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Construction of Tri-parametric derivative free fourth order with and without memory iterative method

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Abstract

We have given two general methods of converting with derivative two-step methods to without derivative two-step methods. It can also be observed that this conversion not only retain the optimal order of convergence of the two-step methods but the resulting derivative free families of iterative methods are also extendable to with memory class. The with-memory methods show greater acceleration in the order of convergence. In this way, order of convergence is accelerated from 4 to 7.53 at the most. An extensive comparison of our methods is done with the recent methods of respective domain.

Keywords: With and without memory methods, derivative free, self accelerating parameters, accelerated order of convergence.
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1. Introduction

Let us consider the problem of finding simple root of a nonlinear equation $f(x) = 0$, where f is a real valued function defined on an interval $I \subseteq \mathbb{R}$. Newton-Raphson method has always remained a widely applicable method for solving this problem, however, it has a drawback that it requires derivative evaluation of the function involved. Many higher order with derivative iterative method have been developed in the recent past [2, 10]. In order to remove this drawback, Steffensen [6] defined a derivative free method by removing the derivative evaluation with forward difference approximation. Steffensen's method is given as:

$$w_n = x_n + f(x_n), \qquad x_{n+1} = x_n - \frac{f(x_n)}{f[w_n, x_n]}.$$

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It is important to mention here that the order of Steffensen's method is also quadratic. Traub (see [8], pp. 185–187) was the first to introduce the parameter based method of Steffensen's type which he further extended to the class of with memory methods. Traub's modification of Steffensen's method is given as :

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f[w_n, x_n]}, \\ N_1(x) &= f(x_n) + (x - x_n) f[x_n, w_n], \\ \gamma_{n+1} &= -\frac{1}{N_1'(x_n)}, \\ w_{n+1} &= x_{n+1} + \gamma_{n+1} f(x_{n+1}), \gamma_n \neq 0, \end{aligned}$$

where w_0, γ_0 and x_0 are suitably chosen. The parameter γ_n is called self accelerating parameter and the method has order of convergence 2.41.

In the recent past, many such attempts has been made to define without derivative methods of Steffensen's like involving parameters through which the order of the iterative methods can further be accelerated [9]. Recently, A. Cordero and J. R. Torregrosa [1] gave a general method of converting with derivative two and three step methods to derivative free Steffensen's like methods by preserving the order of convergence but their methods were not extendable to with memory methods.

In this paper, we give a general technique of constructing derivative free two-step methods involving parameters from with derivative two-step methods that are also extendable to with memory class. The construction and order of convergence is discussed in Section 2. In Section 3, we extend the without memory iterative methods to with memory methods by approximating the accelerating parameters involved at each step by use of Newton's interpolating polynomials. In Section 4, we give some weight functions and some particular methods. In Section 5, we give numerical comparisons of our new methods with the existing efficient methods.

2. Two-Step derivative free optimal method with order four

Let us consider the iterative methods by R. F. King [3] and Kung-Traub [4]. We extend it to derivative free family of methods by using the approximation $f'(x_n) \approx f[w_n, x_n] + pf(w_n)$,where $w_n = x_n + qf(x_n)$ at the first step and $f'(x_n) \approx \frac{f[w_n, y_n] + pf(w_n)}{G(u_n, v_n)}$ at the second step with

$$u_n = \frac{f(y_n)}{f(x_n)}, v_n = \frac{f(y_n)}{f(w_n)}.$$

Thus, we obtain two-step derivative free method in the following form:

$$\begin{aligned} w_n &= x_n + qf(x_n) \\ y_n &= x_n - \frac{f(x_n)}{f[w_n, x_n] + pf(w_n)}, \\ x_{n+1} &= y_n - G(u_n, v_n) \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)} \frac{f(y_n)}{f[w_n, y_n] + pf(w_n)}. \end{aligned} \tag{2.1}$$

In the similar manner, for the Kung and Traub method [4] given by:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{1}{(1 - \frac{f(y_n)}{f(x_n)})^2} \frac{f(y_n)}{f'(x_n)}, \end{aligned}$$

using the similar approximation in the method to make it derivative free as follows:

$$\begin{aligned} w_n &= x_n + qf(x_n), \\ y_n &= x_n - \frac{f(x_n)}{f[w_n, x_n] + pf\,(w_n)}, \\ x_{n+1} &= y_n - G(u_n, v_n)\frac{1}{(1-\frac{f(y_n)}{f(x_n)})^2}\frac{f(y_n)}{f[w_n, y_n] + pf\,(w_n)}, \end{aligned} \tag{2.2}$$

where, $G(u_n, v_n)$ is the weight function. It should be found in such a way that the above method gives optimal order 4.

