Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2959–2970
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Auxiliary principle and iterative algorithms for generalized mixed nonlinear variational-like inequalities

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Communicated by Sh. Wu

Abstract

The aim of this paper is to study the solvability of a class of generalized mixed nonlinear variational-like inequalities in Hilbert spaces. Using the auxiliary principle technique, the Banach fixed-point theorem and an inequality due to Chang and Xiang, we construct two iterative algorithms for finding approximate solutions of the generalized mixed nonlinear variational-like inequality. Under some conditions we prove the existence and uniqueness of solution for the generalized mixed nonlinear variational-like inequality and establish the strong convergence of approximate solutions to the exact solution of the generalized mixed nonlinear variational-like inequality. Our results extend, improve and unify some known results in the literature.

Keywords: Generalized mixed nonlinear variational-like inequality, Banach fixed-point theorem, auxiliary principle technique, iterative algorithm with errors.
2010 MSC: 47J20, 49J40.

1. Introduction

It is well known that the development of an efficient and implementable iterative algorithm to compute approximate solutions of a variational inequality has been one of the most difficult, interesting and important problems in the variational inequality theory.

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Received 2015-11-27

It is worth mentioning that the auxiliary principle technique suggested by Glowinski, Lions and Tremolieres in [7] has become a useful, important and powerful tool for solving various variational-like inequalities. Ansari and Yao [1], Chang and Xiang [3], Ding and Luo [4], Huang and Deng [8], Huang and Fang [9], Liu, Chen, Kang and Ume [11], Zeng [19], Zeng, Lin and Yao [20] and others used the auxiliary principle technique to construct some iterative methods for finding the exact solutions of the variational and variational-like inequalities in [1, 3, 4, 8, 9, 11, 12, 19, 20], and discussed the existence and uniqueness of solutions for the variational and variational-like inequalities in [1, 3, 4, 8, 9, 11, 12, 19, 20] and the convergence of iterative sequences generated by the iterative methods.

Motivated and inspired by the results in [1]–[9], [11]–[20], we introduce and study a new class of generalized mixed nonlinear variational-like inequalities in Hilbert spaces. By applying the auxiliary principle technique, the Banach contraction principle and an inequality due to [3], we show the existence and uniqueness theorems of solution for auxiliary problem relative to the generalized mixed nonlinear variational-like inequality. For finding the approximate solutions of the generalized mixed nonlinear variational-like inequality, we suggest two iterative algorithms with errors by the auxiliary problem. Under certain conditions, we get the existence and uniqueness results of solution for the generalized mixed nonlinear variational-like inequality and prove the convergence of iterative sequences generated by the iterative algorithms with errors. Our results improve and generalize many known results.

2. Preliminaries

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, H be a real Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle$ and a norm $\|\cdot\|$, respectively. Let K be a nonempty closed convex subset of H. Let $a : K \times K \to \mathbb{R}$ be a coercive continuous bilinear form, that is, there exist positive constants $c, d > 0$ such that

(a1) $a(v, v) \geq c\|v\|^2$, $\forall v \in K$;
(a2) $|a(u, v)| \leq d\|u\|\|v\|$, $\forall u, v \in K$.

Let $b : K \times K \to \mathbb{R}$ satisfy the following conditions:

(b1) b is linear in the first argument;
(b2) b is convex in the second argument;
(b3) b is bounded, that is, there exists a constant $l > 0$ satisfying

$$|b(u, v)| \leq l\|u\|\|v\|, \quad \forall u, v \in K;$$

(b4) $b(u, v) - b(u, w) \leq b(u, v - w)$, $\forall u, v, w \in K$.

Remark 2.1. It follows from (a1) and (a2) that $c \leq d$.

Remark 2.2. It follows (b1) and (b2) that

$$|b(u, v) - b(u, w)| \leq l\|u\|\|v - w\|, \forall u, v, w \in K,$$

which implies that b is continuous in the second argument.

Let $f \in H$ and $A, B, C, D : K \to H$, $N, M : H \times H \to H, \eta : K \times K \to H$ be mappings. Now we consider the following problem:

Find $u \in K$ such that

$$\langle N(Au, Bu) - M(Cu, Du) - f, \eta(v, u)\rangle + a(u, v - u) \geq b(u, u) - b(u, v), \quad \forall v \in K, \tag{2.1}$$

which is called *a generalized mixed nonlinear variational-like inequality.*

Special cases

Case 1. If $f = Bx = Dx = a(x,y) = 0, b(x,y) = \varphi(y), N(x,y) = M(x,y) = x$ for all $x, y \in H$, then problem (2.1) reduces to the mixed variational-like inequality studied by Ansari and Yao [1] and Zeng [19]: find $u \in K$ such that

$$\langle Au - Cu, \eta(v,u)\rangle + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K. \tag{2.2}$$

Case 2. If $f = Bx = Cx = Dx = a(x,y) = b(x,y) = M(x,y) = 0, N(x,y) = x$ for all $x, y \in H$, then problem (2.1) reduces to the variational-like inequality studied by Yang and Chen [16]: find $u \in K$ such that

$$\langle Au, \eta(v,u)\rangle \geq 0, \quad \forall v \in K. \tag{2.3}$$

Case 3. If $f = M(x,y) = Cx = Dx = 0, N(x,y) = -x - y$ and $\eta(x,y) = y - x$ for all $x, y \in H$, then problem (2.1) is equivalent to seeking $u \in K$ such that

$$a(u, v-u) + b(u,v) - b(u,u) \geq \langle Au + Bu, v - u\rangle, \quad \forall v \in K, \tag{2.4}$$

which was introduced and studied by Chang and Xiang [3].

