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Full state hybrid projective synchronization of variable-order fractional chaotic/hyperchaotic systems with nonlinear external disturbances and unknown parameters

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Abstract

The full state hybrid projective synchronization (FSHPS) definition for variable-order fractional chaotic/hyperchaotic systems with nonlinear external disturbances and unknown parameters is firstly presented. Then by introducing a compensator and a nonlinear controller, the FSHPS scheme is generated to eliminate the influence of nonlinear external disturbances effectively. Moreover, the parameters are estimated validly. Based on these control methods, appropriate parameters and controller to achieve FSHPS for the variable-order fractional chaotic/hyperchaotic systems are chosen impactfully. Simulations of variable-order fractional Chen and Lü system and fractional order hyperchaotic Lorenz system in the sense of FSHPS are performed and results show the effectiveness of our method.

Keywords: Variable-order fractional systems, synchronization, external disturbance, unknown parameters.
2010 MSC: 34H10, 34H15, 34D06.

1. Introduction

Since ĹHôpital and Leibniz introduced the fractional-order derivative in an exchange of letters in 1695, many nonlinear fractional-order systems have been proposed and studied in many different fields, such as

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in transaction system [4, 26] and nonlinear science[23, 25]. There are many interesting and vital results for different fractional-order systems. For instance, solvability and numerical methods for the solution of nonlinear integral equations were introduced under some assumptions [2, 27]. In addition, dynamical behavior like chaos and hyperchaos were generated in fractional-order systems and its complex properties were also considered [1, 6, 11, 15]. At the same time, many control methods were used to control uncertainties using sliding control, fuzzy control and boundary control for nonlinear fractional-order systems [3, 17, 29].

Recently, researchers have focused on the study of chaos synchronization in fractional-order systems, such as projective synchronization [31, 35, 36, 39], the anti synchronization [19, 24, 32, 33] and lag synchronization [10, 30, 40], etc. Nowadays, full state hybrid projective synchronization (FSHPS) [7] has been presented and has been used in real chaotic systems [8, 16, 28, 37], complex chaotic systems [18, 38] and fractional-order chaotic systems [28, 34].

Importantly, the order in fractional-order systems which means memory effect usually varies in reality with nonlinear external disturbance [21]. However, there is little work focusing on the control and synchronization of variable-order fractional systems. In addition, the parameters of practical systems are hardly known. So how to effectively synchronize two different or identical variable-order fractional chaotic systems with nonlinear external disturbances and unknown parameters is most essential and useful in real applications.

Inspired by the above discussions, the definition of FSHPS for variable-order fractional chaotic systems is given firstly. Then a controller to show the synchronization for variable-order fractional chaotic systems with nonlinear external disturbance and unknown parameters is designed. Moreover, the estimate of parameters for different variable-order fractional chaotic systems is proposed. Simulations for variable-order different and identical fractional chaotic/hyperchaotic systems illustrate the effectiveness of the synchronization. The rest of this paper is organized as follows: Sect. 2 formulates the basic definitions and lemmas about variable-order fractional chaotic systems and FSHPS problem. In Sect. 3, the controller is proposed and results about the FSHPS for two different fractional chaotic systems and two identical fractional chaotic systems with variable-order are given, respectively. Next, in Sect. 4, two examples to realize the synchronization in the sense of FSHPS for variable-order fractional chaotic/hyperchaotic systems are taken. The corresponding simulations illustrate the effectiveness of our method. Finally, the conclusion in Sect. 5 is given.

2. Problem formulation

The following definition of the variable-order derivative type is used by changing the definition in [9]:

$$ {}_{a}^{C}D_{0+}^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t \frac{f^{(')}(\tau)}{(t-\tau)^{\alpha(t)}} d\tau, \tag{2.1} $$

where $0 < \alpha(t) \leq 1$ represents variable-order as time t changes, $f(t)$ means an arbitrary integrable function and $\Gamma(\cdot)$ denotes the Gamma function.

Define $D_t^{\alpha(t)} X = {}_{a}^{C} D_{0+}^{\alpha(t)} X$. Then the variable-order fractional dynamic system is considered as following:

$$ D_t^{\alpha(t)} X = f(t, X), \tag{2.2} $$

with initial condition $X(t_0)$, where $\alpha(t) \in (0, 1]$, $f : [t_0, \infty] \times \Omega^n \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in X on $[t_0, \infty] \times \Omega^n$ and $\Omega^n \subseteq \mathbb{R}^n$.

The following definitions and Lemmas for the next consideration are easily based on the results of Ref. [12, 13, 14, 20].

Definition 2.1. The constant X_0 is an equilibrium point of variable-order fractional dynamic system (2.2), if and only if $f(t, X_0) = 0$.

Lemma 2.2. *Let $x = 0$ be an equilibrium point for system (2.2) and $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Let $V(t, x(t)) : [0, \infty) \times \mathbb{D} \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x such that*

$$\alpha_1 \|x\|^a \leq V(t, x(t)) \leq \alpha_2 \|x\|^{ab},$$

$$D_t^{\alpha(t)} V(t, x(t)) \leq -\alpha_3 \|x\|^{ab},$$

where $t \geq 0$, $x \in \mathbb{D}$, $\alpha \in (0, 1)$, $\alpha_1, \alpha_2, \alpha_3, a$ and b are arbitrary positive constants. Then $x = 0$ is Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x = 0$ is globally Mittag-Leffler stable.

Definition 2.3. For two variable-order fractional chaotic/hyperchaotic systems as following :

$$\begin{cases} D_t^{\alpha(t)} X = f(X) \leftarrow \textit{ drive system}, \\ D_t^{\alpha(t)} Y = g(Y) + u(x, y) \leftarrow \textit{ response system}, \end{cases}$$

we call the synchronization as full state hybrid projective synchronization (FSHPS) if there exists a constant matrix $H = diag(h_1, h_2, \ldots, h_n)$ satisfying the following condition:

$$\lim_{t \to +\infty} \| Y - HX \| = 0,$$

where $X = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ and $Y = [y_1(t), y_2(t), \ldots, y_n(t)]^T \in \mathbb{R}^n$ are the state vectors and T means transpose. $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are vector functions and $u(x, y)$ is a controller.

