Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 1952–1956
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Remark on fundamentally non-expansive mappings in hyperbolic spaces

Cholatis Suanoom, Chakkrid Klin-eam*

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand.

Communicated by Y. J. Cho

Abstract

In this paper, we prove some properties of fixed point set of fundamentally non-expansive mappings and derive the existence of fixed point theorems as follows results of Salahifard et al. [H. Salahifard, S. M. Vaezpour, S. Dhompongsa, J. Nonlinear Anal. Optim., **4** (2013), 241–248] in hyperbolic spaces.

Keywords: Fixed point set, fundamentally non-expansive mappings, $\triangle$-closed set, convex set, hyperbolic spaces.
2010 MSC: 47H05, 47H10, 47J25.

1. Introduction and Preliminaries

There are many nonlinear mappings which are more general than the non-expansive ones. The existence problem of fixed point of those mappings is very useful in studying the theory of equations in science and applied science. Let X be a real Banach space and let K be a nonempty closed convex subset of X. A mapping $T : K \to K$ is said to be nonexpansive, if $||Tx - Ty|| \leq ||x - y||$, for all $x, y \in K$. In 2008, Suzuki [8] introduced condition C as follows.

Let T be a mapping on a subset K of a Banach space X. Then T is said to satisfy condition C (or Suzuki's generalized non-expansive) if

$$\frac{1}{2}||x - Tx|| \leq ||x - y|| \ \ implies \ \ ||Tx - Ty|| \leq ||x - y||$$

*Corresponding author
Email addresses: `Cholatis.Suanoom@gmail.com` (Cholatis Suanoom), `chakkridk@nu.ac.th` (Chakkrid Klin-eam)

Received 2015-09-18

for all $x, y \in K$.
It is obvious that every non-expansive mapping satisfies condition C, but the converse is not true. The next simple example can show this fact.

Example 1.1 ([2])**.** Define a mapping T on [0, 3] by

$$Tx = \begin{cases} 0 & x \neq 3, \\ 1 & x = 3. \end{cases}$$

Then T satisfies condition C, but T is not non-expansive.

In 2014, Ghoncheh and Razani [2], introduced the following definition and recalled some other conditions which generalize the Suzuki and studied fixed point for some generalized non-expansive mappings in Ptolemy spaces as follows.

Let X be a metric space and K be a subset of X. A mapping $T : K \to K$ is said to be fundamentally non-expansive if

$$d(T^2x, Ty) \leq d(Tx, y) \tag{1.1}$$

for all $x, y \in K$.

Proposition 1.2. *Every mapping which satisfies condition C is fundamentally non-expansive, but the converse is not true.*

Example 1.3. Suppose $X = \{(0, 0), (0, 1), (1, 1), (1, 2)\}$. Define

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Define T on X by $T(0, 0) = (1, 2),\ T(0, 1) = (0, 0),\ T(1, 1) = (1, 1),\ T(1, 2) = (0, 1)$.

Then T is fundamentally nonexpansive, but T dose not satisfy condition C.

In 2013, Salahifard et al. [7], introduced the fundamentally non-expansive mappings in complete $CAT(0)$ space and proved for some theorems as follows,

Theorem 1.4. *Let K be a bounded closed convex subset of complete $CAT(0)$ space X. Let $T : K \to K$ be fundamentally non-expansive and $F(T) \neq \emptyset$, then $F(T)$ is $\triangle$-closed and convex.*

Throughout this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [3]. A hyperbolic space is a metric space (X, d) with a mapping $W : X^2 \times [0, 1] \to X$ satisfying the following conditions.

(*i*) $d(u, W(x, y, \alpha)) \leq (1 - \alpha)d(u, x) + \alpha d(u, y)$;
(*ii*) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;
(*iii*) $W(x, y, \alpha) = W(y, x, 1 - \alpha)$;
(*iv*) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

Example 1.5. Let X be a real Banach space which is equipped with norm $|| \cdot ||$. Define the function $d : X^2 \to [0, \infty)$ by

$$d(x, y) = ||x - y||$$

as a meter on X. Let K be a nonempty bounded closed convex subset of Banach space. We see that (X, d) is a hyperbolic space with mapping $W : X^2 \times [0, 1] \to X$ which is defined by

$$W(x, y, \alpha) = (1 - \alpha)x + \alpha y.$$

Definition 1.6 ([3],[4],[6])**.** Let X be a hyperbolic space with a mapping $W : X^2 \times [0,1] \to X$.

