle z - z_{n_i}, z_{n_i} - w_{n_i}\rangle \geq F(z, z_{n_i}), \quad \forall z \in C.$$

By use of (2.5) and (2.8), we find that

$$F(z, x) \leq 0, \quad \forall z \in C.$$

For each t with $0 < t \leq 1$, let $z_t = tz + (1 - t)x$, where $z \in C$. It follows that $z_t \in C$ and hence $F(z_t, x) \leq 0$. It follows that

$$0 = F(z_t, z_t) \leq tF(z_t, z) + (1 - t)F(z_t, x) \leq tF(z_t, z),$$

which yields that $F(z_t, z) \geq 0$, $\forall z \in C$. Letting $t \downarrow 0$, we obtain that $F(x, z) \geq 0$, $\forall z \in C$. This implies that $x \in EP(F)$. Since $x \in \Omega$, we find that

$$\begin{aligned}\|x_1 - P_\Omega x_1\| \leq \|x_1 - x\| &\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \leq \|x_1 - P_\Omega x_1\|,\end{aligned}$$

which yields that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - x\| = \|x_1 - P_\Omega x_1\|.$$

Since H is a Hilbert space, we get that $\{x_{n_i}\}$ converges strongly to $P_\Omega x_1$. Therefore, we can conclude that $\{x_n\}$ converges strongly to $P_\Omega x_1$. The proof is completed. □

If T is an identity, we have the following result.

Corollary 2.2. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : H \rightrightarrows H$ be a maximal monotone mapping such that $Dom(B) \subset C$. Assume that $\Omega = (A + B)^{-1}(0) \cap EP(F)$ is not empty. Let $\{r_n\}$ and $\{s_n\}$ be two positive real number sequences. Let $\{\alpha_n\}$ be a real number sequence in $(0, 1)$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ F(z_n, z) + \frac{1}{s_n}\langle z - z_n, z_n - J_{r_n}(x_n - r_nAx_n + e_n)\rangle \geq 0, \quad \forall z \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)z_n, \\ C_{n+1} = \{\lambda \in C_n : \|y_n - \lambda\| \leq \|x_n - \lambda\| + \|e_n\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1, \end{cases}$$

where $\{e_n\}$ is a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Assume that the control sequences satisfy the following restrictions: $0 \leq \alpha_n \leq a < 1$, $\liminf_{n\to\infty} s_n > 0$ and $0 < r \leq r_n \leq r' < 2\alpha$, where a, r, r' are real constants. Then $\{x_n\}$ converges strongly to $P_\Omega x_1$.

Corollary 2.3. *Let C be a nonempty closed convex subset of H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : H \rightrightarrows H$ be a maximal monotone mapping such that $D(B) \subset C$. Let $T : C \to C$ be an asymptotically κ-strict pseudocontraction. Assume that $\Omega = F(T) \cap (A+B)^{-1}(0)$ is not empty and bounded. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ z_n = J_{r_n}(x_n - r_n A x_n + e_n), \\ y_n = \alpha_n x_n + (1-\alpha_n)\beta_n z_n + (1-\beta_n)(1-\alpha_n)T^n z_n, \\ C_{n+1} = \{\lambda \in C_n : \|y_n - \lambda\| \le \|x_n - \lambda\| + (\sqrt{k_n} - 1)\Theta_n + \sqrt{k_n}\|e_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1, \end{cases}$$

where $\{e_n\}$ is a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\Theta_n = \sup\{\|x_n - q\| : q \in \Omega\}$. Assume that the control sequences satisfy the following restrictions: $0 \le \alpha_n \le a < 1$, $\kappa \le \beta_n \le b < 1$, $0 < r \le r_n \le r' < 2\alpha$, where a, b, r, r' are real constants. Then $\{x_n\}$ converges strongly to $P_\Omega x_1$.

Proof. Set $F(x,y) = 0$, for any $x, y \in C$ and $s_n = 1$. Since $Dom(B) \subset C$, we see that $z_n = J_{r_n}\big(x_n - r_n A x_n + e_n\big)$. This completes the proof. □

If A and B are zero mappings, we find from Theorem 2.1 the following result immediately.

Corollary 2.4. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)-(A4). Let $T : C \to C$ be an asymptotically κ-strict pseudocontraction. Assume that $\Omega = F(T) \cap EP(F)$ is not empty and bounded. Let $\{s_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ F(z_n, z) + \frac{1}{s_n}\langle z - z_n, z_n - x_n - e_n\rangle \ge 0, \quad \forall z \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\beta_n z_n + (1-\beta_n)(1-\alpha_n)T^n z_n, \\ C_{n+1} = \{\lambda \in C_n : \|y_n - \lambda\| \le \|x_n - \lambda\| + (\sqrt{k_n} - 1)\Theta_n + \sqrt{k_n}\|e_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \ge 1, \end{cases}$$

where $\{e_n\}$ is a sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\Theta_n = \sup\{\|x_n - q\| : q \in \Omega\}$. Assume that the control sequences satisfy the following restrictions: $0 \le \alpha_n \le a < 1$, $\kappa \le \beta_n \le b < 1$, $\liminf_{n\to\infty} s_n > 0$, where a and b are real constants. Then $\{x_n\}$ converges strongly to $P_\Omega x_1$.