Moreover, we consider again the two methods and apply the second technique to make them derivative free as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)}\frac{f(y_n)}{f'(x_n)}. \end{aligned} \tag{2.3}$$

By using the approximation $f'(x_n) \approx f[w_n, x_n] + pf(w_n)$, where $w_n = x_n + qf(w_n)$ at the first step and $f'(x_n) \approx \frac{f[w_n,y_n]+pf(w_n)+s(y_n-w_n)(y_n-x_n)}{H(u_n)}$ at the second step with

$$u_n = \frac{f(y_n)}{f(x_n)},$$

we get,

$$\begin{aligned} w_n &= x_n + qf(x_n), \\ y_n &= x_n - \frac{f(x_n)}{f[w_n, x_n] + pf(w_n)}, \\ x_{n+1} &= y_n - H(u_n)\frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)}\frac{f(y_n)}{f[w_n, y_n] + pf(w_n) + s(y_n - w_n)(y_n - x_n)}. \end{aligned} \tag{2.4}$$

In the similar manner, Kung and Traub method [4] is given as:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{1}{(1-\frac{f(y_n)}{f(x_n)})^2}\frac{f(y_n)}{f'(x_n)}. \end{aligned} \tag{2.5}$$

Using the similar approximation in this method, we make it derivative free as follows:

$$\begin{aligned} w_n &= x_n + qf(x_n), \\ y_n &= x_n - \frac{f(x_n)}{f[w_n, x_n] + pf(w_n)}, \\ x_{n+1} &= y_n - H(u_n)\frac{1}{(1-\frac{f(y_n)}{f(x_n)})^2}\frac{f(y_n)}{f[w_n, y_n] + pf(w_n) + s(y_n - w_n)(y_n - x_n)}. \end{aligned} \tag{2.6}$$

The weight function $H(u_n)$ should be found in such a way that the above methods give optimal order four.

Following theorems shows that the methods (2.2) and (2.4) have convergence order four.

Theorem 2.1. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a real valued, differentiable function. Let function has a simple root α in the open interval denoted by I. Let x_0 be the initial guess sufficiently close to α. Then method (2.2) has order of convergence 4 if the weight function $G(u, v)$ satisfies the following conditions:*

$$G(0,0) = 1, G_u(0,0) = -1, G_v(0,0) = 0$$

and

$$|G_{u^2}(0,0)|, |G_{v^2}(0,0)|, |G_{uv}(0,0)| < \infty.$$

The error equation of (2.2) for all values of p and q is given as follows:

$$e_{n+1} = -\frac{1}{2}(c_2 + p)(1 + qf'(\alpha))A_2 e_n^4 + O(e_n^5),$$

where A_2 involves $G_{u^2}(0,0), G_{v^2}(0,0)$ and $G_{uv}(0,0)$.

Proof. Let $f(x)$ is a non-linear function and α be its real zero. That is $f(\alpha) = 0$, and let e_n denote the error at the n^{th} step as follows:

$$e_n = x_n - \alpha.$$

Now using Taylor series to expand $f(x)$ about α. we have,

$$f(x) = f'(\alpha)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O(e_n^5)\right], \tag{2.7}$$

whereas,

$$c_r = \frac{f^{(r)}(\alpha)}{r! f'(\alpha)}, r = 2, 3, ...$$

We have the error term of $w_n = x_n + qf(x_n)$ by using the Taylor's series,

$$e_{n,w} = (1 + qf'(\alpha))e_n + c_2 q f'(\alpha)e_n^2 + c_3 q f'(\alpha)e_n^3 + ...O(e_n^5), \tag{2.8}$$

where,

$$e_{n,w} = w_n - \alpha.$$

Thus,

$$f(w_n) = f'(\alpha)\left((1 + qf'(\alpha))e_n + (3 + qf'(\alpha)) + 1)c_2 e_n^2\right) + ...O(e_n^5). \tag{2.9}$$

Hence,

$$\begin{aligned} f[x_n, w_n] + pf(w_n) = & f'(\alpha) + f'(\alpha)[p(1 + qf'(\alpha)) + c_2(2 + qf'(\alpha))]e_n \\ & + f'(\alpha)[f'(\alpha)q(3 + qf'(\alpha))(c_3 + pc_2) + 3c_3 \\ & + c_2(p + qc_2^2 f'(\alpha))]e_n^2 + ... + O(e_n^5). \end{aligned} \tag{2.10}$$

Also,

$$\begin{aligned} \frac{f(x_n)}{f[x_n, w_n] + pf(w_n)} = & e_n - (c_2 + p)(1 + qf'(\alpha))e_n^2 + [((1 + qf'(\alpha)) \\ & (2c_2(c_2 + p) + (1 + f'(\alpha))q^2) - c_3(2 + 3qf'(\alpha)) \\ & + qc_1^2(qc_2(c_2 + q) - c_3)]e_n^3 + ... + O\left(e_n^5\right). \end{aligned} \tag{2.11}$$

Now, by using (2.9) and (2.11) we have the error term of y_n as follows:

$$e_{n,y} = (c_2 + p)(1 + qf'(\alpha))e_n^2 + ... + O(e_n^5). \tag{2.12}$$

Now, with $G(0,0) = 1$

$$e_{n+1} = -(c_2+p)(1+qf^{'}(\alpha))(c_1 q G_u(0,0) + qc_1 + G_u(0,0) + 1 + G_v(0,0))e_n^3 + O(e_n^4).$$

Substituting,

$$G_u(0,0) = -1, G_v(0,0) = 0,$$

we finally get the required result. □

Theorem 2.2. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a real valued, differentiable function. Let function has a simple root α in the open interval denoted by I. Let x_0 be the initial guess sufficiently close to α. If the weight function $H(u)$ satisfies the following conditions,*

$$H(0) = 1, H^{'}(0) = -1$$

and

$$\left|H^{''}(0)\right| < \infty,$$

then the method defined by (2.4) converges to α for all values of p, q, s and β with order four and the error equation of this method (2.4) is given as follows:

$$e_{n+1} = (1+qf^{'}(\alpha))^2(c_2+p)A_3 e_n^4 + O(e_n^5),$$

where, A_3 involves $H^{''}(0)$.