(i) A nonempty subset $K \subseteq X$ is said to be convex, if $W(x,y,\alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0,1]$.

(ii) A hyperbolic space is said to be uniformly convex if for any $r > 0$ and $\epsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that for all $u, x, y \in X$

$$d(W(x,y,\frac{1}{2}),u) \leq (1-\delta)r,$$

provided $d(x,u) \leq r, d(y,u) \leq r$ and $d(x,y) \geq \epsilon r$.

(iii) A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\epsilon)$ for given $r > 0$ and $\epsilon \in (0,2]$, is known as a modulus of uniform convexity of X. η is said to be monotone, if it decreases with r (for a fixed ϵ), i.e., $\forall \epsilon > 0, \forall r_1 \geq r_2 > 0$ $[\eta(r_2,\epsilon) \leq \eta(r_1,\epsilon)]$.

Definition 1.7. Let (X,d) be a metric space and let K be a nonempty subset of X. We shall denote the fixed point set of a mapping T by $F(T) = \{x \in K : Tx = x\}$.

Definition 1.8. Let $\{x_n\}$ be a bounded sequence in a hyperbolic space (X,d). For $x \in X$, we define a continuous functional $r(\cdot,x_n) : X \to [0,\infty)$ by

$$r(x,x_n) = \limsup_{n\to\infty} d(x,x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x,x_n) : x \in X\}.$$

The asymptotic center $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $K \subseteq X$ is the set

$$A_K(\{x_n\}) = \{x \in X : r(x,x_n) \leq r(y,x_n), \quad \forall y \in K\}.$$

This implies that the asymptotic center is the set of minimizer of the functional $r(\cdot,x_n)$ in K. If the asymptotic center is taken with respect to X, then it is simply denoted by $A_K(\{x_n\})$. It is known that uniformly convex Banach spaces and CAT(0) spaces enjoy the property that bounded sequences have unique asymptotic centers with respect to closed convex subsets.

Lemma 1.9 ([1],[5])**.** *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in K has a unique asymptotic center in K.*

Lemma 1.10 ([1])**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \quad \forall n > 1. \tag{1.2}$$

If $\Sigma_{n=1}^{\infty}\delta_n < \infty$ and $\Sigma_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.
If there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.11 ([1])**.** *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a,b]$ for some $a, b \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n,x) \leq c, \quad \limsup_{n\to\infty} d(y_n,x) \leq c \;\; \textit{and} \;\; \limsup_{n\to\infty} d(W(x_n,y_n,\alpha_n),x) = c$$

for some $c \geq 0$. Then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

In this paper, we prove some properties of the fixed point set of fundamentally non-expansive mappings and derive the existence of fixed point theorems as follows results of Salahifard et al. [7] in hyperbolic spaces.

2. Main results

In this section, we shall prove some lemmas for fundamentally non-expansive mappings in a hyperbolic space.

Definition 2.1. Let X be a hyperbolic space and K be a nonempty bounded closed strictly convex subset of X. A mapping $T : K \to K$ is said to be fundamentally non-expansive if

$$d(T^2x, Ty) \leq d(Tx, y) \tag{2.1}$$

for all $x, y \in K$.

Lemma 2.2. *Let K be a nonempty bounded closed strictly convex subset of complete hyperbolic space X. Let $T : K \to K$ be fundamentally non-expansive and $F(T) \neq \emptyset$, then $F(T)$ is $\triangle$-closed and convex.*