3. FSHPS controller design with nonlinear external disturbance and unknown parameters

The purpose of our paper is to design the controller u such that the response variable-order fractional chaotic system can synchronize the drive fractional system asymptotically in sense of the FSHPS with nonidentical initial vectors. Namely, the error vector $e = Y - HX$ will satisfy

$$\begin{aligned} \lim_{t \to +\infty} \|e\|^2 &= \lim_{t \to +\infty} \|Y - HX\|^2 \\ &= \lim_{t \to +\infty} (\textstyle\sum_{l=1}^{n} \|y_l(t) - h_l x_l(t)\|)^2 \to 0, \end{aligned}$$

where

$$y_0 = [y_1(0), y_2(0), \ldots, y_n(0)]^T$$

is initial value of the derive chaotic fractional-order system,

$$x_0 = [x_1(0), x_2(0), \ldots, x_n(0)]^T$$

is initial value of the response fractional-order chaotic system and the initial values of errors are

$$e_0 = [y_1(0) - h_1 x_1(0), y_2(0) - h_2 x_2(0), \ldots, y_n(0) - h_n x_n(0)]^T.$$

Then the main results are obtained for the following class of drive variable-order fractional chaotic/hyperchaotic system with nonlinear external disturbances and unknown parameters based on the definitions and lemma in Sect. 2:

$$\begin{cases} D_t^{\alpha(t)} X = f(X) + F(X)A + d_1 \leftarrow \textit{ drive system}, \\ D_t^{\alpha(t)} Y = g(Y) + G(Y)B + u + d_2 \leftarrow \textit{ response system}, \end{cases} \tag{3.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are vector functions, $F : \mathbb{R}^n \to \mathbb{R}^{n \times p}$ and $G : \mathbb{R}^n \to \mathbb{R}^{n \times p}$ are function matrixs, $A = [a_1, a_2, \ldots, a_n]^T$ and $B = [b_1, b_2, \ldots, b_n]^T$ are constant parameter vectors. $u \in \mathbb{R}^N$ stands for control input vector. d_1 and d_2 are $n \times 1$ matrices and they represent the nonlinear external disturbances, in which $|h_l d_{1l}| \leq \rho_{1l}$, $|d_{2l}| \leq \rho_{2l}$, where ρ_{1l} and ρ_{2l} are given, $\rho_1 = [\rho_{11}, \rho_{12}, \ldots, \rho_{1n}]^T$ and $\rho_2 = [\rho_{21}, \rho_{22}, \ldots, \rho_{2n}]^T$.

Theorem 3.1. *Suppose that the controller vector u is designed as*

$$\begin{aligned} u &= Hf(X) + HF(X)\hat{A} - g(Y) - G(Y)\hat{B} - ke - \delta \\ &= Hf(x_1, x_2, \dots, x_n) + HF(x_1, x_2, \dots, x_n)\hat{A} - g(y_1, y_2, \dots, y_n) \\ &\quad -G(y_1, y_2, \dots, y_n)\hat{B} - ke - \delta. \end{aligned} \tag{3.2}$$

The compensator δ is taken as

$$\begin{aligned} \delta &= (\delta_1, \delta_2, \dots, \delta_n) \\ &= \tfrac{1}{2\xi}[(\rho_{11} + \rho_{21})^2 e_1, (\rho_{12} + \rho_{22})^2 e_2, \dots, (\rho_{1n} + \rho_{2n})^2 e_n]^T. \end{aligned}$$

$$D_t^{\alpha(t)} \hat{A} = -F(X)He,$$

$$D_t^{\alpha(t)} \hat{B} = G(Y)e.$$

Then the response variable-order fractional chaotic/hyperchaotic system of (3.1) *will asymptotically synchronize the drive variable-order fractional chaotic/hyperchaotic system as FSHPS.*

Proof. Define the error vector as $e = Y - HX$. Then the error between the drive variable-order fractional system and the response variable-order fractional system in (3.1) is acquired based on the Definition 2.3 as following:

$$\begin{aligned} D_t^{\alpha(t)} e &= D_t^{\alpha(t)}(Y - HX) \\ &= D_t^{\alpha(t)} Y - HD_t^{\alpha(t)} X \\ &= g(Y) + G(Y)B + u + d_2 - H(f(X) + F(X)A + d_1) \\ &= g(y_1, y_2, \dots, y_n) + G(y_1, y_2, \dots, y_n)B \\ &\quad + d_2 - Hf(x_1, x_2, \dots, x_n) - HF(x_1, x_2, \dots, x_n)A - Hd_1 + u. \end{aligned} \tag{3.3}$$

After substituting (3.2) into (3.3), the error system is obtained as follows:

$$D_t^{\alpha(t)} e = -ke - \delta + d_2 - Hd_1 - HF(X)(A - \hat{A}) + G(Y)(B - \hat{B}). \tag{3.4}$$

Define a Lyapunov candidate as

$$V = \frac{1}{2}[e^T e + (\hat{A} - A)^T(\hat{A} - A) + (\hat{B} - B)^T(\hat{B} - B)].$$

Then the fractional-order time derivative of V is satisfied as follows based on the results of Ref. [22]:

$$D_t^{\alpha(t)} V(t) \le e^T(D_t^{\alpha(t)} e) + (\hat{A} - A)^T(D_t^{\alpha(t)} \hat{A}) + (\hat{B} - B)^T(D_t^{\alpha(t)} \hat{B}). \tag{3.5}$$

After inserting Eq. (3.4) into Eq. (3.5), we obtain the following result:

$$D_t^{\alpha(t)} V(t) \le -ke^T e + (d_2 - Hd_1)^T e - \delta e. \tag{3.6}$$

When there exist the following conditions that

$$|d_{2i}| \le \rho_{2i},$$

$$|h_i d_{1i}| \le \rho_{1i},$$

we get that

$$|d_{2i} - h_i d_{1i}| \le |d_{2i}| + |h_i d_{1i}| \le \rho_{2i} + \rho_{1i}.$$

Because

$$\delta_i + \tfrac{\xi}{2} = \tfrac{1}{2\xi}[(\rho_{1i} + \rho_{2i})^2 + \tfrac{\xi}{2} \ge \rho_{1i} + \rho_{2i} \ ,$$

then

$$-(\delta_i + \tfrac{\xi}{2}) \leq \rho_{1i} + \rho_{2i}.$$

So we have

$$\begin{array}{ll} D_t^{\alpha(t)}V(t) & < -ke^T e + (\rho_1 + \rho_2)^T e - \sum_{l=1}^{n}(\rho_{1l} + \rho_{2l})e + n\xi \\ & < -ke^T e + n\xi. \end{array} \tag{3.7}$$