Proof. Let $f(x)$ be a non-linear function and α be its real zero. That is $f(\alpha) = 0$, and let e_n denote the error at the n^{th} step as follows:

$$e_n = x_n - \alpha.$$

By Taylor series, expanding f(x) about α. we have

$$f(x_n) = f^{'}(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + ... + c_8e_n^8) + O(e_n^9), \tag{2.13}$$

whereas,

$$c_r = \frac{f^r(\alpha)}{r!f^{'}(r)}.$$

We have the error term of $w_n = x_n + q_n f(x_n)$, by using the Taylor's series,

$$e_{n,w} = (1+qf^{'}(\alpha))e_n + qf^{'}(\alpha)c_2e_n^2 + qf^{'}(\alpha)c_3e_n^3 + ... + O(e_n^5), \tag{2.14}$$

where,

$$e_{n,w} = w_n - \alpha.$$

Thus,

$$f(w_n) = f^{'}(\alpha)(e_n + c_2e_n^2 + c_2e_n^2 + c_3e_n^3 + ... + c_8e_n^8) + O(e_n^5). \tag{2.15}$$

Hence,

$$\begin{aligned} f[x_n, w_n] + pf(w_n) =& f^{'}(\alpha) + f^{'}(\alpha)((2+qf^{'}(\alpha))(c_2+p))e_n \\ &+ f^{'}(\alpha)(pc_2(1+3qf^{'}(\alpha)) + c_2qf^{'}(\alpha)(c_2+pqf^{'}(\alpha)) \\ &+ c_3(3+qf^{'}(\alpha)(3+qf^{'}(\alpha))e_n^2 + ... + O(e_n^5). \end{aligned} \tag{2.16}$$

Now, by using (2.13) and (2.16)

$$e_{n,y} = (c_2+p)(1+qf^{'}(\alpha))e_n^2 + ... + O(e_n^5). \tag{2.17}$$

Also,

$$\begin{aligned}
&\frac{f(x_n)}{f[y_n,w_n]+p_nf(w_n)+s(y_n-w_n)(y_n-x_n)}\\
&\quad= e_n-(c_2+p)(1+qf^{'}(\alpha))e_n^2+((s(1+qf^{'}(\alpha))(1+qf^{'}(\alpha))(-2qc_2)\\
&\quad+pc_2(2c_2-qpf^{'}(\alpha))f^{'}(\alpha)+c_3(2+qf^{'}(\alpha)(3+qf^{'}(\alpha))\\
&\quad-(2+q^2f^{'}(\alpha))(c_2^2+p^2qc_1))e_n^3+...+O(e_n^5),
\end{aligned}$$

implies

$$e_{n+1}=(1-H(0))(1+qf^{'}(\alpha))(c_2+p)e_n^2+...+O(e_n^5).$$

Thus, with $H\,(0)=1$,

$$e_{n+1}=(1+H^{'}(0))(1+qf^{'}(\alpha))^2(c_2+p)^2e_n^3+...+O(e_n^5).$$

Substituting,

$$H^{'}(0)=-1,$$

finally, the error term is,

$$e_{n+1}=(1+qf^{'}(\alpha))^2(c_2+p)A_3e_n^4+...+O(e_n^5), \tag{2.18}$$

where, A_3 involves $H^{''}\,(0)$. □

Remark 2.3. Similarly, we can also prove that (2.1) and (2.6) is also fourth order convergent for the same choices of the values of weight function. It is also apparent from the error analysis of (2.2) and (2.4) that the derivative free iterative methods are extendable to with memory methods.

3. The development of Tri-parametric iterative method with memory

We, now, choose the three parameters q, p and s in without memory method (2.4) to make it with memory method. The optimal order of without memory method (2.4) can be increased by choosing the accelerating parameters at each step as:

$$\begin{aligned}
q=q_n&=\frac{-1}{N_3^{'}(x_n)}=\frac{-1}{\widetilde{f}^{'}(\alpha)}\approx\frac{-1}{f^{'}(\alpha)},\\
p=p_n&=\frac{-N_4^{''}(w_n)}{N_4^{'}(w_n)}=-\widetilde{c_2}\approx -c_2,\\
s=s_n&=\frac{N_5^{'''}(y_n)}{6}=\widetilde{c_1}\widetilde{c_3}=c_1c_3,
\end{aligned}\tag{3.1}$$

where $\widetilde{f}^{'}(\alpha)\approx f^{'}(\alpha)$. We use, Newton's Interpolating polynomial passing through the given points at each step to calculate $f^{'}(\alpha)$. Estimation of $N_3(\varsigma)$ and $N_4(\varsigma)$ is given below:

$$\begin{aligned}
N_3(\varsigma;x_n,y_{n-1},w_{n-1},x_{n-1})=&f(x_n)+(\varsigma-x_n)f[x_n,y_{n-1}]\\
&+(\varsigma-x_n)(\varsigma-y_{n-1})f[x_n,y_{n-1},w_{n-1}]\\
&+(\varsigma-x_n)(\varsigma-y_{n-1})(\varsigma-w_{n-1})f[x_n,y_{n-1},w_{n-1},x_{n-1}].
\end{aligned}$$