Case 4. If $f = N(x,y) = Ax = Bx = Dx = 0, M(x,y) = x$ and $\eta(x,y) = y - x$ for all $x, y \in H$, then problem (2.1) is equivalent to finding $u \in K$ such that

$$a(u, v-u) + b(u,v) - b(u,u) \geq \langle Cu, v - u\rangle, \quad \forall v \in K, \tag{2.5}$$

which was introduced and studied by Bose [2].

Case 5. If $f = M(x,y) = Bx = Cx = Dx = 0, M(x,y) = x$ and $\eta(x,y) = y - x$ for all $x, y \in H$, then problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in K, \tag{2.6}$$

which was introduced and studied by Yao [17] and others.

For appropriate and suitable choices of the mappings $N, M, A, B, C, D, \eta, a$ and b, one can obtain a lot of variational and variational-like inequalities in [1]–[9], [11]–[20] as special cases of the generalized mixed nonlinear variational-like inequality (2.1).

Recall the below concepts and lemmas.

Definition 2.3. Let $A, B, g : K \to H$, $N : H \times H \to H$ and $\eta : K \times K \to H$ be mappings.

(1) A is said to be *Lipschitz continuous* if there exists a constant $t > 0$ such that

$$\|Ax - Ay\| \leq t\|x - y\|, \qquad \forall x, y \in K;$$

(2) A is said to be *η-strongly monotone* if there exists a constant $\tau > 0$ such that

$$\langle Ax - Ay, \eta(x,y)\rangle \geq \tau\|x - y\|^2, \quad \forall x, y \in K;$$

(3) N is said to be *η-strongly monotone* with respect to A in the first argument if there exists a constant $\beta > 0$ such that

$$\langle N(Au,y) - N(Av,y), \eta(u,v)\rangle \geq \beta\|u - v\|^2, \quad \forall u, v \in K, \ \forall y \in H;$$

(4) N is said to be *η-monotone* with respect to A and B in the first and second arguments if

$$\langle N(Au, Bu) - N(Av, Bv), \eta(u, v)\rangle \geq 0, \quad \forall u, v \in K;$$

(5) N is said to be *η-relaxed Lipschitz* with respect to B in the second argument if there exists a constant $\lambda > 0$ such that

$$\langle N(y, Bu) - N(y, Bv), \eta(u, v)\rangle \leq -\lambda\|u - v\|^2, \quad \forall u, v \in K, \ y \in H;$$

(6) N is said to be *η-generalized pseudocontrative* with respect to A in the first argument if there exists a constant $\xi > 0$ such that

$$\langle N(Au, y) - N(Av, y), \eta(u, v)\rangle \leq \xi\|u - v\|^2, \quad \forall u, v \in K, \ y \in H;$$

(7) N is said to be *g-relaxed Lipschitz* with respect to B in the second argument if there exists a constant $\mu > 0$ such that

$$\langle N(y, Bu) - N(y, Bv), gu - gv)\rangle \leq -\mu\|u - v\|^2, \quad \forall u, v \in K, \ y \in H;$$

(8) N is said to be *Lipschitz continuous* in the second argument if there exists a constant $\gamma > 0$ such that

$$\|N(y, u) - N(y, v)\| \leq \gamma\|u - v\|, \quad \forall u, v, y \in H;$$

(9) η is said to be *Lipschitz continuous* if there exists a constant $\delta > 0$ such that

$$\|\eta(u, v)\| \leq \delta\|u - v\|, \quad \forall u, v \in K. \tag{2.7}$$

Similarly we can define that N is *η-relaxed Lipschitz* with respect to B in the first argument.

Lemma 2.4 ([3]). *Let X be a nonempty closed convex subset of a Hausdorff linear topological space E, and $\phi, \psi : X \times X \to \mathbb{R}$ be mappings satisfying the following conditions:*

(a) $\psi(x, y) \leq \phi(x, y)$, $\forall x, y \in X$, and $\psi(x, x) \geq 0$, $\forall x \in X$;

(b) for each $x \in X$, $\phi(x, \cdot)$ is upper semicontinuous on X;

(c) for each $y \in X$, the set $\{x \in X : \psi(x, y) < 0\}$ is a convex set;

(d) there exists a nonempty compact set $Y \subset X$ and $x_0 \in Y$ such that $\psi(x_0, y) < 0$, $\forall y \in X \setminus Y$.

Then there exists $\hat{y} \in Y$ such that $\phi(x, \hat{y}) \geq 0$, $\forall x \in X$.

Lemma 2.5 ([10]). *Let $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}$ and $\{c_n\}_{n\geq 0}$ be nonnegative sequences satisfying*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n b_n + c_n, \quad \forall n \geq 0,$$

where

$$\{\lambda_n\}_{n\geq 0} \subset [0, 1], \quad \sum_{n=0}^{\infty} \lambda_n = +\infty, \quad \sum_{n=0}^{\infty} c_n < +\infty, \quad \lim_{n\to\infty} b_n = 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

Assumption 2.6. Let $f \in H$ and $A, B, C, D, g : K \to H$, $N, M : H \times H \to H$ and $\eta : K \times K \to H$ be mappings such that

(i) $\eta(x, y) = -\eta(y, x)$, $\forall x, y \in K$;

(ii) for each $u, w \in K$, the mappings $v \mapsto \langle gw - gu, \eta(v, w)\rangle$ and $v \mapsto \langle N(Aw, Bv) - M(Cw, Du) - f, \eta(v, w)\rangle$ are convex;

(iii) for each $u, v \in K$, the mappings $w \mapsto \langle gw - gu, \eta(v, w)\rangle$ and $w \mapsto \langle N(Aw, Bw) - M(Cw, Du) - f, \eta(v, w)\rangle$ are upper semicontinuous.

