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A new Househölder's method free from second derivatives for solving nonlinear equations and polynomiography

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Abstract

In this paper, we describe the new Husehölder's method free from second derivatives for solving nonlinear equations. The new Husehölder's method has convergence of order five and efficiency index $5^{\frac{1}{3}} \approx 1.70998$, which converges faster than the Newton's method, the Halley's method and the Husehölder's method. The comparison table demonstrate the faster convergence of our method. Polynomiography via the new Husehölder's method is also presented.

Keywords: Nonlinear equation, Newton's method, Halley's method, Husehölder's method, polynomiography.
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1. Introduction

One of the most frequently problems in Sciences and more specifically in Mathematics is solving a nonlinear equation

$$f(x) = 0 \tag{1.1}$$

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with $f : D \subset \mathbb{R} \to \mathbb{R}$, where D is an open connected set. Except special cases, the solutions of these kind of equations cannot be solved in a direct way. That is why most of the methods for solving these equations are iterative.

Equation (1.1) is solvable iteratively by the Newton's method and a range of its variants [17] as well as by other techniques. The Newton's method defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0, \tag{1.2}$$

converge quadratically in some neighborhood of α.

Some Newton-type methods with third-order convergence that do not require the computation of second-order derivatives have been developed in [2, 3, 7, 8, 13, 16, 18]. Other classes of those iterative methods invoke the Adomian decomposition method as in [1] and He's homotopy perturbation method [5]. One class of those methods have been derived based on quadrature formulas for the computation of the integral

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt, \tag{1.3}$$

arising from the Newton's method. In [18], by solving (1.3), Weerakoon and Fernando derived the following modified Newton's method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) + f'(x_n - \frac{f(x_n)}{f'(x_n)})}, \tag{1.4}$$

which converge cubically. In [3, 16], solving (1.3), the authors yields method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - \frac{f(x_n)}{2f'(x_n)})}. \tag{1.5}$$

The method (1.5) has also been derived by Homeier in [7]. A further multivariate version of this method has been discussed in [4, 6].

By applying Newton's method to the inverse function $x = f(y)$ instead $y = f(x)$, in [7], Homeier derived the following cubically convergent iteration scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - \frac{f(x_n)}{f'(x_n)})}\right). \tag{1.6}$$

The method leading to (1.6) has also been derived by Özban in [16].

Finally, in [13], Kou et al. considered the Newton's method on a new interval of integration and arrived at the following cubically convergent iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n - \frac{f(x_n)}{f'(x_n)})}{f'(x_n)}. \tag{1.7}$$

Any of the aforementioned methods require only first order derivative of the given function. The iterative methods with a higher-order convergence are important which do not require second derivatives from the practical point of view and is an area of current active research.

In this paper, we present a new Househölder's method free from second derivatives having fifth-order convergence. Its efficiency is demonstrated by numerical examples and polynomiography of few complex polynomials is also presented.

2. New Househölder's method

The iterative methods with higher-order convergence are presented in some literature [2, 3, 7, 8, 13, 16, 18]. In [8], Househölder gives an iterative method, called the Househölder's method provisionally, which is expressed as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \frac{f(x_n)f''(x_n)}{2f'^2(x_n)}\right), \quad n \geq 0. \tag{2.1}$$

The Househölder's method has third-order convergence and it requires the evaluation of first and second derivatives of the function $f(x)$. As many functions do not have second derivatives, so the Househölder's method do not work for such functions.

If we use

$$f''(x) = \frac{f'(y) - f'(x)}{y - x} \tag{2.2}$$

in (2.1), then we have our new Househölder's method free from second derivatives having fifth-order convergence as follows:

Algorithm 2.1.

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, ..., \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)}\left(1 - \frac{f'(y_n)f'(x_n)f(y_n) - f'^2(x_n)f(x_n)}{2f'^2(y_n)f(x_n)}\right). \end{aligned}$$

3. Convergence analysis

In this section we consider the convergence criteria of Algorithm 2.1.

