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The dynamics and solution of some difference equations

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Abstract

In this paper, we study solution and periodic nature of the following difference equations

$$x_{n+1} = \frac{x_{n-1}x_{n-5}}{x_{n-3}(\pm 1 \pm x_{n-1}x_{n-5})}, \quad n = 0, 1, ...,$$

where the initial conditions $x_{-5},\ x_{-4},\ x_{-3},\ x_{-2},\ x_{-1},\ x_0$ are arbitrary positive real numbers. we studied the equilibrium points of the given equation. Some qualitative properties such as the global stability, and the periodic character of the solutions in each case have been studied. We presented some numerical examples by using random initial values and the coefficients of each case. Some figures have been given to explain the behavior of the obtained solutions by using MATLAB to confirm the obtained results.

Keywords: Periodicity, stability, rational difference equations.
2010 MSC: 39A10.

1. Introduction

This paper deals with the behavior of the solutions of the recursive sequences

$$x_{n+1} = \frac{x_{n-1}x_{n-5}}{x_{n-3}(\pm 1 \pm x_{n-1}x_{n-5})}, \quad n = 0, 1, ..., \tag{1.1}$$

where the initial conditions $x_{-5},\ x_{-4},\ x_{-3}, x_{-2},\ x_{-1}, x_0$, are arbitrary positive real numbers. Also we obtain the form and study the solution of some special equations.

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The study and solution of nonlinear rational recursive sequence of high order is quite challenging and rewarding. Recently, there has been a lot of interest in studying the qualitative properties of rational recursive sequences, Furthermore diverse nonlinear trend occurring in science and engineering can be modeled by such equations and the solution about such equations offer prototypes towards the development of the theory. However, there have not been any suitable general method to deal with the global behavior of rational difference equations of high order so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Aloqeili [2] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [4] investigated the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}.$$

Ibrahim [16] studied the solutions of the rational recursive sequence

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + b x_n x_{n-2)}}.$$

Karatas et al [17] gave the solution of the following difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

Simsek et al. [26] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

Yalçınkaya [31] has studied the boundedness, global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{a.x_{n-k}}{b + c x_n^p}.$$

Yalçınkaya [30] has studied the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

See also [1]-[19]. Other related work on rational difference equations see in Refs. [20]-[34].

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I \tag{1.2}$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ... \tag{1.3}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.1 (Equilibrium Point)**.** A point $\overline{x} \in I$ is called an equilibrium point of (1.3) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of (1.3), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.2 (Periodicity)**.** A Sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 1.3 (Stability)**.**

(i) The equilibrium point $\overline{x}$ of (1.3) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{ for all } \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of (1.3) is locally asymptotically stable if $\overline{x}$ is locally stable solution of (1.3) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of (1.3) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of (1.3) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of (1.3).

(v) The equilibrium point $\overline{x}$ of (1.3) is unstable if $\overline{x}$ is not locally stable.

(vi) The linearized equation of (1.3) about the equilibrium $\overline{x}$ is the linear difference equation.

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem A ([18]). Assume that $p, q \in R$ and $k \in \{0, 1, 2, ...\}$. Then

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

Remark 1.4. Theorem A can be easily extended to a general linear equations of the form

$$x_{n-k} + p_1 x_{n+k-1} + \cdots + p_k x_n = 0, n = 0, 1, 2, \cdots, \tag{1.4}$$

where $p_1, p_2, ..., p_k \in R$ and $k \in \{1, 2, \cdots\}$. Then Eq. (1.4) is asymptotically stable provided that

$$\sum_{i=1}^{k} | \, p_i \, | < 1.$$

2. On the Equation $x_{n+1} = x_{n-1}x_{n-5}/(x_{n-3}(1 + x_{n-1}x_{n-5}))$

In this section, we give a specific form of the solution of the first equation in the form

$$x_{n+1}=\frac{x_{n-1}x_{n-5}}{x_{n-3}(1+x_{n-1}x_{n-5})},\qquad n=0,1,..., \tag{2.1}$$

where the initial values are arbitrary non-zero real numbers.