3. Auxiliary Problem and Iterative Algorithms

In this section, we give two existence and uniqueness theorems of solution for the auxiliary problem with respect to the generalized mixed nonlinear variational-like inequality (2.1). Based on these existence and uniqueness theorems, we construct two iterative algorithms with errors for the generalized mixed nonlinear variational-like inequality (2.1).

Let $g : K \to H$ be a mapping. Now we consider the following auxiliary problem with respect to the generalized mixed nonlinear variational-like inequality (2.1): for each $u \in K$, find $\hat{w} \in K$ such that

$$\begin{aligned} \langle g\hat{w}, \eta(v, \hat{w})\rangle \geq \langle gu, \eta(v, \hat{w})\rangle - \rho\langle N(A\hat{w}, B\hat{w}) - M(C\hat{w}, Du) - f, \eta(v, \hat{w})\rangle \\ - \rho a(\hat{w}, v - \hat{w}) - \rho b(u, v) + \rho b(u, \hat{w}), \quad \forall v \in K, \end{aligned} \tag{3.1}$$

where $\rho > 0$ is a constant.

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H, $f \in H$ and $\eta : K \times K \to H$ be Lipschitz continuous with constant δ. Assume that $a : K \times K \to \mathbb{R}$ is a coercive continuous bilinear form satisfying* (a1) *and* (a2) *and $b : K \times K \to \mathbb{R}$ satisfies* (b1)-(b4). *Let $g : K \to H$ be Lipschitz continuous and η-strongly monotone with constants ξ and τ, respectively. Let $A, B, C, D : K \to H$ and $N, M : H \times H \to H$ be mappings such that N is η-strongly monotone with respect to A in the first argument with constant α, η-relaxed Lipschitz with respect to B in the second argument with constant β, η-monotone with respect to A and B in the first and second arguments and M is η-relaxed Lipschitz with respect to C in the first argument with constant λ. Assume that* Assumption 2.6 *holds. Then for each $u \in K$, the auxiliary problem* (3.1) *has a unique solution in K.*

Proof. Let u be in K. Define two functionals ϕ and $\psi : K \times K \to \mathbb{R}$ by

$$\begin{aligned} \phi(v, w) = \langle gw - gu, \eta(v, w)\rangle + \rho\langle N(Aw, Bw) - M(Cw, Du) - f, \eta(v, w)\rangle \\ + \rho a(w, v - w) - \rho b(u, w) + \rho b(u, v), \end{aligned}$$

and

$$\begin{aligned} \psi(v, w) = \langle gw - gu, \eta(v, w)\rangle + \rho\langle N(Aw, Bv) - M(Cw, Du) - f, \eta(v, w)\rangle \\ + \rho a(w, v - w) - \rho b(u, w) + \rho b(u, v) \end{aligned}$$

for all $v, w \in K$.

We now prove that the functionals ϕ and ψ satisfy the conditions of Lemma 2.4 in the weak topology. Indeed, it is easy to see for all $v, w \in K$,

$$\begin{aligned} \phi(v, w) - \psi(v, w) &= -\rho\langle N(Aw, Bv) - N(Aw, Bw), \eta(v, w)\rangle \\ &\geq \rho\beta\|v - w\|^2 \\ &\geq 0 \end{aligned}$$

and

$$\psi(v, v) = 0,$$

which imply that ϕ and ψ satisfy the condition (a) of Lemma 2.4. Since a is a coercive continuous bilinear form, b is convex and continuous in the second argument, and for each $u, w \in K$, the mappings $v \mapsto \langle gw - gu, \eta(v, w)\rangle$ and $v \mapsto \langle N(Aw, Bv) - M(Cw, Du) - f, \eta(v, w)\rangle$ are convex, for each $u, v \in K$, the mappings $w \mapsto \langle gw-gu, \eta(v, w)\rangle$ and $w \mapsto \langle N(Aw, Bw)-M(Cw, Du)-f, \eta(v, w)\rangle$ are upper semicontinuous, it follows that for each $v \in K, \phi(v, \cdot)$ is weakly upper semicontinuous in the second argument and the set $\{v \in K : \psi(v, w) < 0\}$ is convex for each $w \in K$. That is, the conditions (b) and (c) of Lemma 2.4 hold. Finally we prove that condition (d) of Lemma 2.4 holds. Let $\bar{v} \in K$ and

$$Y = \{w \in K : \|w - \bar{v}\| \leq E\},$$

where

$$E = \big[\tau + \rho(\alpha + c + \lambda)\big]^{-1}\big[\xi\delta\|u - \bar{v}\| + \rho\delta\|M(C\bar{v}, Du) - N(A\bar{v}, B\bar{v}) + f\| + \rho d\|\bar{v}\| + \rho l\|u\| + 1\big].$$