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References

[1] A. S. Antipin, *Solution methods for variational inequalities with coupled constraints*, Comput. Math. Math. Phys., **40** (2000), 1239–1254. 1

[2] B. A. Bin Dehaish, A. Latif, H. O. Bakodah, X. Qin, *A regularization projection algorithm for various problems with nonlinear mappings in Hilbert spaces,* J. Inequal. Appl., **2015** (2015), 14 pages. 1

[3] B. A. Bin Dehaish, X. Qin, A. Latif, H. O. Bakodah, *Weak and strong convergence of algorithms for the sum of two accretive operators with applications,* J. Nonlinear Convex Anal., **16** (2015), 1321–1336. 1

[4] E. Blum, W. Oettli, *From optimization and variational inequalities to equilibrium problems,* Math. Stud., **63** (1994), 123–145. 1.1

[5] F. E. Browder, W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space,* J. Math. Anal. Appl., **20** (1967), 197–228. 1

[6] P. Cheng, H. Wu, *On asymptotically strict pseudocontractions and equilibrium problems,* J. Inequal. Appl., **2013** (2013), 16 pages. 1

[7] S. Y. Cho, X. Qin, *On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems*, Appl. Math. Comput., **235** (2014), 430–438.

[8] S. Y. Cho, X. Qin, L. Wang, *Strong convergence of a splitting algorithm for treating monotone operators*, Fixed Point Theory Appl., **2014** (2014), 15 pages. 1

[9] W. Cholamjiak, P. Cholamjiak, S. Suantai, *Convergence of iterative schemes for solving fixed point problems for multi-valued nonself mappings and equilibrium problems*, J. Nonlinear Sci. Appl., **8** (2015), 1245–1256.

[10] B. S. Choudhury, S. Kundu, *A viscosity type iteration by weak contraction for approximating solutions of generalized equilibrium problem,* J. Nonlinear Sci. Appl., **5** (2012), 243–251. 1

[11] K. Goebel, W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings,* Proc. Amer. Math. Soc., **35** (1972), 171–174. 1

[12] R. H. He, *Coincidence theorem and existence theorems of solutions for a system of Ky Fan type minimax inequalities in FC-spaces,* Adv. Fixed Point Theory, **2** (2012), 47–57. 1

[13] C. Huang, X. Ma, *On generalized equilibrium problems and strictly pseudocontractive mappings in Hilbert spaces,* Fixed Point Theory Appl., **2014** (2014), 11 pages. 1

[14] S. M. Kang, S. Y. Cho, Z. Liu, *Convergence of iterative sequences for generalized equilibrium problems involving inverse-strongly monotone mappings,* J. Inequal. Appl., **2010** (2010), 16 pages. 1

[15] J. K. Kim, *Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-ϕ-nonexpansive mappings,* Fixed Point Theory Appl., **2011** (2011), 15 pages. 1

[16] J. K. Kim, P. N. Anh, Y. M. Nam, *Strong convergence of an extended extragradient method for equilibrium problems and fixed point problems*, J. Korean Math. Soc., **49** (2012), 187–200. 1

[17] M. Liu, S. S. Chang, *An iterative method for equilibrium problems and quasi-variational inclusion problems,* Nonlinear Funct. Anal. Appl., **14** (2009), 619–638. 1, 1

[18] P. E. Mainge, *A hybird extragradient-viscosity method for monotone operators and fixed point problems*, SIAM J. Control Optim., **47** (2008), 1499–1515. 1, 1

[19] S. Park, *Applications of the KKM principle on abstract convex minimal spaces*, Nonlinear Funct. Anal. Appl., **13** (2008), 179–191. 1

[20] L. Qihou, *Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings,* Nonlinear Anal., **26** (1996), 1835–1845. 1

[21] X. Qin, S. Y. Cho, L. Wang, *A regularization method for treating zero points of the sum of two monotone operators*, Fixed Point Theory Appl., **2014** (2014), 10 pages. 1

[22] X. Qin, S. Y. Cho, L. Wang, *Convergence of splitting algorithms for the sum of two accretive operators with applications*, Fixed Point Theory Appl., **2014** (2014), 12 pages. 1

[23] Y. Qing, S. Lv, *A strong convergence theorem for solutions of equilibrium problems and asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense*, Fixed Point Theory Appl., **2013** (2013), 14 pages. 1

[24] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., **149** (1970), 75–88. 1

[25] D. R. Sahu, H. K. Xu, J. C. Yao, *Asymptotically strict pseudocontractive mappings in the intermediate sense,* Nonlinear Anal., **70** (2009), 3502–3511. 1.2

[26] J. Shen, L. P. Pang, *An approximate bundle method for solving variational inequalities,* Commun. Optim. Theory, **1** (2012), 1–18. 1

[27] T. V. Su, *Second-order optimality conditions for vector equilibrium problems,* J. Nonlinear Funct. Anal., **2015** (2015), 31 pages. 1

[28] L. Sun, *Hybrid methods for common solutions in Hilbert spaces with applications,* J. Inequal. Appl., **2014** (2014), 16 pages. 1

[29] A. Tada, W. Takahashi, *Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem,* J. Optim. Theory Appl., **133** (2007), 359–370. 1

[30] J. Zhao, *Strong convergence theorems for equilibrium problems, fixed point problems of asymptotically nonexpansive mappings and a general system of variational inequalities*, Nonlinear Funct. Appl. Appl., **16** (2011), 447–464. 1

[31] L. C. Zhao, S. S. Chang, *Strong convergence theorems for equilibrium problems and fixed point problems of strict pseudo-contraction mappings*, J. Nonlinear Sci. Appl., **2** (2009), 78–91. 1