Theorem 2.1. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq.* (2.1)*. Then for $n=0,1,...$,*

$$\begin{aligned}
x_{8n-5} &= f\prod_{i=1}^{n-1}\left(\frac{1+4ibf}{1+(4i+2)bf}\right), & x_{8n-4} &= e\prod_{i=1}^{n-1}\left(\frac{1+4iae}{1+(4i+2)ae}\right),\\
x_{8n-3} &= d\prod_{i=1}^{n-1}\left(\frac{1+(4i+1)bf}{1+(4i+3)bf}\right), & x_{8n-2} &= c\prod_{i=1}^{n-1}\left(\frac{1+(4i+1)ae}{1+(4i+3)ae}\right),\\
x_{8n-1} &= b\prod_{i=1}^{n-1}\left(\frac{1+(4i+2)bf}{1+(4i+4)bf}\right), & x_{8n} &= a\prod_{i=1}^{n-1}\left(\frac{1+(4i+2)ae}{1+(4i+4)ae}\right),\\
x_{8n+1} &= \frac{bf}{d(1+bf)}\prod_{i=1}^{n-1}\left(\frac{1+(4i+3)bf}{1+(4i+5)bf}\right), & x_{8n+2} &= \frac{ae}{c(1+ae)}\prod_{i=1}^{n-1}\left(\frac{1+(4i+3)ae}{1+(4i+5)ae}\right),
\end{aligned}$$

where $x_{-5}=f,\ x_{-4}=e,\ x_{-3}=d,\ x_{-2}=c,\ x_{-1}=b,\ x_0=a.$

Proof. For $n=0$, the result holds. Now, suppose that $n>0$ and that our assumption holds for $n-1$. That is,

$$\begin{aligned}
x_{8n-13} &= f\prod_{i=1}^{n-2}\left(\frac{1+4ibf}{1+(4i+1)bf}\right), & x_{8n-12} &= e\prod_{i=1}^{n-2}\left(\frac{1+4iae}{1+(4i+2)ae}\right),\\
x_{8n-11} &= d\prod_{i=1}^{n-2}\left(\frac{1+(4i+1)bf}{1+(4i+3)bf}\right), & x_{8n-10} &= c\prod_{i=1}^{n-2}\left(\frac{1+(4i+1)ae}{1+(4i+3)ae}\right),\\
x_{8n-9} &= b\prod_{i=1}^{n-2}\left(\frac{1+(4i+2)bf}{1+(4i+4)bf}\right), & x_{8n-8} &= a\prod_{i=1}^{n-2}\left(\frac{1+(4i+2)ae}{1+(4i+4)ae}\right),\\
x_{8n-7} &= \frac{bf}{d(1+bf)}\prod_{i=1}^{n-2}\left(\frac{1+(4i+3)bf}{1+(4i+5)bf}\right), & x_{8n-6} &= \frac{ae}{c(1+ae)}\prod_{i=1}^{n-2}\left(\frac{1+(4i+3)ae}{1+(4i+5)ae}\right).
\end{aligned}$$

Now, it follows from (2.1) that,

$$\begin{aligned}
x_{8n-5} &= \frac{x_{8n-7}x_{8n-11}}{x_{8n-9}(1+x_{8n-7}x_{8n-11})}\\
&= \frac{\frac{bf}{d(1+bf)}\prod_{i=1}^{n-2}\left(\frac{1+(4i+3)bf}{1+(4i+5)bf}\right)d\prod_{i=1}^{n-2}\left(\frac{1+(4i+1)bf}{1+(4i+3)bf}\right)}{b\prod_{i=1}^{n-2}\left(\frac{1+(4i+2)bf}{1+(4i+4)bf}\right)\left(1+\frac{bf}{d(1+bf)}\prod_{i=1}^{n-2}\left(\frac{1+(4i+3)bf}{1+(4i+5)bf}\right)d\prod_{i=1}^{n-2}\left(\frac{1+(4i+1)bf}{1+(4i+3)bf}\right)\right)}\\
&= \frac{\frac{bf}{d(1+bf)}\prod_{i=1}^{n-2}\left(\frac{1+(4i+1)bf}{1+(4i+5)bf}\right)}{b\prod_{i=1}^{n-2}\left(\frac{1+(4i+2)bf}{1+(4i+4)bf}\right)\left(1+\frac{bf}{d(1+bf)}\prod_{i=1}^{n-2}\left(\frac{1+(4i+1)bf}{1+(4i+5)bf}\right)\right)}\\
&= \frac{\left(\frac{bf}{1+(4n+1)bf}\right)}{b\prod_{i=1}^{n-2}\left(\frac{1+(4i+2)bf}{1+(4i+4)bf}\right)\left(1+\frac{bf}{1+(4n+1)bf}\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{f}{\prod_{i=1}^{n-2}\left(\frac{1+(4i+2)bf}{1+(4i+4)bf}\right)(1+(4n+1)bf+bf)} \\
&= f\prod_{i=1}^{n-2}\left(\frac{1+(4i+4)bf}{1+(4i+2)bf}\right)\frac{1}{(1+(4n+2)bf)}.
\end{aligned}
$$