Clearly, Y is a weakly compact subset of K. For each $w \in K \setminus Y$, we infer that

$$\begin{aligned}\psi(\bar{v}, w) &= \langle gw - gu, \eta(\bar{v}, w)\rangle + \rho\langle N(Aw, B\bar{v}) - M(Cw, Du) - f, \eta(\bar{v}, w)\rangle\\ &\quad + \rho a(w, \bar{v} - w) - \rho b(u, w) + \rho b(u, \bar{v})\\ &= -\langle gw - g\bar{v}, \eta(w, \bar{v})\rangle + \langle gu - g\bar{v}, \eta(w, \bar{v})\rangle\\ &\quad - \rho\langle N(Aw, B\bar{v}) - N(A\bar{v}, B\bar{v}), \eta(w, \bar{v})\rangle\\ &\quad + \rho\langle M(Cw, Du) - M(C\bar{v}, Du), \eta(w, \bar{v})\rangle\\ &\quad + \rho\langle M(C\bar{v}, Du) - N(A\bar{v}, B\bar{v}) + f, \eta(w, \bar{v})\rangle\\ &\quad - \rho a(w - \bar{v}, w - \bar{v}) - \rho a(\bar{v}, w - \bar{v}) + \rho b(u, \bar{v} - w)\\ &\le -\tau\|w - \bar{v}\|^2 + \xi\delta\|u - \bar{v}\|\|w - \bar{v}\| - \rho\alpha\|w - \bar{v}\|^2 - \rho\lambda\|w - \bar{v}\|^2\\ &\quad + \rho\delta\|M(C\bar{v}, Du) - N(A\bar{v}, B\bar{v}) + f\|\|w - \bar{v}\| - \rho c\|w - \bar{v}\|^2\\ &\quad + \rho d\|\bar{v}\|\|w - \bar{v}\| + \rho l\|u\|\|w - \bar{v}\|\\ &= -\|w - \bar{v}\|\{[\tau + \rho(\alpha + c + \lambda)]\|w - \bar{v}\| - \xi\delta\|u - \bar{v}\|\\ &\quad - \rho\delta\|M(C\bar{v}, Du) - N(A\bar{v}, B\bar{v}) + f\| - \rho d\|\bar{v}\| - \rho l\|u\|\}\\ &< 0,\end{aligned}$$

which means that the condition (d) of Lemma 2.4 holds. Thus Lemma 2.4 ensures that there exists $\hat{w} \in Y \subseteq K$ such that $\phi(v, \hat{w}) \ge 0$ for all $v \in K$, that is,

$$\begin{aligned}\langle g\hat{w}, \eta(v, \hat{w})\rangle &\ge \langle gu, \eta(v, \hat{w})\rangle - \rho\langle N(A\hat{w}, B\hat{w}) - M(C\hat{w}, Du) - f, \eta(v, \hat{w})\rangle\\ &\quad - \rho a(\hat{w}, v - \hat{w}) - \rho b(u, v) + \rho b(u, \hat{w}), \quad \forall v \in K.\end{aligned}$$

That is, $\hat{w} \in K$ is a solution of the auxiliary problem (3.1). Now we prove the uniqueness of solution for the auxiliary problem (3.1). Suppose that $w_1, w_2 \in K$ are two solutions of the auxiliary problem (3.1) with respect to u. It follows that

$$\begin{aligned}\langle gw_1, \eta(v, w_1)\rangle &\ge \langle gu, \eta(v, w_1)\rangle - \rho\langle N(Aw_1, Bw_1) - M(Cw_1, Du) - f, \eta(v, w_1)\rangle\\ &\quad - \rho a(w_1, v - w_1) - \rho b(u, v) + \rho b(u, w_1), \quad \forall v \in K\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\langle gw_2, \eta(v, w_2)\rangle &\ge \langle gu, \eta(v, w_2)\rangle - \rho\langle N(Aw_2, Bw_2) - M(Cw_2, Du) - f, \eta(v, w_2)\rangle\\ &\quad - \rho a(w_2, v - w_2) - \rho b(u, v) + \rho b(u, w_2), \quad \forall v \in K.\end{aligned} \tag{3.3}$$

Taking $v = w_2$ in (3.2) and $v = w_1$ in (3.3), we get that

$$\begin{aligned}\langle gw_1, \eta(w_2, w_1)\rangle &\ge \langle gu, \eta(w_2, w_1)\rangle - \rho\langle N(Aw_1, Bw_1) - M(Cw_1, Du) - f, \eta(w_2, w_1)\rangle\\ &\quad - \rho a(w_1, w_2 - w_1) - \rho b(u, w_2) + \rho b(u, w_1)\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\langle gw_2, \eta(w_1, w_2)\rangle &\ge \langle gu, \eta(w_1, w_2)\rangle - \rho\langle N(Aw_2, Bw_2) - M(Cw_2, Du) - f, \eta(w_1, w_2)\rangle\\ &\quad - \rho a(w_2, w_1 - w_2) - \rho b(u, w_1) + \rho b(u, w_2).\end{aligned} \tag{3.5}$$

Adding (3.4) and (3.5), we deduce that

$$\begin{aligned}\tau\|w_1 - w_2\|^2 &\le -\rho\langle N(Aw_1, Bw_1) - N(Aw_2, Bw_2), \eta(w_1, w_2)\rangle\\ &\quad + \rho\langle M(Cw_1, Du) - M(Cw_2, Du), \eta(w_1, w_2)\rangle - \rho a(w_1 - w_2, w_1 - w_2)\\ &\le -\rho(c + \lambda)\|w_1 - w_2\|^2,\end{aligned}$$

which yields that $w_1 = w_2$. That is, $\hat{w}$ is the unique solution of the auxiliary problem (3.1). This completes the proof. □

The proof of the below result is similar to that of Theorem 3.1 and is omitted.

Theorem 3.2. *Let $K, H, f, \eta, a, b, g, A, B, C, D$ and N be as in* Theorem 3.1 *and* Assumption 2.6 *hold. Let $M : H \times H \to H$ be η-generalized pseudocontrative with respect to C in the first argument with constant λ. If there exists a positive constant ρ satisfying*

$$\rho(\lambda - c) < \tau, \tag{3.6}$$

then for each $u \in K$, the auxiliary problem (3.1) *has a unique solution in K.*

Based on Theorems 3.1 and 3.2, we suggest the following iterative algorithms with errors for solving the generalized mixed nonlinear variational-like inequality (2.1).