When we take $k > n\xi/\phi$, where $e^T e = \phi$, then $D_t^{\alpha(t)}V(t) < 0$ for (3.7). So the error dynamical system (3.3) will be asymptotically stabilized to nearly the zero equilibrium, which means that the response variable-order fractional chaotic/hyperchaotic system will asymptotically synchronize the drive variable-order fractional chaotic/hyperchaotic system in sense of FSHPS. This completes the proof. □

Corollary 3.2. *If the structures of general variable-order fractional chaotic systems without external disturbances are identical, i.e., $F(X) = G(Y)$ and $f(X) = g(Y)$, then the controller vector is designed as*

$$\begin{array}{ll} u & = [HF(X) - F(Y)]\hat{A} + Hf(X) - f(Y) - ke - \delta \\ & = [HF(x_1, x_2, \ldots, x_n) - F(y_1, y_2, \ldots, y_n)]\hat{A} + Hf(x_1, x_2, \ldots, x_n) \\ & \quad -f(y_1, y_2, \ldots, y_n) - ke - \delta, \end{array} \tag{3.8}$$

where $k > 0$ is a constant and the compensator δ is taken as

$$\delta = \frac{1}{2\xi}[(\rho_{11} + \rho_{21})^2 e_1, (\rho_{12} + \rho_{22})^2 e_2, \ldots, (\rho_{1n} + \rho_{2n})^2 e_n]^T.$$

Thus, the FSHPS of two identical general uncertain chaotic fractional systems is also achieved with an arbitrarily small error bound.

Next, we take some famous variable-order fractional chaotic systems as examples to verify the presented method in Sect. 3.

4. Applications

In order to illustrate the effectiveness of the synchronization effects, we consider two variable-order fractional chaotic/hyperchaotic systems with numerical simulations using Matlab.

4.1. FSHPS of two different variable-order fractional chaotic systems

In this subsection, we consider the synchronization between variable-order fractional Chen and L*ü* system with nonlinear external disturbances and unknown parameters as an example. The variable-order fractional Chen system is defined as follows

$$\begin{cases} D_t^{\alpha(t)}x_1 = a_1(x_2 - x_1); \\ D_t^{\alpha(t)}x_2 = (a_3 - a_1)x_1 + a_3 x_2 - x_1 x_3; \\ D_t^{\alpha(t)}x_3 = x_1 x_2 - a_2 x_3, \end{cases} \tag{4.1}$$

where x_1, x_2 and x_3 are state variables, and a_1, a_2, a_3 are system parameters.

The variable-order fractional L*ü* system is expressed by the following form:

$$\begin{cases} D_t^{\alpha(t)}y_1 = b_1(y_2 - y_1); \\ D_t^{\alpha(t)}y_2 = b_3 y_2 - y_1 y_3; \\ D_t^{\alpha(t)}y_3 = y_1 y_2 - b_2 y_3, \end{cases} \tag{4.2}$$

where y_1, y_2 and y_3 are state variables, and b_1, b_2, b_3 are system parameters.

The drive system of the variable-order fractional Chen system with nonlinear external disturbances is written as follows:

$$\begin{cases} D_t^{\alpha(t)}x_1 = a_1(x_2 - x_1) + d_{11}; \\ D_t^{\alpha(t)}x_2 = (a_3 - a_1)x_1 + a_3x_2 - x_1x_3 + d_{12}; \\ D_t^{\alpha(t)}x_3 = x_1x_2 - a_2x_3 + d_{13}, \end{cases} \tag{4.3}$$

where $d_1 = [d_{11}, d_{12}, d_{13}]^T$ responses nonlinear external disturbances.

The response system of the variable-order fractional L$\ddot{u}$ system with nonlinear external disturbances is described as follows:

$$\begin{cases} D_t^{\alpha(t)}y_1 = b_1(y_2 - y_1) + d_{21} + u_1; \\ D_t^{\alpha(t)}y_2 = b_3y_2 - y_1y_3 + d_{22} + u_2; \\ D_t^{\alpha(t)}y_3 = y_1y_2 - b_2y_3 + d_{23} + u_3, \end{cases} \tag{4.4}$$

where $d_2 = [d_{21}, d_{22}, d_{23}]^T$ is nonlinear external disturbances and $u = [u_1, u_2, u_3]^T$ stands for the controller.

Then, according to Theorem 3.1, we choose the following controller for variable-order fractional Chen and L$\ddot{u}$ systems:

$$\begin{cases} u_1 = h_1(x_2 - x_1)\hat{a_1} - (y_2 - y_1)\hat{b_1} - ke_1 - \frac{1}{2\xi}(\rho_{11} + \rho_{21})^2; \\ u_2 = -h_2x_3x_1 - h_2x_1\hat{a_1} + h_2(x_1 + x_2)\hat{a_3} + y_3y_1 - y_2\hat{b_3} - ke_2 - \frac{1}{2\xi}(\rho_{12} + \rho_{22})^2; \\ u_3 = -h_3x_3\hat{a_2} - y_2y_1 + y_3\hat{b_2} + h_2x_1x_2 - ke_3 - \frac{1}{2\xi}(\rho_{13} + \rho_{23})^2, \end{cases} \tag{4.5}$$

where

$$\begin{cases} D_t^{\alpha(t)}\hat{a_1} = h_1(x_1 - x_2)e_1; \\ D_t^{\alpha(t)}\hat{a_2} = h_1x_1e_1 - h_3(x_1 + x_2)e_3; \\ D_t^{\alpha(t)}\hat{a_3} = h_2x_3e_2, \end{cases}$$

$$\begin{cases} D_t^{\alpha(t)}\hat{b_1} = (y_2 - y_1)e_1; \\ D_t^{\alpha(t)}\hat{b_2} = y_2e_3; \\ D_t^{\alpha(t)}\hat{b_3} = -y_3e_2. \end{cases}$$

And $e = (e_1, e_2, e_3)$ is the error defined as following:

$$e = Y - HX.$$

Hence, the two variable-order fractional chaotic systems will be synchronized in sense of FSHPS asymptotically according to different relationships shown in Sect. 3.