Theorem 3.1. *Let $f : X \subset \mathbb{R} \to \mathbb{R}$ for an open interval X and consider that the nonlinear equation $f(x) = 0$ has a simple root $\alpha \in X$, where $f(x)$ be sufficiently smooth in the neighborhood of α. Then the convergence order of new Househölder's method given in* Algorithm 2.1 *is at least five.*

Proof. If α is the root and e_n be the error at nth iteration, then $e_n = x_n - \alpha$ and using Taylor series expansion, we have

$$f(x_n) = e_n f'(\alpha) + \frac{e_n^2}{2!}f''(\alpha) + \frac{e_n^3}{3!}f'''(\alpha) + \frac{e_n^4}{4!}f^{(4)}(\alpha) + \cdots + \frac{e_n^7}{7!}f^{(7)}(\alpha) + O(e_n^8), \tag{3.1}$$

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots + c_7e_n^7 + O(e_n^8)], \tag{3.2}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots + 7c_7e_n^6 + O(e_n^7)], \tag{3.3}$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, $k = 2, 3, 4, ...$ and $e_n = x_n - \alpha$.

From (3.2) and (3.3), we have

$$\begin{aligned} y_n = \alpha &+ c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 \\ &+ (-6c_3^2 + 20c_2^2c_3 - 10c_2c_4 + 4c_5 - 8c_2^4)e_n^5 \\ &+ (-17c_4c_3 + 28c_4c_2^2 - 13c_2c_5 + 5c_6 + 33c_2c_3^2 - 52c_3c_2^3 + 16c_2^5)e_n^6 \\ &+ (-22c_3c_5 + 36c_5c_2^2 + 6c_7 - 16c_2c_6 - 12c_4^2 + 92c_2c_3c_4 \\ &- 72c_4c_2^3 + 18c_3^3 - 126c_2^2c_3^2 + 128c_3c_2^4 - 32c_2^6)e_n^7 + O(e_n^8). \end{aligned} \tag{3.4}$$

Using Taylor's series, we have,

$$\begin{aligned} f(y_n) = f'(\alpha)[&c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 \\ &+ (-6c_3^2 + 24c_2^2c_3 - 10c_2c_4 + 4c_5 - 12c_2^4)e_n^5 \\ &+ (-17c_4c_3 + 34c_4c_2^2 - 13c_2c_5 + 5c_6 + 37c_2c_3^2 - 73c_3c_2^3 + 28c_2^5)e_n^6 \\ &+ (-22c_3c_5 + 44c_5c_2^2 + 6c_7 - 16c_2c_6 - 12c_4^2 + 104c_2c_3c_4 \\ &- 104c_4c_2^3 + 18c_3^3 + 160c_2^2c_3^2 + 206c_3c_2^4 - 64c_2^6)e_n^7 + O(e_n^8), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} f'(y_n) = f'(\alpha)[1 &+ 2c_2^2e_n^2 + (4c_2c_3 - 4c_2^3)e_n^3 \\ &+ (6c_2c_4 - 11c_2^2c_3 + 8c_2^4)e_n^4 + (28c_3c_2^3 - 20c_2^2c_4 + 8c_2c_5 - 16c_2^5)e_n^5 \\ &+ (-16c_2c_4c_3 - 68c_3c_2^4 + 12c_3^3 + 60c_4c_2^3 - 26c_5c_2^2 + 10c_2c_6 + 32c_2^6)e_n^6 \\ &+ (-84c_3^3c_2 + 112c_3c_4c_2^2 - 64c_2^7 - 20c_3c_2c_5 + 160c_3c_2^5 + 36c_3^2c_4 \\ &+ 72c_5c_2^3 + 12c_2c_7 - 32c_2^2c_6 - 24c_2c_4^2 - 168c_4c_2^4)e_n^7 + O(e_n^8). \end{aligned} \tag{3.6}$$