Hence we have

$$x_{8n-5} = f\prod_{i=1}^{n-1}\left(\frac{1+4ibf}{1+(4i+2)bf}\right).$$

Similarly we see that,

$$
\begin{aligned}
x_{8n+1} &= \frac{x_{8n-1}x_{8n-5}}{x_{8n-3}(1+x_{8n-17}x_{8n-5})} \\
&= \frac{b\prod_{i=1}^{n-2}\left(\frac{1+(4i+2)bf}{1+(4i+4)bf}\right) f\prod_{i=1}^{n-2}\left(\frac{1+4ibf}{1+(4i+2)bf}\right)}{d\prod_{i=1}^{n-2}\left(\frac{1+(4i+1)bf}{1+(4i+3)bf}\right)\left(1+b\prod_{i=1}^{n-2}\left(\frac{1+(4i+2)bf}{1+(4i+4)bf}\right) f\prod_{i=1}^{n-2}\left(\frac{1+4ibf}{1+(4i+2)bf}\right)\right)} \\
&= \frac{bf\prod_{i=1}^{n-2}\left(\frac{1+(4i)bf}{1+(4i+4)bf}\right)}{d\prod_{i=1}^{n-2}\left(\frac{1+(4i+1)bf}{1+(4i+3)bf}\right)\left(1+bf\prod_{i=1}^{n-2}\left(\frac{1+(4i)bf}{1+(4i+4)bf}\right)\right)} \\
&= \frac{\left(\frac{bf}{1+4nbf}\right)}{d\prod_{i=1}^{n-2}\left(\frac{1+(4i+1)bf}{1+(4i+3)bf}\right)\left(1+\frac{bf}{1+4nbf}\right)} \\
&= \frac{bf}{d\prod_{i=1}^{n-2}\left(\frac{1+(4i+1)bf}{1+(4i+3)bf}\right)(1+4nbf+bf)} \\
&= \prod_{i=1}^{n-2}\left(\frac{1+(4i+3)bf}{1+(4i+1)bf}\right)\frac{bf}{d\,(1+(4n+1)bf)}.
\end{aligned}
$$

Hence we have,

$$x_{8n+1} = \frac{bf}{d\,(1+bf)}\prod_{i=1}^{n-1}\left(\frac{1+(4i+3)bf}{1+(4i+5)bf}\right).$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

Theorem 2.2. *Eq.* (2.1) *has a unique equilibrium point which is* $\overline{x} = 0$, *and is not locally asymptotically stable.*

Proof. From Eq. (2.1), we see that

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}(1+\overline{x}^2)},$$

or

$$\overline{x}^2(1+\overline{x}^2-1) = 0,$$

$$\overline{x}^4 = 0.$$

Thus the equilibrium point of Eq. (2.1) is $\overline{x} = 0$.

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a continuously differentiable function defined by

$$f(u, v, w) = \frac{uw}{v(1 + uw)}.$$

Therefore at $\overline{x} = 0$, we get

$$\left(\frac{\partial f}{\partial u}\right)_{\overline{x}} = 1, \quad \left(\frac{\partial f}{\partial v}\right)_{\overline{x}} = 1, \quad \left(\frac{\partial f}{\partial w}\right)_{\overline{x}} = 1.$$

The proof follows by using Theorem A. □

Now, we consider some numerical examples which represent different types of solutions to Eq. (2.1).

Example 2.3. We assume $x_{-5} = 13,\ x_{-4} = 7,\ x_{-3} = 19,\ x_{-2} = 10, x_{-1} = 15,\ x_0 = 10$. (See Figure 1).