In the simulation, parameters for variable-order fractional Chen system are chosen as $\alpha(t) = 0.09sin(1/t)+0.9$, $a_1 = 32$, $a_2 = 5$, $a_3 = 24$, $d_1 = [0.1, 0.1, 0.1]^T$, the initial value is taken as $x_1 = -1$, $x_2 = 3$, $x_3 = -4$ and the iteration number is 20000. As for the variable-order fractional L$\ddot{u}$ system, we take $\alpha(t) = 0.09sin(1/t) + 0.9$, $b_1 = 29$, $b_2 = 21$, $b_3 = 2$, $d_2 = [0.1, 0.1, 0.1]^T$, the initial value is taken as $y_1 = 5$, $y_2 = 2$, $y_3 = 0.5$ and the iteration number is 20000. So the variable-order fractional Chen and L$\ddot{u}$ systems have chaotic attractors respectively as shown in Figs. 1 and 2. Fig. 3 shows the FSHPS between the variable-order fractional drive-response system and it is clear that the trajectories of the response system synchronize asymptotically those of the drive system with controller.

In order to see the effectiveness of this synchronization method, we take controller as shown in Sect. 3 and the nonlinear external disturbances are chosen as $\alpha(t) = 0.09sin(1/t) + 0.9$, $e_1 = 5.02$, $e_2 = 1.82$, $e_3 = 0.82$, , $k = 2$, $\xi = 0.1$, $d_{11} = 0.3e^{2/n}$, $d_{12} = 0.2cos(1.5\pi/n) + 0.5sin(0.25\pi/n)$, $d_{13} = 0.2/cos(6.5\pi/n)$, $\rho_{11} = 0.5$, $\rho_{12} = 0.8$, $\rho_{13} = 0.2$, and $\rho_{21} = 0.7$, $\rho_{22} = 0.8$, $\rho_{23} = 0.8$, $\hat{a_1} = 31.8$, $\hat{a_2} = 4.9$, $\hat{a_3} = 23.8$, $\hat{b_1} = 28.8$, $\hat{b_2} = 20.7$, $\hat{b_3} = 1.8$. We assume that the controllers activate at $t = 0$s. The simulation results are shown in Fig. 4. Fig. 4 shows that each error signal converges to nearly zero quickly by the controller. Fig 5 illustrates the estimate of parameters for variable-order fractional Chen and L$\ddot{u}$ systems, respectively.

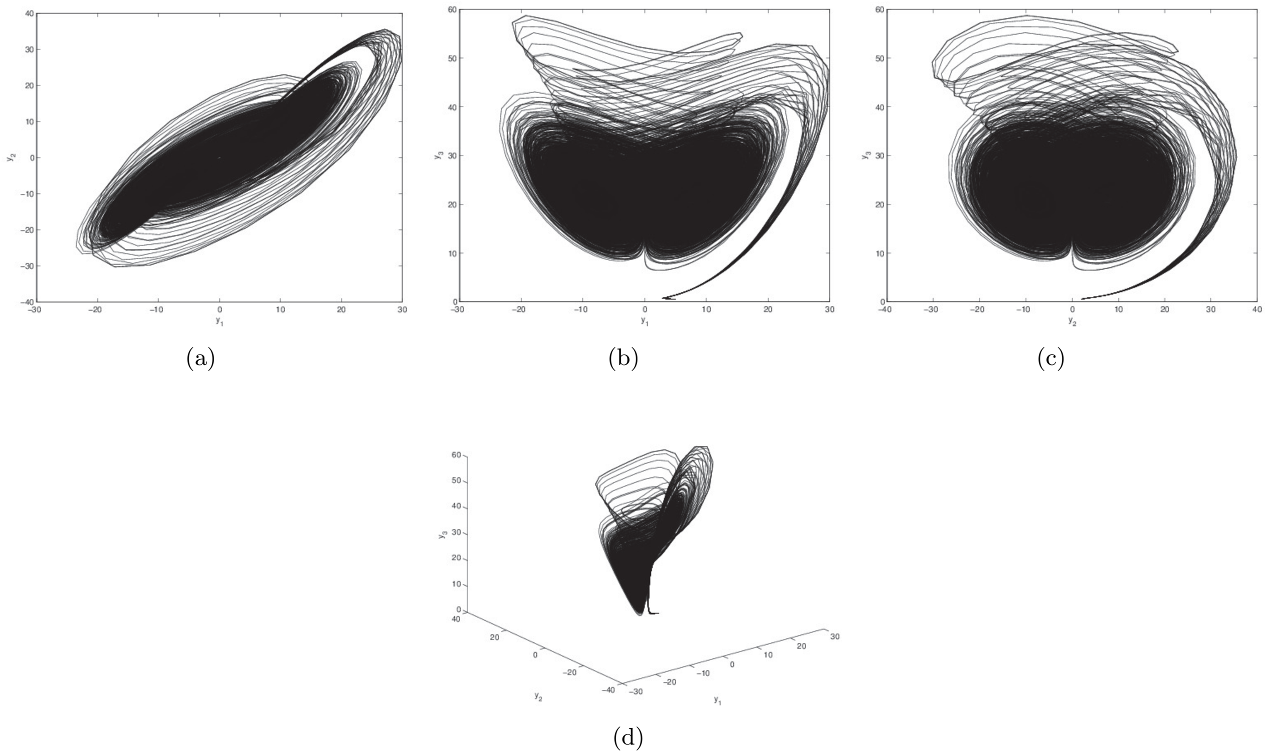


Figure 1: Chaotic behavior and attractors of the variable-order fractional Chen system (4.3) in different planes and projections.