Using (3.2)-(3.6) in Algorithm 2.1, we have

$$x_{n+1} = \alpha + \left(- \frac{3}{2}c_2^2c_3\right)e_n^5 + O(e_n^6),$$

$$e_{n+1} = \left(- \frac{3}{2}c_2^2c_3\right)e_n^5 + O(e_n^6),$$

which shows that Algorithm 2.1 is fifth-order convergence. □

4. Numerical Examples

We present some examples to illustrate the efficiency of the developed two-step iterative method in this paper. We compare the Newton method, the Halley's method [15], the Househölder's method [8] and our new Househölder's method (Algorithm 2.1) introduced in this present paper. We used $\varepsilon = 10^{-15}$. The following stopping criteria is used for computer programs:

1. $|x_{n+1} - x_{n+1}| < \varepsilon$.
2. $|f(x_{n+1})| < \varepsilon$.

Table 1. Comparison of various iterative methods
($f(x) = x^3 + 4x^2 - 10$, $x_0 = 2$)

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
NM	5	10	$2.040551e-18$	
HM	3	9	$8.907717e-17$	1.3652300134140968
HHM	4	12	$6.184837e-41$	
NHHM	3	9	$5.427683e-18$	

Table 2. Comparison of various iterative methods
($f(x) = \sin^2 x - x^2 + 1$, $x_0 = 1$)

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
NM	6	12	$1.819126e-25$	
HM	4	12	$2.527247e-38$	1.4044916482153412
HHM	5	15	$9.230984e-28$	
NHHM	3	9	$4.576466e-20$	

Table 3. Comparison of various iterative methods
($f(x) = x - e^x - 3x + 2$, $x_0 = 0.8$)

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
NM	4	8	$1.253473e-25$	
HM	3	9	$6.088025e-22$	0.2575302854398609
HHM	3	9	$2.374454e-22$	
NHHM	2	6	$4.813705e-16$	

Table 4. Comparison of various iterative methods
($f(x) = x^3 + 4x^2 - 15$, $x_0 = 1$)

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
NM	6	12	$5.386529e-30$	
HM	4	12	$1.765952e-44$	1.6319808055660635
HHM	4	12	$8.204846e-21$	
NHHM	3	9	$3.167116e-24$	

Table 5. Comparison of various iterative methods
($f(x) = x - \frac{1}{x}$, $x_0 = -2$)

Method	N	N_f	$\lvert f(x_{n+1})\rvert$	x_{n+1}
NM	6	12	$2.158638e-15$	
HM	4	12	$9.020661e-39$	−1.0000000000000000
HHM	4	12	$2.166186e-22$	
NHHM	3	9	$3.557950e-21$	

Tables 1-5. Shows the numerical comparisons of the Newton's method (NM), the Halley's method (HM), the Househölder's method (HHM) and the new Househölder's method (Algorithm 2.1) (NHHM). The columns represent the number of iterations N and the number of functions or derivatives evaluations N_f required to meet the stopping criteria, and the magnitude $|f(x)|$ of $f(x)$ at the final estimate x_n.

5. Polynomiographs

Polynomials are one of the most significant objects in many fields of mathematics. Polynomial root-finding has played a key role in the history of mathematics. It is one of the oldest and most deeply studied mathematical problems. The last interesting contribution to the polynomials root finding history was made by Kalantari [11], who introduced the polynomiography. As a method which generates nice looking graphics, it was patented by Kalantari [10] in 2005. Polynomiography is defined to be "the art and science of visualization in approximation of the zeros of complex polynomials, via fractal and non fractal images created using the mathematical convergence properties of iteration functions" [11]. An individual image is called a "polynomiograph". Polynomiography combines both art and science aspects.