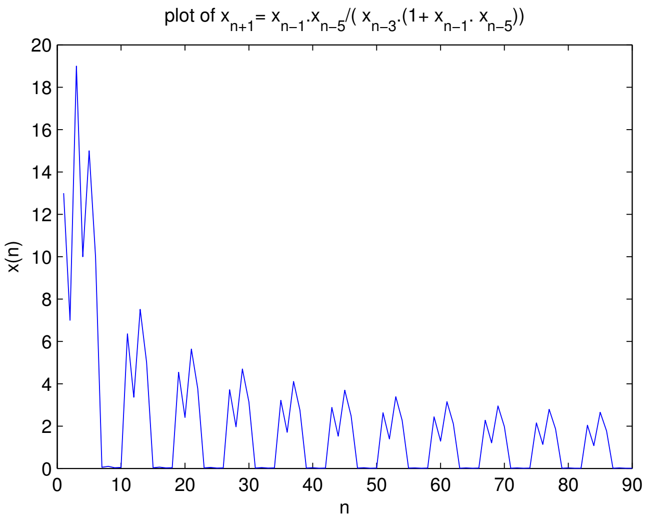

Figure 1

Example 2.4. See Figure 2, where we put $x_{-5} = 4,\ x_{-4} = 7,\ x_{-3} = 2,\ x_{-2} = 6, x_{-1} = 9,\ x_0 = 1$.

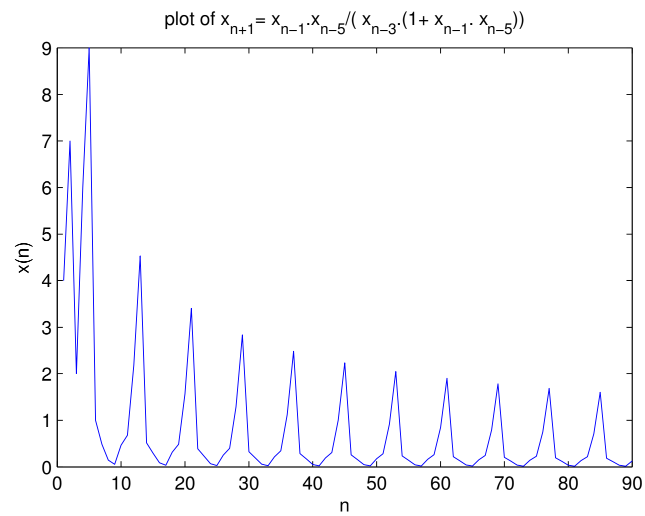

Figure 2

3. On the Equation $x_{n+1} = x_{n-1}x_{n-5}/(x_{n-3}(-1 + x_{n-1}x_{n-5}))$

In this section, we give a specific form of the solution of the second equation in the form

$$x_{n+1} = \frac{x_{n-1}x_{n-5}}{x_{n-3}(-1 + x_{n-1}x_{n-5})}, \qquad n = 0, 1, ..., \tag{3.1}$$

where the initial values are arbitrary non zero real numbers with $x_{-1}x_{-5}$, $x_0x_{-4} \neq 1$.

Theorem 3.1. *Let* $\{x_n\}_{n=-5}^{\infty}$ *be a solution of Eq.* (3.1) *Then every solution of Eq.* (3.1) *is periodic with period 8. Moreover,* $\{x_n\}_{n=-5}^{\infty}$ *takes the form*

$$\left\{ f, e, d, c, b, a, \frac{bf}{d(-1+bf)}, \frac{ae}{c(-1+ae)}, f, e, d, c, b, a, ... \right\},$$

or,

$$\begin{aligned} x_{8n-5} &= f, & x_{8n-4} &= e, & x_{8n-3} &= d, & x_{8n-2} &= c, \\ x_{8n-1} &= b, & x_{8n} &= a, & x_{8n+1} &= \frac{bf}{d(-1+bf)}, & x_{8n+2} &= \frac{ae}{c(-1+ae)}. \end{aligned}$$

Proof. As proof of Theorem 2.1 so will be omitted. □

Theorem 3.2. *Eq.* (3.1) *has two equilibrium points which are* $0, \pm\sqrt{2}$ *and these equilibrium points are not locally asymptotically stable.*

Proof. For the equilibrium points of Eq. (3.1), we can write

$$\overline{x} = \frac{\overline{x}^2}{\overline{x}(-1+\overline{x}^2)},$$

or

$$\overline{x}^2(\overline{x}^2 - 2) = 0.$$

Thus, the equilibrium points of Eq. (3.1) are 0 and $\pm\sqrt{2}$.

Let $f : (0, \infty)^3 \longrightarrow (0, \infty)$ be a continuously differentiable function defined by

$$f(u, v, w) = \frac{uw}{v(1 + uw)}.$$

Therefore at $\overline{x} = 0$

$$\left(\frac{\partial f}{\partial u}\right)_{\overline{x}} = -1, \quad \left(\frac{\partial f}{\partial v}\right)_{\overline{x}} = \pm 1, \quad \left(\frac{\partial f}{\partial w}\right)_{\overline{x}} = -1.$$

The proof follows by using Theorem A. □

Example 3.3. We assume $x_{-5} = 3,\ x_{-4} = 7,\ x_{-3} = 9,\ x_{-2} = 10, x_{-1} = 7,\ x_0 = 4/9$. (See Figure 3).