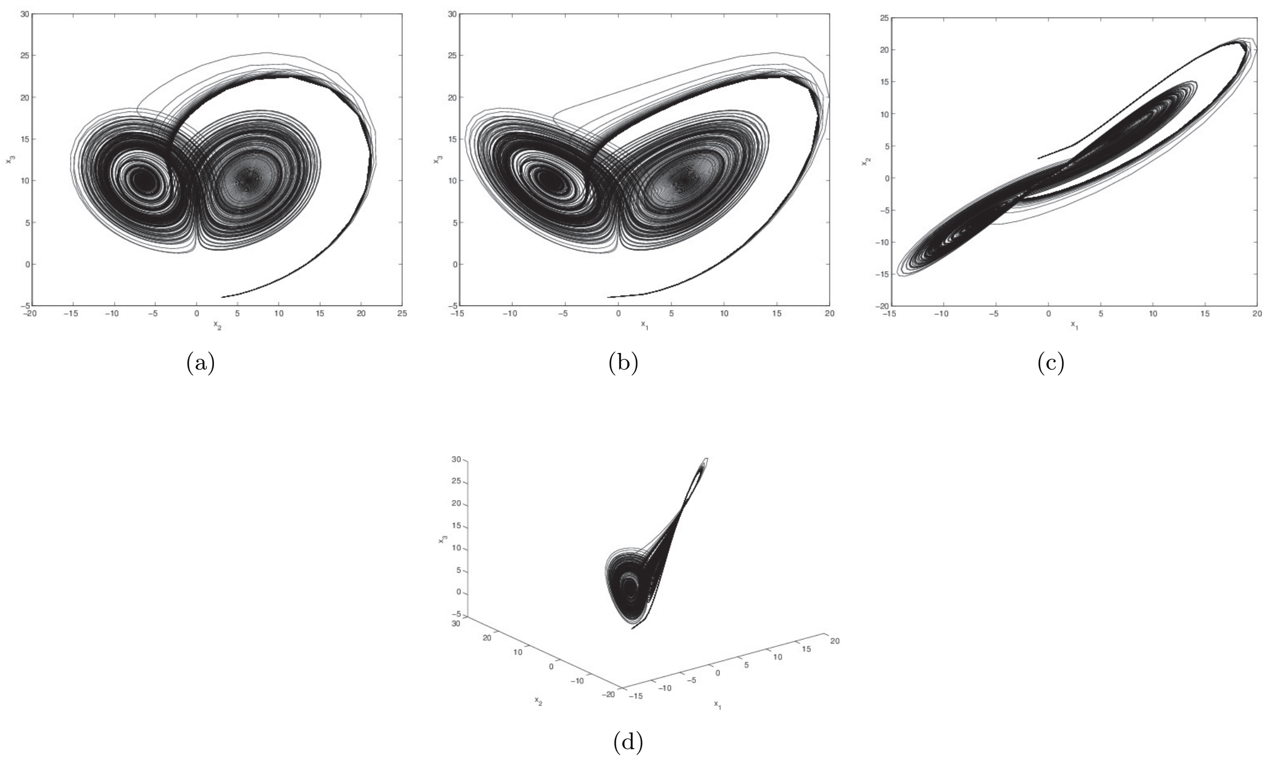
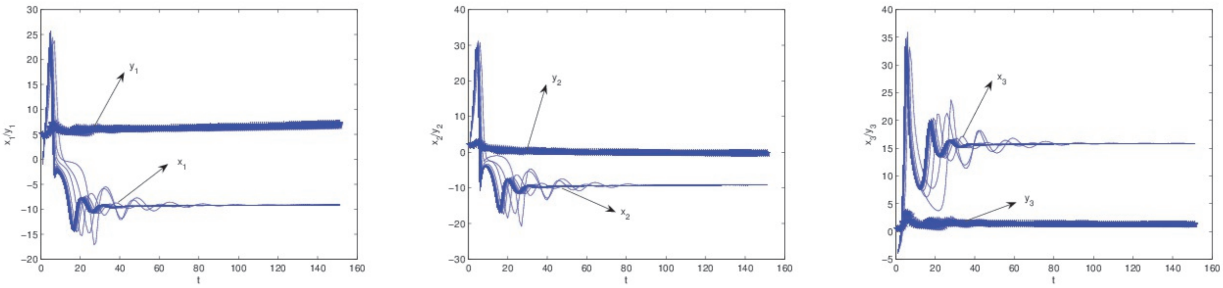
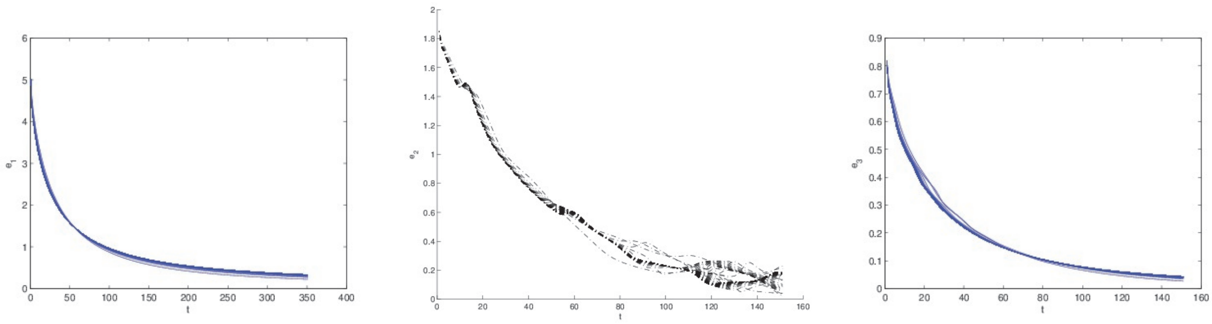


Figure 2: Chaotic behavior and attractors of the variable-order fractional Lü system (4.4) in different planes and projections.



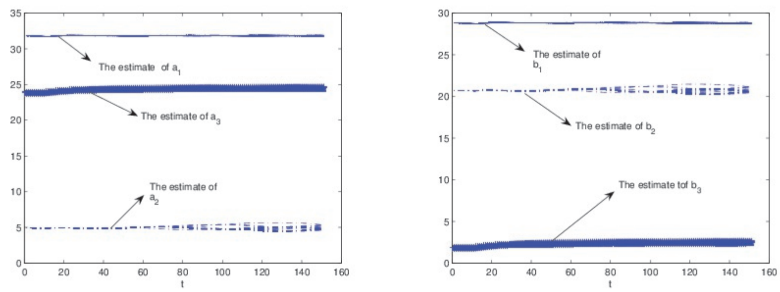
(a) (b) (c)

Figure 3: Chaotic FSHP of variable-order fractional Chen system (4.3) and L$\ddot{u}$ system (4.4).