Algorithm 3.3. For given $u_0 \in K$, compute sequence $\{u_n\}_{n\geq 0} \subset K$ by the following iterative scheme:

$$\begin{aligned} \langle gu_{n+1}, \eta(v, u_{n+1})\rangle \geq{} & \langle gu_n, \eta(v, u_{n+1})\rangle \\ & - \rho\langle N(Au_{n+1}, Bu_{n+1}) - M(Cu_{n+1}, Du_n) - f, \eta(v, u_{n+1})\rangle \\ & - \rho a(u_{n+1}, v - u_{n+1}) - \rho b(u_n, v) + \rho b(u_n, u_{n+1}) \\ & + \langle e_n, \eta(v, u_{n+1})\rangle, \quad \forall v \in K, n \geq 0, \end{aligned} \tag{3.7}$$

where $\rho > 0$ is a constant and $\{e_n\}_{n\geq 0}$ is an arbitrary sequence in K introduced to take into account possible inexact computation and satisfies that

$$\lim_{n\to\infty} \|e_n\| = 0. \tag{3.8}$$

Algorithm 3.4. For given $u_0 \in K$, compute sequence $\{u_n\}_{n\geq 0} \subset K$ by the below iterative schemes:

$$\begin{aligned} \langle gw_n, \eta(v, w_n)\rangle \geq{} & (1 - \alpha_n)\langle gu_n, \eta(v, w_n)\rangle \\ & + \alpha_n\langle gw_n - \rho N(Aw_n, Bw_n) + \rho M(Cw_n, Du_n) + \rho f, \eta(v, w_n)\rangle \\ & - \alpha_n\rho a(w_n, v - w_n) - \alpha_n\rho b(u_n, v) + \alpha_n\rho b(u_n, w_n) \\ & + \langle r_n, \eta(v, w_n)\rangle, \quad \forall v \in K,\ n \geq 0 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \langle gu_{n+1}, \eta(v, u_{n+1})\rangle \geq{} & (1 - \beta_n)\langle gw_n, \eta(v, u_{n+1})\rangle \\ & + \beta_n\langle gu_{n+1} - \rho N(Au_{n+1}, Bu_{n+1}) + \rho M(Cu_{n+1}, Dw_n) + \rho f, \eta(v, u_{n+1})\rangle \\ & - \beta_n\rho a(u_{n+1}, v - u_{n+1}) - \beta_n\rho b(w_n, v) + \beta_n\rho b(w_n, u_{n+1}) \\ & + \langle s_n, \eta(v, u_{n+1})\rangle, \quad \forall v \in K,\ n \geq 0, \end{aligned} \tag{3.10}$$

where $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0} \subset [0, 1]$ and $\{r_n\}_{n\geq 0}$, $\{s_n\}_{n\geq 0}$ are two arbitrary sequences in K introduced to take into account possible inexact computation and satisfy that

$$\lim_{n\to\infty} \|r_n\| = \lim_{n\to\infty} \|s_n\| = 0. \tag{3.11}$$

Remark 3.5. Algorithm 3.4 is an Ishikawa iterative methods with errors of inequality type and it is different from the algorithms in [1]–[9], [11]–[20].

4. Existence and Convergence

Now we show the existence and uniqueness of solution for the generalized mixed nonlinear variational-like inequality (2.1) and discuss the convergence of the iterative sequences generated by Algorithms 3.3 and 3.4, respectively.

Theorem 4.1. *Let K, H, f, η, a, b and g be as in* Theorem 3.1 *Let $A, B, C, D : K \to H$ and $N, M : H \times H \to H$ be mappings such that D is Lipschitz continuous with constant μ, N is η-strongly monotone with respect to A in the first argument with constant α, η-relaxed Lipschitz with respect to B in the second argument with constant β, η-monotone with respect to A and B in the first and second arguments, M is η-relaxed Lipschitz with respect to C in the first argument with constant λ, g-relaxed Lipschitz with respect to D in the second argument with constant γ and Lipschitz continuous in the second argument with constant q. Let $T = \delta^2\gamma + \tau(c + \lambda - l)$, $Q = \delta^2\xi^2 - \tau^2$, $P = \delta^2\mu^2q^2 - (c + \lambda - l)^2$ and $l + \delta\mu q < c + \lambda$. Assume that and* Assumption 2.6 *holds. If there exists a constant $\rho > 0$ satisfying*

$$T^2 > PQ, \qquad \left|\rho - \frac{T}{P}\right| > \frac{\sqrt{T^2 - PQ}}{-P}, \tag{4.1}$$

then the generalized mixed nonlinear variational-like inequality (2.1) *possesses a unique solution $u \in K$ and the iterative sequence $\{u_n\}_{n\geq 0}$ generated by* Algorithm 3.3 *converges strongly to u.*