Example 3.4. See Figure 4 when we take $x_{-5} = 4,\ x_{-4} = 7,\ x_{-3} = 2,\ x_{-2} = 6, x_{-1} = 9,\ x_0 = 1$.

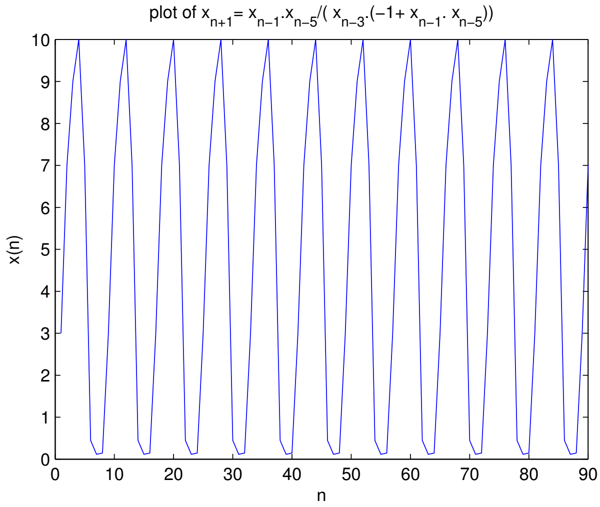

Figure 3

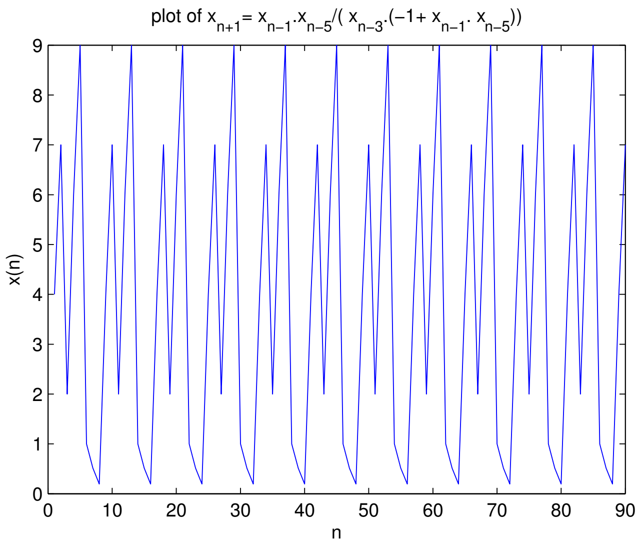

Figure 4

The following cases can be proved similarly.

4. On the Equation $x_{n+1} = x_{n-1}x_{n-5}/(x_{n-3}(1 - x_{n-1}x_{n-5}))$

In this section, we obtain the form of the solution of the third equation in the form

$$x_{n+1} = \frac{x_{n-1}x_{n-5}}{x_{n-3}(1 - x_{n-1}x_{n-5})}, \qquad n = 0, 1, ..., \tag{4.1}$$

where the initial values are arbitrary non zero real numbers.

Theorem 4.1. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq. (4.1). Then for $n = 0, 1, ...,$ we see that*

$$x_{8n-5} = f\prod_{i=1}^{n-1}\left(\frac{1-4ibf}{1-(4i+2)bf}\right), \qquad x_{8n-4} = e\prod_{i=1}^{n-1}\left(\frac{1-4iae}{1-(4i+2)ae}\right),$$

$$x_{8n-3} = d\prod_{i=1}^{n-1}\left(\frac{1-(4i+1)bf}{1-(4i+3)bf}\right), \qquad x_{8n-2} = c\prod_{i=1}^{n-1}\left(\frac{1-(4i+1)ae}{1-(4i+3)ae}\right),$$

$$x_{8n-1} = b\prod_{i=1}^{n-1}\left(\frac{1-(4i+2)bf}{1-(4i+4)bf}\right), \qquad x_{8n} = a\prod_{i=1}^{n-1}\left(\frac{1-(4i+2)ae}{1-(4i+4)ae}\right),$$

$$x_{8n+1} = \frac{bf}{d(1-bf)}\prod_{i=1}^{n-1}\left(\frac{1-(4i+3)bf}{1-(4i+5)bf}\right), \qquad x_{8n+2} = \frac{ae}{c(1-ae)}\prod_{i=1}^{n-1}\left(\frac{1-(4i+3)ae}{1-(4i+5)ae}\right).$$

Theorem 4.2. *Eq.* (4.1) *has a unique equilibrium point which is* $\overline{x} = 0$, *and is not locally asymptotically stable.*

Example 4.3. We put the initial conditions as follows $x_{-5} = 2,\ x_{-4} = 5,\ x_{-3} = 9,\ x_{-2} = 5, x_{-1} = 1,\ x_0 = 3$. (See Figure 5).