(a) (b) (c)

Figure 4: Variation of errors for variable-order fractional Chen system (4.3) and L$\ddot{u}$ system (4.4) after using the controller in the sense of FSHPS.



(a) (b)

Figure 5: The estimate of uncertain parameters a_1, a_2, a_3 and b_1, b_2, b_3 in variable-order fractional Chen system (4.3) and L$\ddot{u}$ (4.4) system.

4.2. FSHPS of two identical variable-order fractional hyperchaotic Lorenz systems with nonlinear external disturbances

In order to observe the FSHPS behavior between two identical variable-order fractional hyperchaotic Lorenz systems with nonlinear external disturbances, the drive and response systems are defined as follows:

$$\begin{cases} D_t^{\alpha(t)} z_1 = a_1(z_2 - z_1) + z_4 + d_{11}; \\ D_t^{\alpha(t)} z_2 = a_2 z_1 - z_2 - z_1 z_3 - z_4 + d_{12}; \\ D_t^{\alpha(t)} z_3 = -a_3 z_3 + z(1) z_2 + d_{13}; \\ D_t^{\alpha(t)} z_4 = -a_4 z_4 + z(1) z_2 + d_{14}, \end{cases} \tag{4.6}$$

where $z_1,\ z_2,\ z_3,\ z_4$ are fractional state variables; $a_1,\ a_2,\ a_3,\ a_4$ mean system parameters and $d_1 = [d_{11}, d_{12}, d_{13}, d_{14}]^T$ represents nonlinear external disturbances;

$$\begin{cases} D_t^{\alpha(t)} w(1) = a_1(w(2) - w(1)) + w_4 + d_{21} + u_1; \\ D_t^{\alpha(t)} w(2) = a_2 w_1 - w_2 - w_1 w_3 - w_4 + d_{22} + u_2; \\ D_t^{\alpha(t)} w(3) = -a_3 w_3 + 1/2(z(1) w_2 + w_1 z_2) + d_{23} + u_3; \\ D_t^{\alpha(t)} w(4) = -a_4 w_4 + 1/2(z(1) w_2 + w_1 z_2) + d_{24} + u_4, \end{cases} \tag{4.7}$$

where $d_2 = [d_{21}, d_{22}, d_{23}, d_{24}]^T$ is nonlinear external disturbance and $u = [u_1, u_2, u_3, u_4]^T$ is the controller based on Theorem 3.1. As the results in Theorem 3.1, the controller $u = [HF(z) - F(w)]\hat{A} + Hf(z) - f(w) - ke - \delta$ is described as

$$u = \begin{pmatrix} [h_1(z_2 - z_1) - (w_2 - w_1)]\hat{a_1} + (h_1 z_4 - w_4) - ke_1 - \frac{1}{2\xi}(\rho_{11} + \rho_{21})^2 \\ (h_2 z_1 - w_1)\hat{a_2} - (z_2 + z_1 z_3 - z_4)h_2 + w_2 + w_1 w_3 - w_4 - ke_2 - \frac{1}{2\xi}(\rho_{12} + \rho_{22})^2 \\ (w_3 - h_3 z_3)\hat{a_3} + h_3 z_1 z_2 + w_1 w_2 - ke_3 - \frac{1}{2\xi}(\rho_{13} + \rho_{23})^2 \\ (w_4 - h_4 z_4)\hat{a_4} + h_4 z_1 z_2 + w_1 w_2 - ke_4 - \frac{1}{2\xi}(\rho_{14} + \rho_{24})^2, \end{pmatrix}$$

where

$$D_t^{\alpha(t)} \hat{A} = \begin{cases} D_t^{\alpha(t)} \hat{a_1} = [y_2 - y_1 - h_1(x_2 - x_1)]e_1; \\ D_t^{\alpha(t)} \hat{a_2} = (y_1 - h_2 x_1)e_2; \\ D_t^{\alpha(t)} \hat{a_3} = (-y_3 + h_3 x_3)e_3; \\ D_t^{\alpha(t)} \hat{a_4} = (-y_4 + h_4 x_4)e_4. \end{cases}$$

In the simulations, the values are selected as $\alpha = 0.09cos(t) + 0.9$, $z_1 = 0.2$, $z_2 = 0.6$, $z_3 = 0.3$, $z_4 = 1$, $w_1 = -0.8$, $w_2 = 0.3$, $w_3 = 0.5$, $w_4 = 1.1$, $d_{11} = sin(0.25\pi t)$, $d_{12} = 0.3sin(\pi t)$, $d_{13} = 0.2sin(0.5\pi t)$, $d_{14} = 0.5cos(0.5\pi t$, $d_{21} = 0.2cos(1.5\pi t)$, $d_{22} = 0.5sin(0.5\pi t)$, $d_{23} = 0.8sin(1.5\pi t)$, $d_{24} = 0.3cos(0.3\pi t)$, $\rho_{11} = 0.5$, $\rho_{12} = 0.8$ $\rho_{13} = 0.2$, $\rho_{14} = 0.25$, $\rho_{21} = 0.7$, $\rho_{22} = 0.8$, $\rho_{23} = 0.8$, $\rho_{24} = 0.3$, $\hat{a_1} = 13$, $\hat{a_2} = 37$, $\hat{a_3} = 5.01$, and $\hat{a_4} = 12.5$. Fig. 6 shows the attractor of the variable-order fractional hyperchaotic Lorenz system. The controller is assumed to activate at $t = 0$s. Fig 7. illustrates the FHSPS between the response-drive variable-order fractional systems. The effectiveness of the errors in sense of FSHPS is shown clearly in Fig. 8, which illustrates that each error signal converges to nearly zero asymptotically and quickly.

Fig. 9 illustrates the estimate of parameters for variable-order fractional hyperchaotic Lorenz system.

Remark 4.1. Based on the definitions of the complete synchronization, antisynchronization and partial synchronization, projective synchronization, it is easy to prove that these three synchronizations are the special cases of FSHPS.