Proof. It follows from Theorem 3.1 that there exists a mapping $F : K \to K$ such that for each $u \in K$, $F(u) = \hat{w}$ is the unique solution of the auxiliary problem (3.1). Next we show that F is a contraction mapping in K. Let u_1 and u_2 be arbitrary elements in K. Using (3.1), we see that

$$\begin{aligned}
&\langle gFu_1, \eta(v, Fu_1)\rangle \\
&\geq \langle gu_1, \eta(v, Fu_1)\rangle - \rho\langle N(AFu_1, BFu_1) - M(CFu_1, Du_1) - f, \eta(v, Fu_1)\rangle \\
&\quad - \rho a(Fu_1, v - Fu_1) - \rho b(u_1, v) + \rho b(u_1, Fu_1), \quad \forall v \in K
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\langle gFu_2, \eta(v, Fu_2)\rangle \\
&\geq \langle gu_2, \eta(v, Fu_2)\rangle - \rho\langle N(AFu_2, BFu_2) - M(CFu_2, Du_2) - f, \eta(v, Fu_2)\rangle \\
&\quad - \rho a(Fu_2, v - Fu_2) - \rho b(u_2, v) + \rho b(u_2, Fu_2), \quad \forall v \in K.
\end{aligned} \tag{4.3}$$

Letting $v = Fu_2$ in (4.2) and $v = Fu_1$ in (4.3), and adding these inequalities, we arrive at

$$\begin{aligned}
&\tau\|Fu_1 - Fu_2\|^2 \\
&\leq \langle gFu_1 - gFu_2, \eta(Fu_1, Fu_2)\rangle \\
&\leq \langle gu_1 - gu_2, \eta(Fu_1, Fu_2)\rangle - \rho\langle N(AFu_1, BFu_1) - M(CFu_1, Du_1) \\
&\quad - N(AFu_2, BFu_2) + M(CFu_2, Du_2), \eta(Fu_1, Fu_2)\rangle \\
&\quad - \rho a(Fu_1 - Fu_2, Fu_1 - Fu_2) + \rho b(u_1 - u_2, Fu_2 - Fu_1) \\
&= -\rho\langle N(AFu_1, BFu_1) - N(AFu_2, BFu_2), \eta(Fu_1, Fu_2)\rangle \\
&\quad + \rho\langle M(CFu_1, Du_1) - M(CFu_2, Du_1), \eta(Fu_1, Fu_2)\rangle \\
&\quad + \langle gu_1 - gu_2 + \rho(M(CFu_2, Du_1) - M(CFu_2, Du_2)), \eta(Fu_1, Fu_2)\rangle \\
&\quad - \rho a(Fu_1 - Fu_2, Fu_1 - Fu_2) + \rho b(u_1 - u_2, Fu_2 - Fu_1) \\
&\leq (\delta\sqrt{\xi^2 - 2\rho\gamma + \rho^2\mu^2q^2} + \rho l)\|u_1 - u_2\|\|Fu_1 - Fu_2\| \\
&\quad - \rho(c + \lambda)\|Fu_1 - Fu_2\|^2,
\end{aligned}$$

that is,

$$\|Fu_1 - Fu_2\| \leq \theta\|u_1 - u_2\|,$$

where

$$\theta = \frac{\delta\sqrt{\xi^2 - 2\rho\gamma + \rho^2\mu^2q^2} + \rho l}{\tau + \rho(c + \lambda)}. \tag{4.4}$$

It is obvious that (4.1) is equivalent to $\theta < 1$. Therefore, $F : K \to K$ is a contraction mapping. It follows from the Banach fixed-point theorem that F has a unique fixed point $u \in K$. In light of (3.1), we get that

$$\begin{aligned}
\langle gu, \eta(v, u)\rangle &\geq \langle gu, \eta(v, u)\rangle - \rho\langle N(Au, Bu) - M(Cu, Du) - f, \eta(v, u)\rangle \\
&\quad - \rho a(u, v - u) - \rho b(u, v) + \rho b(u, u), \quad \forall v \in K,
\end{aligned} \tag{4.5}$$

which implies that

$$\langle N(Au, Bu) - M(Cu, Du) - f, \eta(v, u)\rangle + a(u, v - u) \geq b(u, u) - b(u, v), \quad \forall v \in K,$$

that is, $u \in K$ is a solution of the generalized mixed nonlinear variational-like inequality (2.1).

Now we prove the uniqueness. Suppose that the generalized mixed nonlinear variational-like inequality (2.1) has two solutions $\hat{u}, u_0 \in K$. It follows that

$$\langle N(A\hat{u}, B\hat{u}) - M(C\hat{u}, D\hat{u}) - f, \eta(v, \hat{u})\rangle + a(\hat{u}, v - \hat{u}) \geq b(\hat{u}, \hat{u}) - b(\hat{u}, v), \tag{4.6}$$

$$\langle N(Au_0, Bu_0) - M(Cu_0, Du_0) - f, \eta(v, u_0)\rangle + a(u_0, v - u_0) \geq b(u_0, u_0) - b(u_0, v), \tag{4.7}$$

for all $v \in K$. Taking $v = u_0$ in (4.6) and $v = \hat{u}$ in (4.7), we obtain that

$$\langle N(A\hat{u}, B\hat{u}) - M(C\hat{u}, D\hat{u}) - f, \eta(u_0, \hat{u})\rangle + a(\hat{u}, u_0 - \hat{u}) \geq b(\hat{u}, \hat{u}) - b(\hat{u}, u_0), \tag{4.8}$$

$$\langle N(Au_0, Bu_0) - M(Cu_0, Du_0) - f, \eta(\hat{u}, u_0)\rangle + a(u_0, \hat{u} - u_0) \geq b(u_0, u_0) - b(u_0, \hat{u}). \tag{4.9}$$

Adding (4.8) and (4.9), we deduce that

$$\begin{aligned}
&(c + \lambda - \mu q\delta - l)\|\hat{u} - u_0\|^2 \\
&\leq \langle N(A\hat{u}, B\hat{u}) - N(Au_0, Bu_0), \eta(\hat{u}, u_0)\rangle - \langle M(C\hat{u}, D\hat{u}) - M(Cu_0, Du_0), \eta(\hat{u}, u_0)\rangle \\
&\quad + a(\hat{u} - u_0, \hat{u} - u_0) - b(\hat{u} - u_0, \hat{u} - u_0) \\
&\leq 0,
\end{aligned}$$

which yields that $\hat{u} = u_0$ by $l + \delta\mu q < c + \lambda$. That is, the generalized mixed nonlinear variational-like inequality (2.1) has a unique solution in K.