Example 4.4. See Figure 6, since $x_{-5} = 3,\ x_{-4} = 7,\ x_{-3} = 2,\ x_{-2} = 6, x_{-1} = 8,\ x_0 = 2/5$.

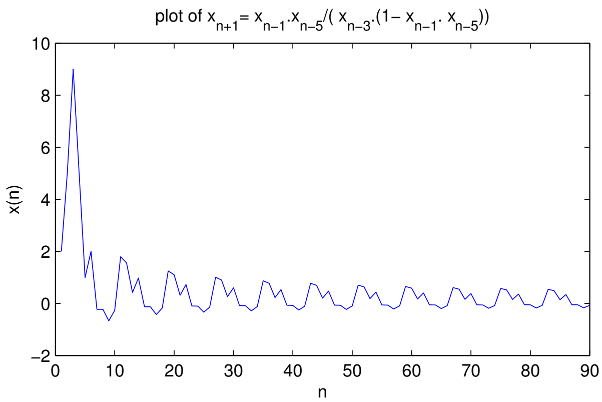

Figure 5

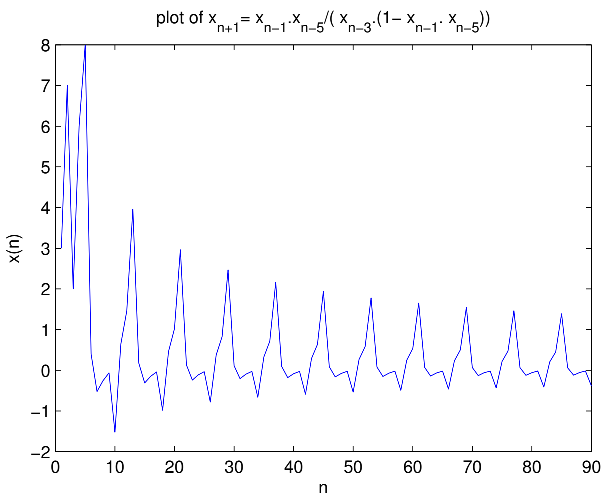

Figure 6

5. On the Equation $x_{n+1} = x_{n-1}x_{n-5}/(x_{n-3}(-1 - x_{n-1}x_{n-5}))$

In this section, we get the solutions form of the fourth difference equation as follows

$$x_{n+1} = \frac{x_{n-1}x_{n-5}}{x_{n-3}(-1 - x_{n-1}x_{n-5})}, \qquad n = 0, 1, ..., \tag{5.1}$$

where the initial values are arbitrary non zero real numbers with $x_{-1}x_{-5},\ x_0x_{-4} \neq -1$.

Theorem 5.1. *Let $\{x_n\}_{n=-5}^{\infty}$ be a solution of Eq. (5.1) Then every solution of it is periodic with period 8. Moreover, $\{x_n\}_{n=-5}^{\infty}$ takes the form*

$$\left\{f, e, d, c, b, a, \frac{bf}{d(-1-bf)}, \frac{ae}{c(-1-ae)}, f, e, d, c, b, a, ...\right\},$$

or,

$$\begin{aligned} x_{8n-5} &= f, & x_{8n-4} &= e, & x_{8n-3} &= d, & x_{8n-2} &= c, \\ x_{8n-1} &= b, & x_{8n} &= a, & x_{8n+1} &= \frac{bf}{d(-1-bf)}, & x_{8n+2} &= \frac{ae}{c(-1-ae)}. \end{aligned}$$

Example 5.2. We suppose that $x_{-5} = 9,\ x_{-4} = 7, x_{-3} = 12, x_{-2} = 3,\ x_{-1} = 2,\ x_0 = 5$. (See Figure 7).

Example 5.3. Figure 8, shows the periodicity of the solution when $x_{-5} = -7,\ x_{-4} = 5,\ x_{-3} = 8, x_{-2} = 6,$ $x_{-1} = 9,\ x_0 = 5/9$.