Polynomiography gives a new way to solve the ancient problem by using new algorithms and computer technology. Polynomiography is based on the use of one or an infinite number of iteration methods formulated for the purpose of approximation of the root of polynomials, for example, the Newton's method, the Halley's method, etc. The word "fractal", which partially appeared in the definition of polynomiography, was coined by the famous mathematician Mandelbrot [14]. Both fractal images and polynomiographs can be obtained via different iterative schemes. Fractals are self-similar has typical structure and independent of scale. On the other hand, polynomiographs are quite different. The "polynomiographer" can be controlled the shape and designed in a more predictable way by using different iteration methods to the infinite variety of complex polynomials. Generally, fractals and polynomiographs belong to different classes of graphical objects. Polynomiography has diverse applications in math, science, education, art and design.

According to Fundamental Theorem of Algebra, any complex polynomial with complex coefficients $\{a_n, a_{n-1}, ..., a_1, a_0\}$:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \tag{5.1}$$

or by its zeros (roots) $\{r_1, r_2, ..., r_{n-1}, r_n\}$:

$$p(z) = (z - r_1)(z - r_2) \cdots (z - r_n) \tag{5.2}$$

of degree n has n roots (zeros) which may or may not be distinct. The degree of polynomial describes the number of basins of attraction and placing roots on the complex plane manually localization of basins can be controlled.

Usually, polynomiographs are colored based on the number of iterations needed to obtain the approximation of some polynomial root with a given accuracy and a chosen iteration method. The description of polynomiography, its theoretical background and artistic applications are described in [9, 10, 12].

6. Iteration

During the last century, the different numerical techniques for solving nonlinear equation $f(x) = 0$ have been successfully applied. Now we define

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, ...,$$
$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}\left(1 - \frac{f'(y_n)f'(x_n)f(y_n) - f'^2(x_n)f(x_n)}{2f'^2(y_n)f(x_n)}\right).$$

This is so-called the Househölder's method free from second derivatives for solving nonlinear equations. Let $p(z)$ be the complex polynomial, then

$$\begin{aligned} y_n &= z_n - \frac{p(z_n)}{p'(z_n)}, n = 0, 1, 2, ..., \\ z_{n+1} &= y_n - \frac{p(y_n)}{p'(y_n)}\left(1 - \frac{p'(y_n)p'(z_n)p(y_n) - p'^2(z_n)p(y_n)}{2p'^2(y_n)p(z_n)}\right), \end{aligned} \tag{6.1}$$

where $z_0 \in \mathbb{C}$ is a starting point, is so-called the Househölder's method free from second derivatives for solving nonlinear complex equations. The sequence $\{z_n\}_{n=0}^{\infty}$ is called the orbit of the point z_0 converges to a root z^* of p then, we say that z_0 is attracted to z^*. A set of all such starting points for which $\{z_n\}_{n=0}^{\infty}$ converges to root z^* is called the basin of attraction of z^*.

7. Convergence test

In the numerical algorithms that are based on iterative processes we need a stop criterion for the process, that is a test that tells us that the process has converged or it is very near to the solution. This type of test is called a convergence test. Usually, in the iterative process that use a feedback, like the root finding methods, the standard convergence test has the following form:

$$|z_{n+1} - z_n| < \varepsilon, \tag{7.1}$$

where z_{n+1} and z_n are two successive points in the iteration process and $\varepsilon > 0$ is a given accuracy. In this paper we also use the stop criterion (7.1).

8. Applications

The applications of the Househölder's method free from second derivatives for solving nonlinear complex equations perturbs the shape of polynomial basins and makes the polynomiographs look more "fractal". The aim of using the Househölder's method free from second derivatives for solving nonlinear complex equations to create images that are quite new, different from images by the Newton's method and interesting from the aesthetic point of view.

In this section we present some examples of polynomiographs for different complex polynomials equation $p(z) = 0$ and some special polynomials. The different colors of a images depend upon number of iterations to reach a root with given accuracy $\varepsilon = 0.001$. One can obtain infinitely many nice looking polynomiographs by changing parameter k, where k is the upper bound of the number of iterations.