Thus,

$$\begin{aligned}
N_3^{'}(\varsigma)=&f[x_n,y_{n-1}]+((\varsigma-x_n)+((\varsigma-y_{n-1}))f[x_n,y_{n-1},w_{n-1}]\\
&+((\varsigma-x_n)(\varsigma-y_{n-1})+(\varsigma-y_{n-1})(\varsigma-w_{n-1})+(\varsigma-x_n)(\varsigma-w_{n-1}))\\
&f[x_n,y_{n-1},w_{n-1},x_{n-1}],
\end{aligned}$$

and

$$N_3'(x_n) = f[x_n, y_{n-1}] + (x_n - y_{n-1})f[x_n, y_{n-1}, w_{n-1}] + (x_n - y_{n-1})(x_n - w_{n-1})f[x_n, y_{n-1}, w_{n-1,}x_{n-1}].$$

Also,

$$\begin{aligned} &N_4(\varsigma; w_n, x_n, y_{n-1}, w_{n-1}, x_{n-1}) \\ &= f(w_n) + (\varsigma - w_n)f[w_n, x_n] + (\varsigma - x_n)(\varsigma - w_n)f[w_n, x_n, y_{n-1}] \\ &\quad + (\varsigma - w_n)(\varsigma - x_n)(\varsigma - y_{n-1})f[w_n, x_n, y_{n-1}, w_{n-1}] \\ &\quad + (\varsigma - w_n)(\varsigma - y_{n-1})(\varsigma - x_n)(\varsigma - w_{n-1})f[w_n, x_n, y_{n-1}, w_{n-1}, x_{n-1}]. \end{aligned}$$

Differentiating the above expression w.r.t. ς and evaluating at w_n, we have:

$$\begin{aligned} N_4'(w_n) = &f[w_n, x_n] + (w_n - x_n)f[w_n, x_n, y_{n-1}] \\ &+ (w_n - x_n)(w_n - y_{n-1})f[w_n, x_n, y_{n-1}, w_{n-1}] \\ &+ (w_n - x_n)(w_n - y_{n-1})(w_n - w_{n-1})(\varsigma - w_{n-1})f[w_n, x_n, y_{n-1}, w_{n-1}, x_{n-1}]. \end{aligned}$$

Also,

$$\begin{aligned} N_4''(w_n) = &2f[w_n, x_n, y_{n-1}] + 2((w_n - x_n) \\ &+ (w_n - y_{n-1}))f[w_n, x_n, y_{n-1}] + 2((w_n - x_n)(w_n - y_{n-1}) \\ &+ (w_n - x_n)(w_n - w_{n-1}) + (w_n - w_{n-1})(w_n - y_{n-1})) \\ &f[w_n, x_n, y_{n-1}, w_{n-1}, x_{n-1}]. \end{aligned}$$

Similarly,

$$\begin{aligned} &N_5(\varsigma; y_n, w_n, x_n, y_{n-1}, w_{n-1}, x_{n-1}) \\ &= f(y_n) + (\varsigma - y_n)f[y_n, w_n] + (\varsigma - w_n)(\varsigma - y_n)f[y_n, w_n, x_n] \\ &\quad + (\varsigma - x_n)(\varsigma - w_n)(\varsigma - y_n)f[y_n, w_n, x_n, y_{n-1}] \\ &\quad + (\varsigma - y_{n-1})(\varsigma - x_n)(\varsigma - w_n)(\varsigma - y_n)f[y_n, w_n, x_n, y_{n-1}, w_{n-1}] \\ &\quad + (\varsigma - w_{n-1})(\varsigma - y_{n-1})(\varsigma - x_n)(\varsigma - w_n)(\varsigma - y_n)f[y_n, w_n, x_n, y_{n-1}, w_{n-1}, x_{n-1}]. \end{aligned}$$

Differentiating three times $N_5(\varsigma)$ w.r.t. ς and evaluating at $\varsigma = y_n$, we get:

$$\begin{aligned} N_5'''(y_n) = &6f[y_n, w_n, x_n, y_{n-1}] + 6((y_n - y_{n-1}) + (y_n - w_n) + (y_n - x_n)) \\ &\times f[y_n, w_n, x_n, y_{n-1}, w_{n-1}] + 6((y_n - w_{n-1})(y_n - y_{n-1}) + (y_n - x_n) \\ &(y_n - w_{n-1}) + (y_n - w_{n-1})(y_n - w_n) + (y_n - w_n)(y_n - y_{n-1}) \\ &+ (y_n - w_n)(y_n - x_n) + (y_n - x_n)(y_n - y_{n-1})) \\ &\times f[y_n, w_n, x_n, y_{n-1}, w_{n-1}, x_{n-1}]. \end{aligned}$$

Now, we define the two-step with memory extension of (2.4) as follows:

$$\begin{aligned} w_n &= x_n + q_n f(x_n), q_n = \frac{-1}{N_3'(x_n)}, \\ y_n &= x_n - \frac{f(x_n)}{f[w_n, x_n] + p_n f(w_n)}, \\ p_n &= \frac{-N_4''(w_n)}{2N_4'(w_n)}, s_n = \frac{N_5'''(y_n)}{6}, \\ x_{n+1} &= y_n - H(u_n)\frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)} \\ &\quad \times \frac{f(y_n)}{f[w_n, y_n] + p_n f(w_n) + s_n(y_n - w_n)(y_n - x_n)}. \end{aligned} \tag{3.2}$$