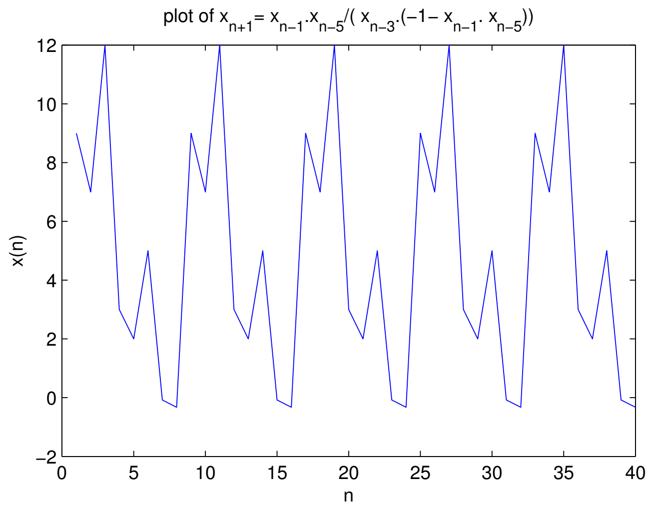

Figure 7

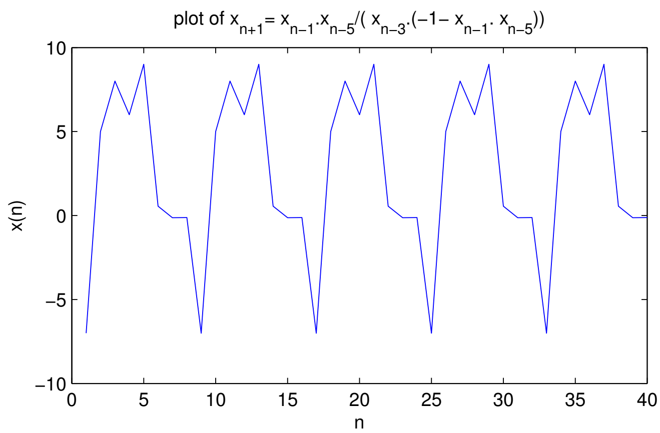

Figure 8

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References

[1] R. Agarwal, *Difference Equations and Inequalities: Theory, Methods and Applications*, Marcel Dekker Inc., New York, (1992). 1

[2] M. Aloqeili, *Dynamics of a rational difference equation*, Appl. Math. Comput., **176** (2006), 768–774. 1

[3] M. B. Bekker, M. J. Bohner, H. D. Voulov, *Asymptotic behavior of solutions of a rational system of difference equations*, J. Nonlinear Sci. Appl., **7** (2014), 379–382.

[4] C. Cinar, *On the positive solutions of the difference equation* $x_{n+1} = \frac{ax_{n-1}}{1+bx_nx_{n-1}}$, Appl. Math. Comp., **156** (2004), 587–590. 1

[5] H. Chen, H. Wang, *Global attractivity of the difference equation* $x_{n+1} = \dfrac{x_n + \alpha x_{n-1}}{\beta + x_n}$, Appl. Math. Comput., **181** (2006), 1431–1438.

[6] Q. Din, E. M. Elsayed, *Stability analysis of a discrete ecological model*, Comput. Ecol. Softw., **4** (2014), 89–103.

[7] E. M. Elabbasy, H. El-Metwally, E. M. Elsayed, *On the difference equations* $x_{n+1} = \dfrac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}$, J. Concr. Appl. Math., **5** (2007), 101–113.

[8] E. M. Elabbasy, H. El-Metwally, E. M. Elsayed, *Qualitative behavior of higher order difference equation*, Soochow J. Math., **33** (2007), 861–873.

[9] E. M. Elsayed, *Qualitative behaviour of difference equation of order two*, Math. Comput. Modelling, **50** (2009), 1130–1141.

[10] E. M. Elsayed, *Solution and attractivity for a rational recursive sequence*, Discrete Dyn. Nat. Soc., **2011** (2011), 17 pages.

[11] E. M. Elsayed, *Behavior and expression of the solutions of some rational difference equations*, J. Comput. Anal. Appl., **15** (2013), 73–81.

[12] E. M. Elsayed, *Solution for systems of difference equations of rational form of order two*, Comput. Appl. Math, **33** (2014), 751–765.