Next we discuss the convergence of the iterative sequence generated by Algorithm 3.3. Taking $v = u_{n+1}$ in (4.5) and $v = u$ in (3.7), and adding these inequalities, we infer that

$$\begin{aligned}
\tau\|u_{n+1} - u\|^2 \leq& -\rho\langle N(Au_{n+1}, Bu_{n+1}) - N(Au, Bu), \eta(u_{n+1}, u)\rangle \\
&+ \rho\langle M(Cu_{n+1}, Du_n) - M(Cu, Du_n), \eta(u_{n+1}, u)\rangle \\
&+ \langle gu_n - gu + \rho M(Cu, Du_n) - M(Cu, Du), \eta(u_{n+1}, u)\rangle \\
&- \rho a(u_{n+1} - u, u_{n+1} - u) + \rho b(u_n - u, u - u_{n+1}) + \langle e_n, \eta(u_{n+1}, u)\rangle \\
\leq& -\rho(c + \lambda)\|u_{n+1} - u\|^2 + (\delta\sqrt{\xi^2 - 2\rho\gamma + \rho^2\mu^2 q^2} \\
&+ \rho l)\|u_{n+1} - u\|\|u_n - u\| + \delta\|e_n\|\|u_{n+1} - u\|, \quad \forall n \geq 0.
\end{aligned}$$

That is,

$$\|u_{n+1} - u\| \leq \theta\|u_n - u\| + \frac{\delta}{\tau + \rho(c + \lambda)}\|e_n\|, \quad \forall n \geq 0. \tag{4.10}$$

It follows from Lemma 2.5, (3.8), (4.1), (4.4) and (4.10) that the iterative sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.3 converges strongly to u. This completes the proof. □

As in the proof of Theorem 4.1, we have,

Theorem 4.2. *Let $K, H, f, \eta, a, b, g, A, B, C, D$ and N be as in* Theorem 4.1 *and* Assumption 2.6 *hold. Let $M : H \times H \to H$ be η-generalized pseudocontrative with respect to C in the first argument with constant λ, g-relaxed Lipschitz with respect to D in the second argument with constant γ and Lipschitz continuous in the second argument with constant q. Let $T = \delta^2\gamma + \tau(c - \lambda - l)$, $Q = \delta^2\xi^2 - \tau^2$, $P = \delta^2\mu^2 q^2 - (c - \lambda - l)^2$ and $c > \lambda + l + \delta\mu q$. If there exists a constant $\rho > 0$ satisfying* (4.1)*, then the generalized mixed nonlinear variational-like inequality* (2.1) *possesses a unique solution $u \in K$ and the iterative sequence $\{u_n\}_{n\geq 0}$ generated by* Algorithm 3.3 *converges strongly to u.*

Theorem 4.3. *Let $K, H, f, \eta, a, b, g, A, B, C, D, N$ and M be as in* Theorem 4.1 *with $c + \lambda > l + \delta\mu q$ and* Assumption 2.6 *hold. Assume that*

$$\inf\{\alpha_n, \beta_n : n \geq 0\} > 0. \tag{4.11}$$

If there exists a constant $\rho > 0$ satisfying

$$\max\left\{\frac{\delta\xi - \tau \inf\{1-\alpha_n, 1-\beta_n : n \geq 0\}}{(c+\lambda)\inf\{\alpha_n, \beta_n : n \geq 0\}},\ \frac{\xi^2}{2\gamma}\right\} \leq \rho < \frac{\delta\xi}{l + \mu q\delta}, \tag{4.12}$$

then the generalized strongly nonlinear variational-like inequality (2.1) *possesses a unique solution $u \in K$ and the iterative sequence $\{u_n\}_{n\geq 0}$ generated by* Algorithm 3.4 *converges strongly to u.*

Proof. Put

$$\theta_1 = \frac{\delta\xi}{\tau \inf\{1-\alpha_n, 1-\beta_n : n \geq 0\} + \rho(c+\lambda)\inf\{\alpha_n, \beta_n : n \geq 0\}},$$

and

$$\theta_2 = \frac{\rho l}{\delta\xi} + \frac{\rho\mu q}{\xi}.$$

In view of (4.4), (4.11) and (4.12), we conclude easily that

$$\begin{aligned}
\theta &= \frac{\delta\sqrt{\xi^2 - 2\rho\gamma + \rho^2\mu^2 q^2} + \rho l}{\tau + \rho(c+\lambda)} \\
&= \frac{\delta\xi}{\tau + \rho(c+\lambda)}\Big(\frac{\rho l}{\delta\xi} + \frac{\sqrt{\xi^2 - 2\rho\gamma + \rho^2\mu^2 q^2}}{\xi}\Big) \\
&\leq \frac{\delta\xi}{\tau + \rho(c+\lambda)}\Big(\frac{\rho l}{\delta\xi} + \frac{\rho\mu q}{\xi}\Big) \\
&\leq \frac{\delta\xi\theta_2}{\tau + \rho(c+\lambda)\inf\{\alpha_n, \beta_n : n \geq 0\}} \\
&\leq \theta_1\theta_2 \leq \theta_2 < 1.
\end{aligned} \tag{4.13}$$