Lemma 3.1. *The following holds,*

$$\begin{aligned} c_2 + p_n \approx & \frac{c_5}{2} e_{n-1,w}\, e_{n-1,y}\, e_{n-1}, \\ 1 + q_n f^{'}(\alpha) \approx & L_{\ n} e_{n-1,y}\, e_{n-1,w}\, e_{n-1}, \\ c_1 c_3 - s_n \approx & \frac{c_6}{6} e_{n-1,y}\, e_{n-1,w}\, e_{n-1}, \end{aligned} \tag{3.3}$$

where,

$$L_n = c_4 - 2c_2 U_n,$$

and

$$V_n = (1 + qf^{'}(\alpha))^2 (c_2 + p) A_3,$$

with the estimation of following accelerating parameters

$$q_n = \frac{-1}{N_3'(x_n)},\ p_n = \frac{-N_4^{''}(w_n)}{2N_4'(w_n)}, s_n = \frac{N_5^{'''}(y_n)}{6}.$$

Proof. Suppose $N_n(\varsigma)$ is Newton interpolating polynomial that represent the approximation of the function $f(\varsigma)$. Its error term is given by

$$f(\varsigma) - N_n(\varsigma) = \frac{f^{(n+1)}(\xi)}{(n+1)!}.\prod_{j=0}^{n}(\varsigma - \varsigma_j).$$

Putting n=3, we have:

$$\begin{aligned} f(\varsigma) - N_3(\varsigma) =& \frac{f^{(3+1)}(\xi)}{4!} \Pi_{j=0}^{3} (\varsigma - \varsigma_j), \\ N_3(\varsigma) =& f(\varsigma) - \frac{f^4(\xi)}{4!}(\varsigma - \varsigma_0)(\varsigma - \varsigma_1)(\varsigma - \varsigma_2)(\varsigma - \varsigma_3), \\ N_3(\varsigma) =& f(\varsigma) - \frac{f^4(\xi)}{4!}(\varsigma - x_n)(\varsigma - y_{n-1})(\varsigma - w_{n-1})(\varsigma - x_{n-1}). \end{aligned}$$

Differentiating at $\varsigma = x_n$, we get,

$$N_3^{'}(x_n) = f(x_n) - \frac{f^4(\xi)}{4!}(x_n - y_{n-1})(x_n - w_{n-1})(x_n - x_{n-1})$$

and

$$N_3^{'}(x_n) \approx f^{'}(\alpha)(1 + 2c_2 e_n + 3c_3 e_n^3 + c_4 e_{n-1,y}\, e_{n-1,w}\, e_{n-1}).$$

Since,

$$1 + q_n f^{'}(\alpha) = 1 - \frac{f^{'}(\alpha)}{N_3^{'}(x_n)}.$$

Thus,

$$\begin{aligned} 1 + q_n f^{'}(\alpha) = 1 - & \frac{1}{1 + 2c_2 e_n + 3c_3 e_n^3 - c_4 e_{n-1,y}\, e_{n-1,w} p e_{n-1}}, \\ \approx & L_n e_{n-1,y}\, e_{n-1,p}\, e_{n-1}, \end{aligned}$$

where

$$L_n = c_4 - 2c_2 V_n,$$

and

$$V_n = (1 + H^{'}(0))(c_2 + p)(1 + qf^{'}(\alpha))e_n^2 + ... + O(e_n^5).$$

For $n = 5$ then we have

$$f(\varsigma) - N_4(\varsigma) = \frac{f^{(4+1)}(\xi)}{5!}\Pi_{j=0}^{4}(\varsigma - \varsigma_j),$$
$$N_4(\varsigma) = f(\varsigma) - \frac{f^5(\xi)}{5!}(\varsigma - \varsigma_0)(\varsigma - \varsigma_1)(\varsigma - \varsigma_2)(\varsigma - \varsigma_3)(\varsigma - \varsigma_4),$$
$$N_4(\varsigma) = f(\varsigma) - \frac{f^5(\xi)}{5!}(\varsigma - w_n)(\varsigma - y_{n-1})(\varsigma - w_{n-1})(\varsigma - x_{n-1}).$$

Differentiating w.r.t. ς at $\varsigma = w_n$, we have

$$N_4'(w_n) = f(w_n) - \frac{f^{(5)}(\xi)}{5!}(w_n - x_n)(w_n - y_{n-1})(w_n - w_{n-1})(w_n - x_{n-1}),$$
$$N_4'(w_n) = f'(w_n) - \frac{f^{(5)}(\xi)}{5!}e_n e_{n-1,y}\, e_{n-1,w}\, e_{n-1}.$$

By using Taylor's series of $f'(w_n)$,

$$N_4'(w_n) = f'(\alpha)(1 + 2c_2 e_{n,p} + 3c_3 e_{n,p}^3 - c_5 e_n e_{n-1,y}\, e_{n-1,w}\, e_{n-1}).$$

Now,

$$N_4''(w_n) = f''(w_n) - \frac{f^{(5)}(\xi)}{5!}e_{n-1,y}\, e_{n-1,w}\, e_{n-1}.$$