Proof. Suppose that $\{x_n\}$ is a sequence in $F(T)$ which $\triangle$-converges to some $y \in K$. To show that $y \in F(T)$, we write

$$d(x_n, Ty) = d(T^2x_n, Ty) \leq d(Tx_n, y) = d(x_n, y),$$

thus

$$\limsup_{n\to\infty} d(x_n, Ty) \leq \limsup_{n\to\infty} d(x_n, y).$$

By the uniqueness of asymptotic center, we get $Ty = y$. Hence $F(T)$ is closed.
Next, we will show that $F(T)$ is convex. Let $x, y \in F(T)$ and each $\alpha \in [0, 1]$. Then,

$$d(x, Tz) = d(T^2x, Tz) \leq d(Tx, z) = d(x, z)$$

and

$$d(y, Tz) = d(T^2y, Tz) \leq d(Ty, z) = d(y, z).$$

For $z = W(x, y, \alpha)$, we have

$$\begin{aligned} d(x, y) &\leq d(x, Tz) + d(Tz, y) \\ &\leq d(x, z) + d(z, y) \\ &= d(x, W(x, y, \alpha)) + d(W(x, y, \alpha), y) \\ &\leq (1-\alpha)d(x, x) + \alpha d(x, y) + (1-\alpha)d(x, y) + \alpha d(y, y) \\ &= d(x, y). \end{aligned} \tag{2.2}$$

Thus $d(x, Tz) = d(x, z)$ and $d(Tz, y) = d(z, y)$, because if $d(x, Tz) < d(x, z)$ or $d(Tz, y) < d(z, y)$, then which the contradiction to $d(x, y) < d(x, y)$, therefore $Tz = W(x, y, \alpha)$ and $Tz = z$, and then $W(x, y, \alpha) \in F(T)$. Hence $F(T)$ is convex.

□

Lemma 2.3. *Let K be a nonempty bounded closed subset of complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T : K \to K$ be fundamentally non-expansive, then $F(T)$ is nonempty.*

Proof. By Lemma 1.9, the asymptotic center of any bounded sequence is in K, particularly, the asymptotic center of approximate fixed point sequence for T is in K. Let $A(\{x_n\}) = \{y\}$, we want to show that y is a fixed point of T. We can consider

$$d(x_n, Ty) \leq d(T^2x_n, Ty) \leq d(Tx_n, y) = d(x_n, y),$$

hence

$$\limsup_{n\to\infty} d(x_n, Ty) \leq \limsup_{n\to\infty} d(x_n, y).$$

By the uniqueness of the asymptotic center $Ty = y$. □

Theorem 2.4. *Let K be a nonempty bounded closed strictly convex subset of complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T : K \to K$ be fundamentally non-expansive, then $F(T)$ is nonempty $\triangle$-closed and convex.*

Proof. By Lemmas 2.2 and 2.3, we get that $F(T)$ is nonempty $\triangle$-closed and convex. □

Acknowledgements

The authors would like to thank Science Achievement Scholarship of Thailand, which provides funding for research.

References

[1] S. Chang, G. Wang, L. Wang, Y. K. Tang, Z. L. Zhao, *Δ-convergence theorems for multi-valued nonexpansive mappings in hyperbolic spaces*, Appl. Math. Comput., **249** (2014), 535–540. 1.9, 1.10, 1.11

[2] S. J. H. Ghoncheh, A. Razani, *Fixed point theorems for some generalized nonexpansive mappings in Ptolemy spaces*, Fixed Point Theory Appl., **2014** (2014), 11 pages. 1.1, 1

[3] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis*, Trans. Amer. Math. Soc., **357** (2005), 89–128. 1, 1.6

[4] L. Leustean, *A quadratic rate of asymptotic regularity for $CAT(0)$-spaces,* J. Math. Anal. Appl., **325** (2007), 386–399. 1.6

[5] L. Leustean, *Nonexpansive iterations in uniformly convex W-hyperbolic spaces*, Nonlinear Anal. Optim., **513** (2010), 193–209. 1.9

[6] T. C. Lim, *Remarks on some fixed point theorems*, Proc. Am. Math. Soc., **60** (1976), 179–182. 1.6

[7] H. Salahifard, S. M. Vaezpour, S. Dhompongsa, *Fixed point theorems for some generalized nonexpansive mappings in $CAT(0)$ spaces*, J. Nonlinear Anal. Optim., **4** (2013), 241–248. 1, 1

[8] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl., **340** (2008), 1088–1095. 1