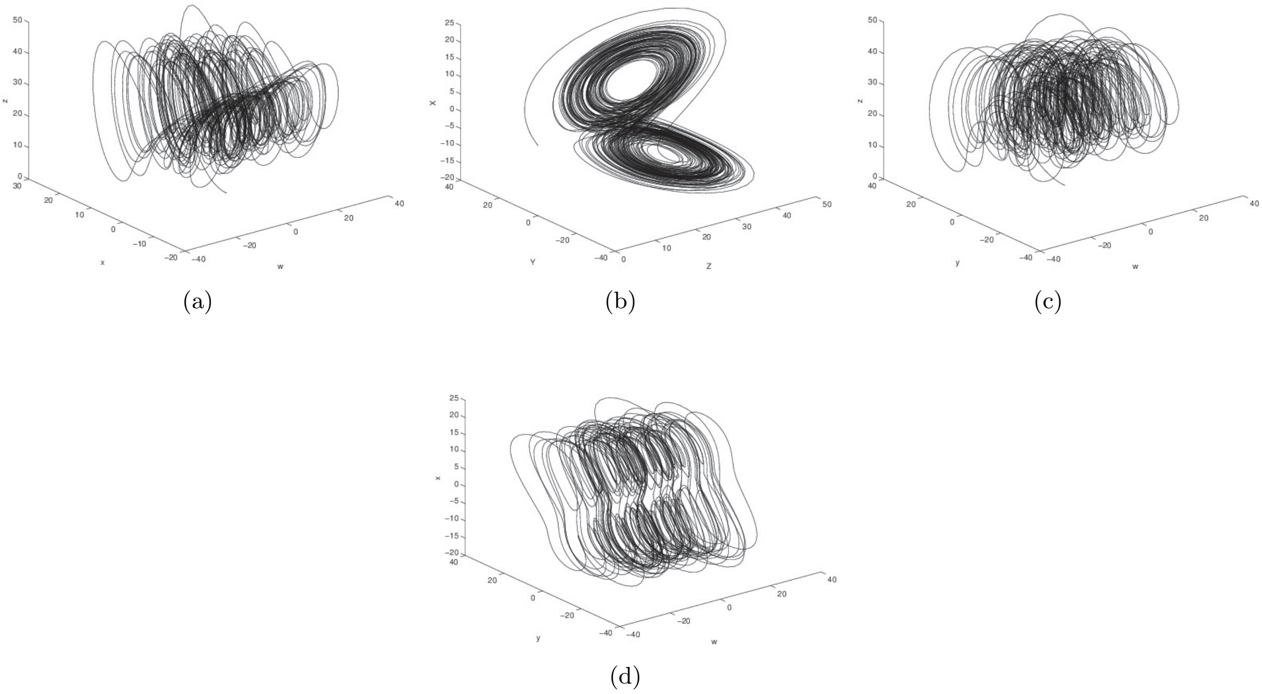


Figure 6: Attractors and hyperchaotic behavior of the variable-order fractional hyperchaotic Lorenz system (4.6) in different phase planes and projections.

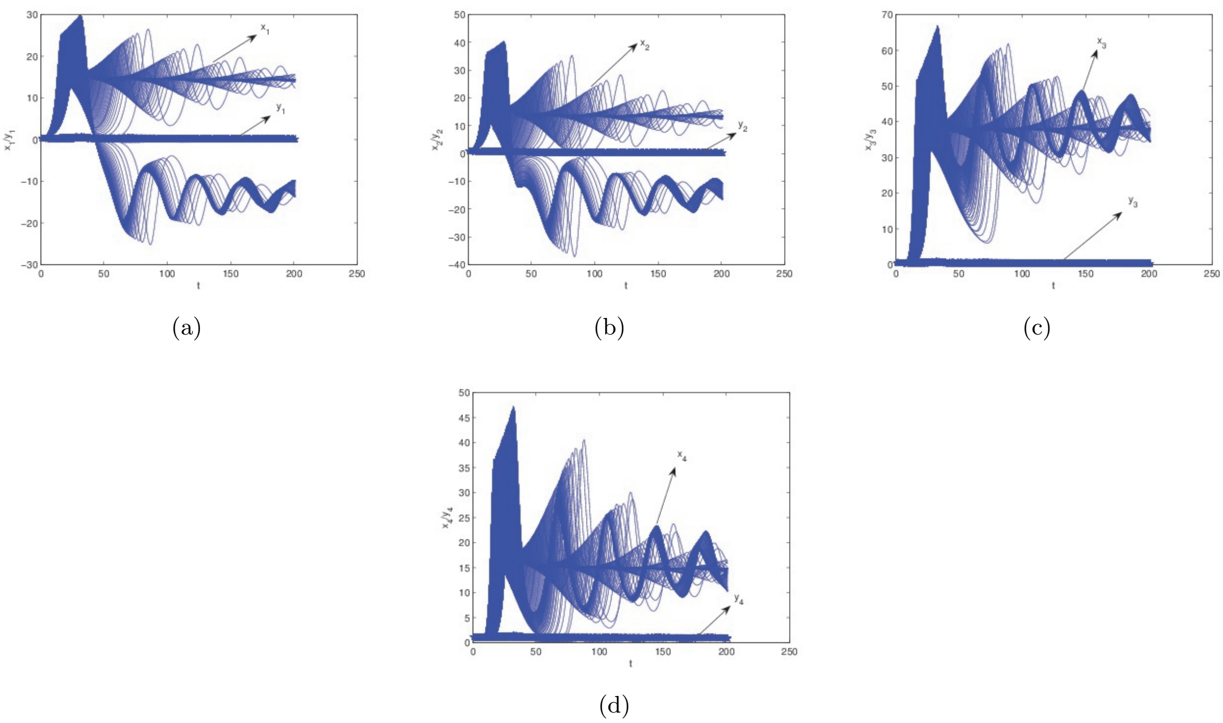
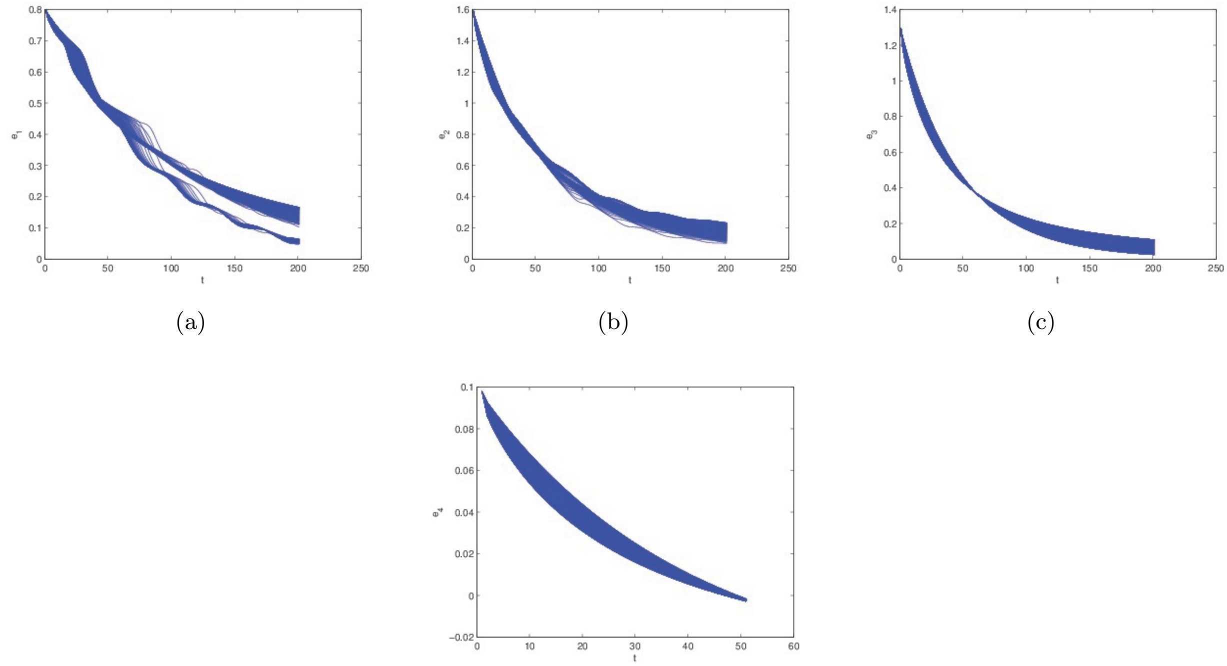


