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Some new coupled fixed point theorems in partially ordered complete probabilistic metric spaces

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Abstract

In this paper, we weaken the notion of Ψ of Luong and Thuan, [V. N. Luong, N. X. Thuan, Nonlinear Anal., **74** (2011), 983–992] and prove some new coupled coincidences and coupled common fixed point theorems for mappings having a mixed g-monotone property in partially ordered complete probabilistic metric spaces. As an application, we discuss the existence and uniqueness for a solution of a nonlinear integral equation.

Keywords: Coupled fixed point, mixed g-monotone mappings, partially ordered, probabilistic metric space.
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1. Introduction

As a generalization of a metric space, the concept of a probabilistic metric space has been introduced by Menger [14],[16]. Fixed point theory in a probabilistic metric space is an important branch of probabilistic analysis, and many results on the existence of fixed points or solutions of nonlinear equations under various types of conditions in Menger PM-spaces have been extensively studied by many scholars (see *e.g.* [18],[19]).

In 2006, Bhaskar and Lakshmikantham [8] introduced the concept of a mixed monotone mapping and proved coupled coincidence and coupled common fixed point theorems in partially ordered complete metric spaces. After that, in 2009, Lakshmikantham and Ćirić [12] introduced the concept of a mixed g-monotone mapping, which is a generalization of a mixed monotone mapping. Their results extend the results of [8]. On

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the other hand, Choudhury and Das [2] gave a generalized unique fixed point theorem by using an altering distance function, which was originally introduced by Khan et al. [10]. For other results in this direction, we refer to [4], [5], [7], [9], [15], [17]. In 2009, Dutta et al. [6] gave some fixed point results in Menger spaces using a control function. Moreover, Kutbi and Gopal et al. [11] established some fixed point theorems by revisting the notion of ψ-contractive mapping in Menger PM-spaces.

In this paper, we combine the results of [11] and [13], weaken the notion of Ψ in [1] and establish some new coupled coincidences and coupled common fixed point theorems for mappings having a mixed g-monotone property in partially ordered complete probabilistic metric spaces, by using an altering distance function. Finally, we discuss the existence and uniqueness for a solution of a nonlinear integral equation, as an application to our main results.

2. Preliminaries

Throughout this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{Z}^+$ be the set of all positive integers.

A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is nondecreasing left-continuous with $\sup_{t\in\mathbb{R}} F(t) = 1$ and $\inf_{t\in\mathbb{R}} F(t) = 0$.

We shall denote by $\mathcal{D}$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied:

(1) $\Delta(a, 1) = a$;
(2) $\Delta(a, b) = \Delta(b, a)$;
(3) $a \geq b, c \geq d \Rightarrow \Delta(a, c) \geq \Delta(b, d)$;
(4) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

A typical example of t-norm is Δ_m, where $\Delta_m(a, b) = \min\{a, b\}$, for each $a, b \in [0, 1]$.

Definition 2.1 ([1]). A triplet $(X, \mathcal{F}, \Delta)$ is called a Menger probabilistic metric space (for short, a Menger PM-space) if X is a nonempty set, Δ is a t-norm and $\mathcal{F}$ is a mapping from $X \times X$ into $\mathcal{F}$ satisfying the following conditions (we denote $\mathcal{F}(x, y)$ by $F_{x,y}$):
(MS-1) $F_{x,y}(t) = H(t)$ for all $t \in R$ if and only if $x = y$;
(MS-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in R$;
(MS-3) $F_{x,y}(t + s) \geq \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

Remark 2.2. In [1], it is pointed out that if $(X, \mathcal{F}, \Delta)$ satisfies the condition $\sup_{0<t<1} \Delta(t, t) = 1$, then $(X, \mathcal{F}, \Delta)$ is a Hausdorff topological space in the (ϵ, λ)-topology $\mathcal{T}$, *i.e.*, the family of sets $\{U_x(\epsilon, \lambda) : \epsilon > 0, \lambda \in (0, 1]\}(x \in X)$ is a basis of neighborhoods of a point x for $\mathcal{T}$, where $U_x(\epsilon, \lambda) = \{y \in X : F_{x,y}(\epsilon) > 1 - \lambda)\}$.

By virtue of this topology $\mathcal{T}$, a sequence $\{x_n\}$ is said to be $\mathcal{T}$-convergent to $x \in X$(we write $x_n \xrightarrow{\mathcal{T}} x$) if for any given $\epsilon > 0$ and $\lambda \in (0, 1]$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$, which is equivalent to $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a $\mathcal{T}$-*Cauchy* sequence in $(X, \mathcal{F}, \Delta)$ if for any given $\epsilon > 0$ and $\lambda \in (0, 1]$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geq N$; $(X, \mathcal{F}, \Delta)$ is said to be $\mathcal{T}$-complete if each $\mathcal{T}$-*Cauchy* sequence in X is $\mathcal{T}$-convergent in X. Note that in a Menger PM-space, when we write $\lim_{n\to\infty} x_n = x$, it means that $x_n \xrightarrow{\mathcal{T}} x$.