8.1. Polynomiograph for $z^2 - 1 = 0$

Complex polynomial equation $z^2 - 1 = 0$ having two roots: 1 and -1. The polynomiograph is presented in the following figure with two distinct basins of attraction to the two roots of the polynomial $z^2 - 1 = 0$.

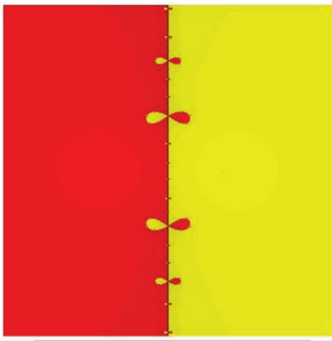

Figure 1. Polynomiograph for $z^2 - 1 = 0$

8.2. Polynomiograph for $z^3 - 1 = 0$

Complex polynomial equation $z^3 - 1 = 0$ having three roots: 1, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$. The polynomiograph is presented in the following figure with three distinct basins of attraction to the three roots of the polynomial $z^3 - 1 = 0$.

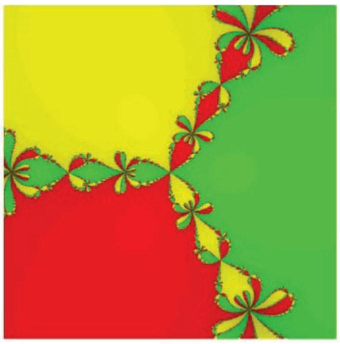

Figure 2. Polynomiograph for $z^3 - 1 = 0$

8.3. Polynomiograph for $z^4 - 1 = 0$

Complex polynomial equation $z^4 - 1 = 0$ having four roots: -1, $-i$, i and 1. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $z^4 - 1 = 0$.

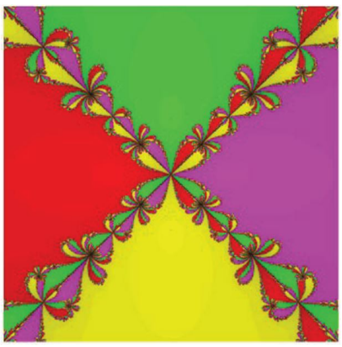

Figure 3. Polynomiograph for $z^4 - 1 = 0$

8.4. Polynomiograph for $z^4 - z^3 + z^2 - z + 1 = 0$

Complex polynomial equation $z^4 - z^3 + z^2 - z + 1 = 0$ having four roots: $-0.309017 - 0.951057i$, $-0.309017 + 0.951057i$, $0.809017 - 0.587785i$ and $0.809017 + 0.587785i$. The polynomiograph is presented in the following figure with four distinct basins of attraction to the four roots of the polynomial $z^4 - z^3 + z^2 - z + 1 = 0$.

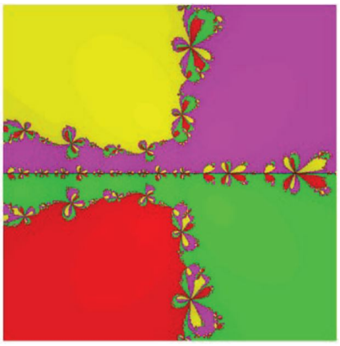

Figure 4. Polynomiograph for $z^4 - z^3 + z^2 - z + 1 = 0$

8.5. Polynomiograph for $z(z^2 + 1)(z^2 + 4) = 0$

Complex polynomial equation $z(z^2 + 1)(z^2 + 4) = 0$ having five roots: 0, $0 - 1i$, $0 + 1i$, $0 - 2i$ and $0 + 2i$. The polynomiograph is presented in the following figure with five distinct basins of attraction to the five roots of the polynomial $z(z^2 + 1)(z^2 + 4) = 0$.