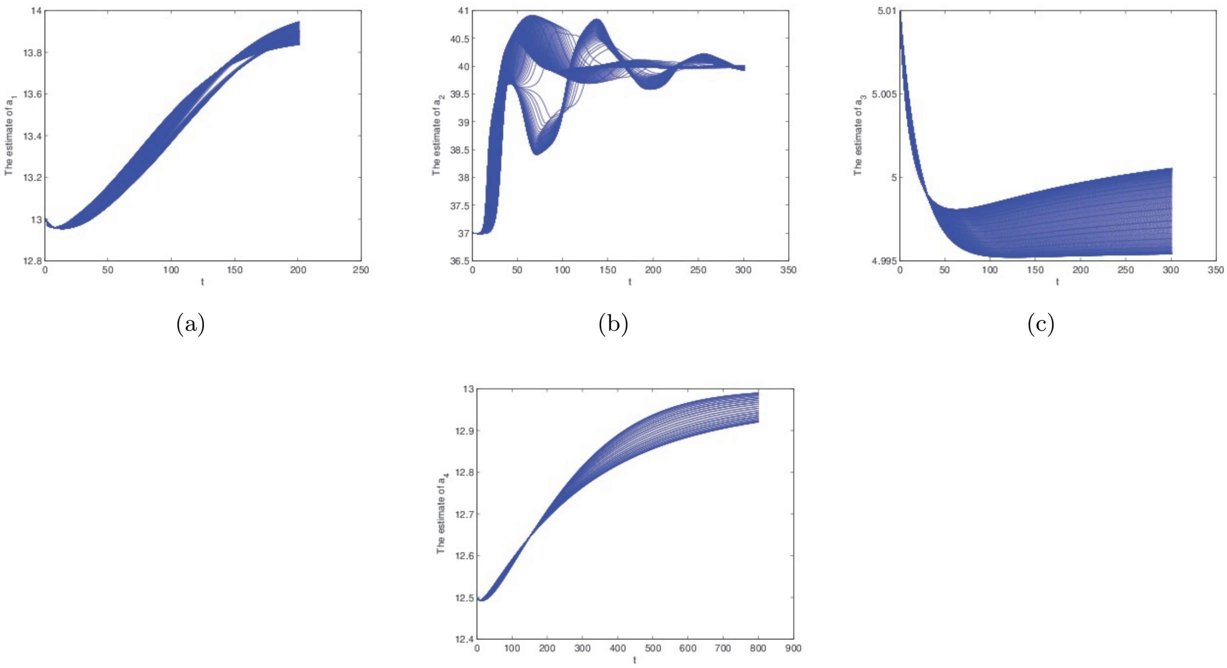
Figure 7: Hyperchaotic FSHPS of the variable-order fractional hyperchaotic Lorenz system (4.6) with the controller (4.5) and the estimation parameters (4.2).



(a) (b) (c)

(d)

Figure 8: Variation of error between the drive-response variable-order fractional hyperchaotic Lorenz system (4.6).



(a) (b) (c)

(d)

Figure 9: The estimate of parameters a_1, a_2, a_3, a_4 of variable-order fractional hyperchaotic Lorenz system (4.6).

5. Conclusions

The synchronization for a class of variable-order fractional chaotic/hyperchaotic systems with nonlinear external disturbances and unknown parameters was studied in the sense of FSHPS. Based on adaptive control and Mittag-Leffler stable theory, the controller was developed by introducing a compensator and it can eliminate the influence of nonlinear external disturbances effectively. Furthermore, the FSHPS of the two different variable-order fractional chaotic/hyperchaotic systems was realized by the controller asymptotically. Meanwhile, the estimate of parameters for variable-order fractional chaotic systems was obtained. Finally, the variable-order fractional chaotic Chen and L*ü* systems and the fractional hyperchaotic Lorenz system were taken as two examples to verify the effectiveness of our proposed control scheme.

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