By using Taylor's series

$$N_4''(w_n) = f''(\alpha)(2c_2 + 6c_3 e_{n,p} + 12c_4 e_{n,p}^2 - c_5 e_n e_{n-1,y}\, e_{n-1,w}\, e_{n-1}),$$
$$N_4''(w_n) = f''(\alpha)(1 + \frac{3c_3}{c_2}e_{n,w} + \frac{6c_4}{c_2}e_{n,w}^2 - \frac{c_5}{c_2}e_n e_{n-1,y}\, e_{n-1,w}\, e_{n-1}),$$
$$\frac{N_4''(w_n)}{N_4'(w_n)} = \frac{f''(\alpha)(1 + \frac{3c_3}{c_2}e_{n,w} + \frac{6c_4}{c_2}e_{n,w}^2 - \frac{c_5}{c_2}e_n e_{n-1,y}\, e_{n-1,w}\, e_{n-1})}{f'(\alpha)(1 + 2c_2 e_{n,p} + 3c_3 e_{n,p}^3 - c_5 e_n e_{n-1,y}\, e_{n-1,w}\, e_{n-1})},$$
$$\approx c_2(1 - \frac{c_5}{c_2}e_{n-1,y}\, e_{n-1,w}\, e_{n-1}).$$

Thus, we have:

$$c_2 + p_n \approx c_2 - \frac{N_4''(w_n)}{2N_4'(w_n)},$$
$$\approx \frac{c_5}{2}e_{n-1,y}\, e_{n-1,w}\, e_{n-1}.$$

Similarly, we have,

$$c_1 c_3 - s_n \approx \frac{c_6}{6}e_{n-1,y}\, e_{n-1,w}\, e_{n-1}.$$

□

Theorem 3.2. *Minimum R-Order of Convergence of Method* (3.2). *Let $f(x)$ be a non linear function, which is differentiable in the neighborhood of simple zero α. Let x_0 is the initial guess, which is sufficiently close to α. Then, minimum $R-$order of convergence of* (3.2) *is at the most 7.53.*

Proof. Let $\{t_n\}$ is an iterative sequence. Let $t_n = \varphi(t_{n-1})$ represents the processes of iterations that generates $\{t_n\}$. Let α be the root of the equation $f(x) = 0$. If $\{x_n\}$ converge to the real root α with at least order R. Then,

$$e_{n+1} \sim \rho_{n,R}\, e_n^R,$$

where,

$$
\begin{aligned}
e_n &= x_n - \alpha, \\
e_{n+1} &= \rho_{n,R}\,(\rho_{n-1,R}\,e_{n-1}^{R})^{R}, \\
e_{n+1} &= \rho_{n,R}\,\rho_{n-1,R}^{R}\,e_{n-1}^{R^2}.
\end{aligned} \tag{3.4}
$$

Let us consider that m and r be the $R-order$ of $\{w_n\}$ and $\{y_n\}$.we have,

$$e_{n,w} = \rho_{n,m} e_n^m = \rho_{n,m}(\rho_{n-1,R}\,e_{n-1}^{R})^m = \rho_{n,m}\rho_{n-1,R}^{m}\,e_{n-1}^{Rm}, \tag{3.5}$$

$$e_{n,y} = \rho_{n,r} e_n^r = \rho_{n,r}(\rho_{n-1,R}\,e_{n-1}^{R})^r = \rho_{n,r}\rho_{n-1,R}^{r}\,e_{n-1}^{Rr}. \tag{3.6}$$

By using (3.5) and (3.6), we get:

$$1 + q_n f'(\alpha) \approx L_n \rho_{n-1,m}\rho_{n-1,r}\, e_{n-1}^{m+r+1}$$

and

$$
\begin{aligned}
c_2 + p_n &\approx \frac{c_5}{2}\rho_{n-1,m}\rho_{n-1,r}\, e_{n-1}^{m+r+1}, \\
c_1 c_3 - s &\approx \frac{c_6}{6}\rho_{n-1,m}\rho_{n-1,r}\, e_{n-1}^{m+r+1}.
\end{aligned}
$$

From (2.14),

$$e_{n,w} = (1 + q_n f'(\alpha))e_n = L_n \rho_{n-1,m}\rho_{n-1,r}\,\rho_{n-1,R} e_{n-1}^{m+r+1+R}. \tag{3.7}$$

From (2.17) we get,

$$e_{n,y} = (c_2 + p_n)(1 + q_n f'(\alpha))e_n^2 = L_n \rho_{n-1,m}\rho_{n-1,r}\,\rho_{n-1,R}^2 e_{n-1}^{2R+2r+2m+2}. \tag{3.8}$$

By (2.18),

$$e_{n+1} = (1 + q_n f'(\alpha))^2(c_2 + p_n)(c_1c_3 - s)e_n^4 = L_n \rho_{n-1,m}\rho_{n-1,r}\,\rho_{n-1,R}^2 e_{n-1}^{4R+4m+4r+4}. \tag{3.9}$$

Comparing the coefficients of (3.5), (3.6) and (3.4) with (3.7), (3.8) and (3.9) respectively, we obtain:

$$
\begin{aligned}
&Rm - R - m - r - 1 = 0, \\
&Rr - 2R - 2m - 2r - 2 = 0, \\
&R^2 - 4R - 4m - 4r - 4 = 0.
\end{aligned}
$$