[13] E. M. Elsayed, *On the solutions and periodic nature of some systems of difference equations*, Int. J. Biomath, **7** (2014), 26 pages.

[14] E. M. Elsayed, *New method to obtain periodic solutions of period two and three of a rational difference equation*, Nonlinear Dynam., **79** (2015), 241–250.

[15] E. M. Elsayed, M. M. El-Dessoky, *Dynamics and global behavior for a fourth-order rational difference equation*, Hacet. J. Math. Stat., **42** (2013), 479–494.

[16] T. F. Ibrahim, *On the third order rational difference equation* $x_{n+1} = \dfrac{x_nx_{n-2}}{x_{n-1}(a + bx_nx_{n-2})}$, Int. J. Contemp. Math. Sci., **4** (2009), 1321–1334. 1

[17] R. Karatas, C. Cinar, D. Simsek, *On positive solutions of the difference equation* $x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}$, Int. J. Contemp. Math. Sci., **1** (2006), 495–500. 1

[18] V. L. Kocic, G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, (1993). 1

[19] M. R. S. Kulenovic, G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall, CRC Press, London, (2001). 1

[20] A. S. Kurbanli, *On the behavior of solutions of the system of rational difference equations*, World Appl. Sci. J., **10** (2010), 1344–1350. 1

[21] H. Ma, H. Feng, J. Wang, W. Ding, *Boundedness and asymptotic behavior of positive solutions for difference equations of exponential form,* J. Nonlinear Sci. Appl., **8** (2015), 893–899.

[22] R. Memarbashi, *Sufficient conditions for the exponential stability of nonautonomous difference equations*, Appl. Math. Lett., **21** (2008), 232–235.

[23] A. Neyrameh, H. Neyrameh, M. Ebrahimi, A. Roozi, *Analytic solution diffusivity equation in rational form*, World Appl. Sci. J., **10** (2010), 764–768.

[24] M. Saleh, M. Aloqeili, *On the difference equation* $y_{n+1} = A + \dfrac{y_n}{y_{n-k}}$ *with* $A < 0$, Appl. Math. Comp., **176** (2006), 359–363.

[25] M. Saleh, M. Aloqeili, *On the difference equation* $y_{n+1} = A + \dfrac{y_n}{y_{n-k}}$, Appl. Math. Comp., **171** (2005), 862–869.

[26] D. Simsek, C. Cinar, I. Yalcinkaya, *On the recursive sequence* $x_{n+1} = \frac{x_{n-3}}{1+x_{n-1}}$, Int. J. Contemp. Math. Sci., **1** (2006), 475–480. 1

[27] D. Simsek, C. Cınar, R. Karatas, I. Yalcinkaya, *On the recursive sequence* $x_{n+1} = \frac{x_{n-5}}{1+x_{n-1}x_{n-3}}$, Int. J. Pure Appl. Math., **28** (2006), 117–124.

[28] T. Sun, H. Xi, *On convergence of the solutions of the difference equation* $x_{n+1} = 1 + \dfrac{x_{n-1}}{x_n}$, J. Math. Anal. Appl., **325** (2007) 1491–1494.

[29] W. Wang, J. Tian, *Difference equations involving causal operators with nonlinear boundary conditions*, J. Nonlinear Sci. Appl., **8** (2015), 267–274.

[30] I. Yalçınkaya, *On the difference equation* $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$, Discrete Dyn. Nat. Soc., **2008** (2008), 8 pages. 1

[31] I. Yalcinkaya, C. Cinar, *On the dynamics of difference equation* $\frac{a.x_{n-k}}{b+cx_n^p}$, Fasciculi Mathematici, **42** (2009), 141–148. 1

[32] E. M. E. Zayed, M. A. El-Moneam, *On the rational recursive sequence* $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+Bx_n+Cx_{n-1}}$, Comm. Appl. Nonlinear Anal., **12** (2005), 15–28.

[33] E. M. E. Zayed, M. A. El-Moneam, *On the rational recursive sequence* $x_{n+1} = \frac{\alpha x_n+\beta x_{n-1}+\gamma x_{n-2}+\delta x_{n-3}}{Ax_n+Bx_{n-1}+Cx_{n-2}+Dx_{n-3}}$, Comm. Appl. Nonlinear Anal., **12** (2005), 15–28.

[34] B. G. Zhang, C. J. Tian, P. J. Wong, *Global attractivity of difference equations with variable delay*, Dynam. Contin. Discrete Impuls. Systems, **6** (1999), 307–317. 1