It follows from Theorem 4.1 that the generalized strongly nonlinear variational-like inequality (2.1) has a unique solution $u \in K$ such that

$$\begin{aligned}
\langle gu, \eta(v,u)\rangle \geq{}& (1-\alpha_n)\langle gu, \eta(v,u)\rangle \\
&+ \alpha_n\langle gu - \rho N(Au, Bu) + \rho M(Cu, Du) + \rho f, \eta(v,u)\rangle \\
&- \alpha_n\rho a(u, v-u) - \alpha_n\rho b(u,v) + \alpha_n\rho b(u,u)
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
\langle gu, \eta(v,u)\rangle \geq{}& (1-\beta_n)\langle gu, \eta(v,u)\rangle \\
&+ \beta_n\langle gu - \rho N(Au, Bu) + \rho M(Cu, Du) + \rho f, \eta(v,u)\rangle \\
&- \beta_n\rho a(u, v-u) - \beta_n\rho b(u,v) + \beta_n\rho b(u,u),
\end{aligned} \tag{4.15}$$

for all $v \in K$ and $n \geq 0$. Taking $v = u$ in (3.9), $v = w_n$ in (4.14) and adding these inequalities, we get that

$$\begin{aligned}
&(1-\alpha_n)\tau\|w_n - u\|^2 \\
&\leq (1-\alpha_n)\langle gu_n - gu, \eta(w_n, u)\rangle - \alpha_n\rho\langle N(Aw_n, Bw_n) - N(Au, Bu), \eta(w_n, u)\rangle \\
&\quad + \alpha_n\rho\langle M(Cw_n, Du_n) - M(Cu, Du_n), \eta(w_n,u)\rangle + \alpha_n\rho\langle M(Cu, Du_n) - M(Cu, Du)), \eta(w_n, u)\rangle \\
&\quad - \alpha_n\rho a(w_n - u, w_n - u) + \alpha_n\rho b(u_n - u, u - w_n) + \langle r_n, \eta(w_n, u)\rangle \\
&\leq (1-\alpha_n)\xi\delta\|u_n - u\|\|w_n - u\| - \alpha_n\rho\lambda\|w_n - u\|^2 \\
&\quad + \alpha_n\delta\rho\mu q\|u_n - u\|\|w_n - u\| + \delta\|r_n\|\|w_n - u\|, \quad \forall n \geq 0,
\end{aligned}$$

which means that

$$\begin{aligned}\|w_n - u\| &\le \frac{\delta\xi}{(1-\alpha_n)\tau + \rho(c+\lambda)\alpha_n}[1-\alpha_n(1-\theta_2)]\|u_n - u\| \\ &\quad + \frac{\delta}{(1-\alpha_n)\tau + \rho(c+\lambda)\alpha_n}\|r_n\| \\ &\le \theta_1[1-\alpha_n(1-\theta_2)]\|u_n - u\| + \frac{\theta_1}{\xi}\|r_n\| \\ &\le [1-\alpha_n(1-\theta_2)]\|u_n - u\| + \frac{1}{\xi}\|r_n\| \\ &\le \|u_n - u\| + \frac{1}{\xi}\|r_n\|, \quad \forall n \ge 0.\end{aligned} \tag{4.16}$$

From (3.10), (4.11), (4.12) and (4.15), we deduce similarly that

$$\begin{aligned}\|u_{n+1} - u\| &\le [1-\beta_n(1-\theta_2)]\|w_n - u\| + \frac{1}{\xi}\|s_n\| \\ &\le [1-\beta_n(1-\theta_2)]\|u_n - u\| + \frac{1}{\xi}\|s_n\| + \frac{1}{\xi}\|r_n\|, \quad \forall n \ge 0.\end{aligned} \tag{4.17}$$

It follows from Lemma 2.5, (3.11), (4.11) and (4.17) that $\lim_{n\to\infty}\|u_{n+1} - u\| = 0$. This completes the proof. □

Similarly we have the following result.

Theorem 4.4. *Let $K, H, f, \eta, a, b, g, A, B, C, D, N$ and M be as in* Theorem 4.2 *with $c > \lambda + l + \delta\mu q$. Let* (4.11) *and* Assumption 2.6 *hold. If there exists a constant $\rho > 0$ satisfying*

$$\max\left\{\frac{\delta\xi - \tau\inf\{1-\alpha_n, 1-\beta_n : n \ge 0\}}{(c-\lambda)\inf\{\alpha_n, \beta_n : n \ge 0\}}, \ \frac{\xi^2}{2\gamma}\right\} \le \rho < \frac{\delta\xi}{l + \mu q\delta},$$

then the generalized strongly nonlinear variational-like inequality (2.1) *possesses a unique solution $u \in K$ and the iterative sequence $\{u_n\}_{n\ge 0}$ generated by* Algorithm 3.4 *converges strongly to u.*

Proof. Put

$$\theta_1 = \frac{\delta\xi}{\tau\inf\{1-\alpha_n, 1-\beta_n : n \ge 0\} + \rho(c-\lambda)\inf\{\alpha_n, \beta_n : n \ge 0\}},$$

and

$$\theta_2 = \frac{\rho l}{\delta\xi} + \frac{\rho\mu q}{\xi}.$$

The rest of the proof is similar to that of Theorem 4.3, and is omitted. This completes the proof. □

Remark 4.5. Theorems 4.1 to 4.4 extend and improve the corresponding results in [1]–[3] and [17, 18].

Acknowledgements

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Science, ICT & Future Planning (2013R1A1A2057665).

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