After solving these equations, we have: $m = 1.88, r = 3.76$ and $R = 7.53$. □

4. Some weight functions and iterative methods

For defining with and without memory methods, we choose $H(u_n)$, in such a way that it satisfies the following conditions,

$$H(0) = 1, H'(0) = -1$$

and examples of such functions are:

$$
\begin{aligned}
H(u_n) &= 1 - u_n, \\
H(u_n) &= \frac{1}{1 + u_n}.
\end{aligned}
$$

Method F1:

$$w_n = x_n + q_n f(x_n),$$

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n] + p_n f(w_n)},\\
q_n &= -\frac{1}{N_3'(x_n)},\ p_n = -\frac{N_4'(w_n)}{2N_4'(w_n)}\ ,\\
x_{\ n+1} &= y_{\ n} - (1-u_n)\frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\\
&\times \frac{f(y_n)}{f[w_n, y_n] + p_n f\,(w_n) + s_n(y_n - w_n)(y_n - x_n)}.\\
s_n &= \frac{N_5'''(y_n)}{6}, u_n = \frac{f(y_n)}{f(x_n)}.
\end{aligned}
\tag{4.1}
$$

Method F2:

$$
\begin{aligned}
w_n &= x_n + q_n f(x_n),\\
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n] + p_n f(w_n)},\\
q_n &= -\frac{1}{N_3'(x_n)},\ p_n = -\frac{N_4''(w_n)}{2N_4'(w_n)}\ ,\\
x_{\ n+1} &= y_{\ n} - (\frac{1}{1+u_n})\frac{1}{(1-\frac{f(y_n)}{f(x_n)})^2}\frac{f(y_n)}{f[w_n, y_n] + p_n f\,(w_n) + s_n(y_n - w_n)(y_n - x_n)}.\\
s_n &= \frac{N_5'''(y_n)}{6}, u_n = \frac{f(y_n)}{f(x_n)}.
\end{aligned}
\tag{4.2}
$$

5. Numerical results

In this section, we compare our proposed methods with some of the existing methods of respective domain. COC is defined as:

$$COC = \frac{\log|f(x_n)/f(x_{n-1})|}{\log|f(x_{n-1})/f(x_{n-2})|}.$$

We observe first three iterations to estimate the accuracy, α is the exact root and x_0 is the initial guess. All the computations are performed on Maple software. Here, we compare our with and without memory methods F1 and F2 with Lotfi et al. [5] and F. Soleymani et al. methods [7] given as:

$$
\begin{aligned}
w_n &= x_n + q_n f(x_n), q_n = -\frac{1}{N_3'(x_n)},\\
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}(1 + p_n\frac{f(w_n)}{f[x_n, w_n]}), p_n = -\frac{N_4''(w_n)}{2N_4'(w_n)},\\
x_{n+1} &= y_n - \frac{f(y_n)}{f[x_n, y_n] + s_n(y_n - x_n)(y_n - w_n)}(B_n + (B_n - 1)^4),\\
B_n &= \frac{f[x_n, w_n]}{f[y_n, w_n]}, s_n = -\frac{1}{4}\frac{N_5''(y_n)^2}{2N_5'(y_n)} + \frac{1}{6}N_5'''(y_n)
\end{aligned}
$$

and

$$w_n = x_n + q_n f(x_n), q_n = -\frac{1}{N_3'(x_n)},$$

$$y_n = x_n - \frac{f(x_n)}{f[w_n, x_n] + p_n f(w_n)}, p_n = -\frac{N_4''(w_n)}{2N_4'(w_n)},$$
$$x_{n+1} = y_n - \frac{f(y_n)}{f[y_n, w_n] + p_n f(w_n) + s_n(y_n - x_n)(y_n - w_n)}(1 + \frac{f(y_n)}{f(x_n)}),$$
$$s_n = \frac{N_4'''(w_n)}{6}.$$

Table 1 shows the test functions, their actual roots and initial guess that have been used for the sake of comparison.

Test Functions	Actual Root	x_0
$f_1(x) = \sin^2 x + x$	$\alpha = 0$	0.2
$f_2(x) = e^{-5x}(x-2)(x^{10} + x + 2)$	$\alpha = 2$	1.8
$f_3(x) = e^{x^2-3x}\sin x + \log(x^2 + 1)$	$\alpha = 0$	0.35
$f_4(x) = \frac{1}{x^4} - x^2 - \frac{1}{x} + 1$	$\alpha = 1$	2

Table 1: Test Functions

$x_0 = 0.2, q = 0.01, p = 0.01, s = 0.01$				
	Lotfi	Soleymani	F1	F2
$\|x_1 - \alpha\|$	1.3274×10^{-3}	1.7283×10^{-3}	2.8395×10^{-4}	9.8039×10^{-4}
$\|x_2 - \alpha\|$	6.2331×10^{-12}	2.5063×10^{-11}	2.0068×10^{-16}	9.5039×10^{-13}
$\|x_3 - \alpha\|$	3.0491×10^{-45}	1.6409×10^{-32}	1.6333×10^{-59}	4.3463×10^{-49}
COC	3.99	2.70	3.54	4.03

Table 2: Comparison Table for without Memory Methods for $f_1(x)$

$x_0 = 1.8, q = 0.01, p = 0.01, s = 0.01$				
	Lotfi	Soleymani	F1	F2
$\|x_1 - \alpha\|$	1.6591×10^{-3}	2.0961×10^{-3}	1.2827×10^{-3}	1.4527×10^{-3}
$\|x_2 - \alpha\|$	3.4867×10^{-13}	4.1733×10^{-10}	4.4338×10^{-14}	7.5553×10^{-14}
$\|x_3 - \alpha\|$	5.9687×10^{-52}	1.4780×10^{-21}	7.4006×10^{-45}	$3.6589\times\times10^{-44}$
COC	4.00	1.70	2.94	2.94