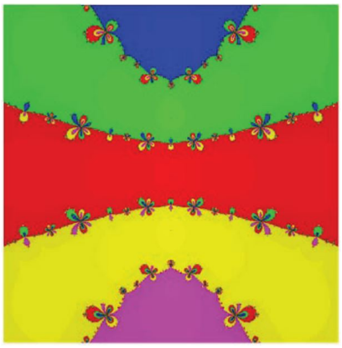

Figure 5. Polynomiograph for $z(z^2 + 1)(z^2 + 4) = 0$

8.6. Polynomiograph for $z^6 - \frac{1}{2}z^5 + \frac{11(1+i)}{4}z^4 - \frac{19+3i}{4}z^3 + \frac{11+5i}{4}z^2 - \frac{11+i}{4}z + \frac{3}{2} - 3i = 0$

Complex polynomial equation $z^6 - \frac{1}{2}z^5 + \frac{11(1+i)}{4}z^4 - \frac{19+3i}{4}z^3 + \frac{11+5i}{4}z^2 - \frac{11+i}{4}z + \frac{3}{2} - 3i = 0$ having six roots: $-1 + 2i$, $-0.5 - 0.5i$, $0 + ii$, $0 - 1.5i$, $1 - 1i$ and 1. The polynomiograph is presented in the following figure with six distinct basins of attraction to the six roots of the polynomial $z^6 - \frac{1}{2}z^5 + \frac{11(1+i)}{4}z^4 - \frac{19+3i}{4}z^3 + \frac{11+5i}{4}z^2 - \frac{11+i}{4}z + \frac{3}{2} - 3i = 0$.

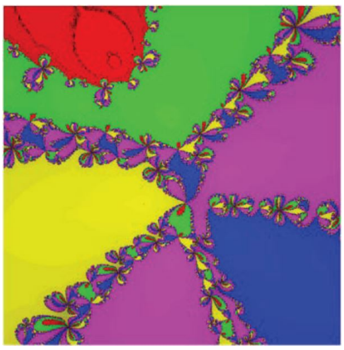

Figure 6. Polynomiograph for $z^6 - \frac{1}{2}z^5 + \frac{11(1+i)}{4}z^4 - \frac{19+3i}{4}z^3 + \frac{11+5i}{4}z^2 - \frac{11+i}{4}z + \frac{3}{2} - 3i = 0$

8.7. Polynomiograph for $z^5 - 1 = 0$

Complex polynomial equation $z^5-1=0$ having five roots: $-0.809017-0.587785i$, $-0.809017+0.587785i$, $0.309017-0.951057i$, $0.309017+0.951057i$ and 1. The polynomiograph is presented in the following figure with five distinct basins of attraction to the five roots of the polynomial $z^5-1=0$.

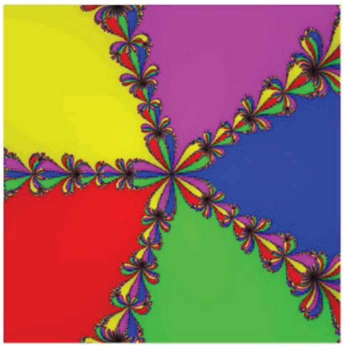

Figure 7. Polynomiograph for $z^5 - 1 = 0$

9. Conclusions

A new Househölder's method free from second derivatives for solving nonlinear equations has been established. We can concluded from Tables 1-5 that

1. The new Househölder's method has an efficiency of $5^{\frac{1}{3}} \approx 1.70998$.
2. The new Househölder's method has convergence of order five.

By using some examples the performance of the new Househölder's method is also discussed. The new Househölder's method is performing very well in comparison to the Newton's method, the Halley's method and the Househölder's method as discussed in Tables 1-5.

We also presented some examples of polynomiographs for different complex polynomials equation $p(z) = 0$ and some special polynomials. We used the new Househölder's method free from second derivatives for solving nonlinear complex equations to create images that are quite new, different from images by the Newton's method and interesting from the aesthetic point of view.

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