Table 3: Comparison Table for without Memory Methods for $f_2(x)$

$x_0 = 0.35, q = 0.01, p = 0.01, s = 0.01$				
	Lotfi	Soleymani	F1	F2
$\|x_1 - \alpha\|$	5.9217×10^{-3}	5.9217×10^{-3}	5.0761×10^{-3}	5.1802×10^{-3}
$\|x_2 - \alpha\|$	6.4072×10^{-9}	6.4226×10^{-9}	6.4507×10^{-9}	2.1821×10^{-9}
$\|x_3 - \alpha\|$	9.3188×10^{-33}	4.0237×10^{-27}	4.5622×10^{-32}	6.3242×10^{-35}
COC	3.99	3.08	4.00	4.00

Table 4: Comparison Table for without Memory Methods for $f_3(x)$

$x_0 = 2, q = 0.01, p = 0.01, s = 0.01$				
	Lotfi	Soleymani	F1	F2
$\|x_1 - \alpha\|$	5.1001×10^{-2}	5.4640×10^{-2}	6.6659×10^{-2}	6.0041×10^{-2}
$\|x_2 - \alpha\|$	1.9103×10^{-5}	5.8620×10^{-5}	1.1638×10^{-4}	7.6161×10^{-6}
$\|x_3 - \alpha\|$	2.5817×10^{-19}	8.8661×10^{-11}	9.2560×10^{-16}	5.5991×10^{-21}
COC	4.00	1.93	3.95	3.83

Table 5: Comparison Table for without Memory Methods for $f_4(x)$

$x_0 = 0.2, q_0 = 0.01, p_0 = 0.01, s_0 = 0.01$				
	Lotfi	Soleymani	F1	F2
$\|x_1 - \alpha\|$	1.3274×10^{-3}	1.7744×10^{-3}	2.8395×10^{-4}	9.8039×10^{-4}
$\|x_2 - \alpha\|$	1.4475×10^{-25}	1.7969×10^{-25}	1.0572×10^{-31}	2.6804×10^{-27}
$\|x_3 - \alpha\|$	6.9029×10^{-185}	2.8321×10^{-181}	9.1786×10^{-240}	4.5628×10^{-206}
COC	7.25	7.08	7.58	7.58

Table 6: Comparison Table for with Memory Methods for $f_1(x)$

$x_0 = 1.8, q_0 = 0.01, p_0 = 0.01, s_0 = 0.01$				
	Lotfi	Soleymani	F1	F2
$\|x_1 - \alpha\|$	1.6591×10^{-3}	2.0505×10^{-3}	1.2827×10^{-3}	1.4527×10^{-3}
$\|x_2 - \alpha\|$	1.7608×10^{-25}	1.0967×10^{-24}	1.8943×10^{-28}	1.0412×10^{-27}
$\|x_3 - \alpha\|$	6.0276×10^{-181}	5.2672×10^{-175}	1.4412×10^{-211}	9.3873×10^{-206}
COC	7.07	7.06	7.37	7.37

Table 7: Comparison Table for with Memory Methods for $f_2(x)$

$x_0 = 0.35, q_0 = 0.01, p_0 = 0.01, s_0 = 0.01$				
	Lotfi	Soleymani	F1	F2
$\|x_1 - \alpha\|$	5.9217×10^{-3}	4.9991×10^{-3}	5.2473×10^{-3}	5.1802×10^{-3}
$\|x_2 - \alpha\|$	6.3159×10^{-18}	8.1145×10^{-19}	3.3581×10^{-20}	6.7080×10^{-22}
$\|x_3 - \alpha\|$	1.0662×10^{-128}	2.1309×10^{-131}	2.3293×10^{-148}	6.1244×10^{-162}
COC	7.40	7.13	7.45	7.41

Table 8: Comparison Table for with Memory Methods for $f_3(x)$

$x_0 = 2, q_0 = 0.01, p_0 = 0.01, s_0 = 0.01$				
	Lotfi	Soleymani	F1	F2
$\|x_1 - \alpha\|$	5.1001×10^{-2}	5.3222×10^{-2}	6.6659×10^{-2}	6.0041×10^{-2}
$\|x_2 - \alpha\|$	1.2486×10^{-9}	7.1748×10^{-9}	4.5920×10^{-9}	3.4781×10^{-9}
$\|x_3 - \alpha\|$	4.7206×10^{-62}	4.5383×10^{-58}	6.2000×10^{-62}	7.7250×10^{-63}
COC	6.85	7.11	7.32	7.36

Table 9: Comparison Table for with Memory Methods for $f_4(x)$

Conclusions

We have given two general techniques of constructing derivative free families of iterative methods without memory which are optimal in the sense of Kung and Traub's conjecture, thus, the order of convergence is preserved. Our proposed methods do not need any derivative. In addition, they contain accelerating parameter due to which convergence order can be raised by extending these methods to with memory class. The second procedure involves three parameters, therefore, it offers maximum acceleration in order of convergence i.e., from 4 to 7.53 using only one previous iterate and an increase in the efficiency index from 1.5874 to 1.96.

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