Definition 2.3 ([3]). Let Φ denotes the class of all functions $\phi : R^+ \to R^+$ satisfying the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$;
(ii) $\phi(t)$ is strictly increasing and $\phi(t) \to \infty$ as $t \to \infty$;
(iii) ϕ is left continuous in $(0, \infty)$;
(iv) ϕ is continuous at 0.

Remark 2.4. Φ-functions play the role of altering distance functions in probabilistic metric spaces.

Definition 2.5. Let Ψ denote the class of all functions $\psi : R^+ \to R^+$ satisfying the following conditions:
(i) ψ is non-decreasing;
(ii) $\psi(t + s) \leq \psi(t) + \psi(s)$ for all $t, s \in [0, 1)$.

Remark 2.6. In [13], Ψ also satisfy: Ψ is continuous and $\Psi(t) = 0$ if and only if $t = 0$. It is obvious that Definition 2.3 is weaker than the notion of Ψ in [13].

We recall the definition of mixed monotone property and mixed g-monotone property.

Definition 2.7 ([8])**.** Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_2) \leq F(x, y_1).$$

Definition 2.8 ([12])**.** Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad g(y_1) \leq g(y_2) \Rightarrow F(x, y_2) \leq F(x, y_1).$$

3. Main results

In this section, we will give some coupled fixed point theorems in partially ordered complete probabilistic metric spaces.

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and (X, F, Δ) be a complete Menger PM-space with a continuous t-norm. Assume $T : X \times X \to X$ and $g : X \to X$ are mappings such that T has the mixed g-monotone property and*

$$\psi(\frac{1}{F_{T(x,y),T(u,v)}(\phi(ct))} - 1) \leq \frac{1}{2}\psi(\frac{1}{F_{g(x),g(u)}(\phi(t))} - 1 + \frac{1}{F_{g(y),g(v)}(\phi(t))} - 1) \tag{3.1}$$

for all $t > 0$ and $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$, where $c \in (0, 1)$, $\psi \in \Psi$ and $\phi \in \Phi$ such that $F_{g(x),g(u)}(\phi(t)) > 0$ and $F_{g(y),g(v)}(\phi(t)) > 0$. Suppose $T(X \times X) \subseteq g(X)$, g is continuous and commutes with T and also suppose either
(a) T is continuous or
(b) X has the following property:
(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,
(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y_n \leq y$ for all n.
If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq T(x_0, y_0) \quad \text{and} \quad g(y_0) \geq T(y_0, x_0),$$

then there exist $x, y \in X$ such that

$$g(x) = T(x, y) \quad \text{and} \quad g(y) = T(y, x),$$

that is, T and g have a coupled coincidence point.

Proof. Let $x_0, y_0 \in X$ such that $g(x_0) \leq T(x_0, y_0)$ and $g(y_0) \geq T(y_0, x_0)$. Since $T(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = T(x_0, y_0)$ and $g(y_1) = T(y_0, x_0)$. Continuing this process we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = T(x_n, y_n) \quad \text{and} \quad g(y_{n+1}) = T(y_n, x_n) \quad \text{for all } n \geq 0. \tag{3.2}$$

We shall show that

$$g(x_n) \leq g(x_{n+1}) \quad \text{for all } n \geq 0, \tag{3.3}$$

$$g(y_n) \geq g(y_{n+1}) \quad \text{for all } n \geq 0. \tag{3.4}$$

We shall use the mathematical induction. Let $n = 0$. Since $g(x_0) \leq T(x_0, y_0)$, $g(y_0) \geq T(y_0, x_0)$, and as $g(x_1) = T(x_0, y_0)$, $g(y_1) = T(y_0, x_0)$, we have $g(x_0) \leq g(x_1)$, $g(y_0) \geq g(y_1)$. Thus (3.3) and (3.4) hold for $n = 0$.

Suppose now that (3.3) and (3.4) hold for some $n \geq 0$. Since $g(x_n) \leq g(x_{n+1})$, $g(y_n) \geq g(y_{n+1})$ and T has the mixed g-monotone property, we have

$$g(x_{n+2}) = T(x_{n+1}, y_{n+1}) \geq T(x_n, y_{n+1}) \geq T(x_n, y_n) = g(x_{n+1}),$$

$$g(y_{n+2}) = T(y_{n+1}, x_{n+1}) \leq T(y_n, x_{n+1}) \leq T(y_n, x_n) = g(y_{n+1}).$$

Thus by the mathematical induction we conclude that (3.3) and (3.4) hold for all $n \geq 0$. Therefore,

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq \cdots \leq g(x_n) \leq g(x_{n+1}) \leq \cdots, \tag{3.5}$$

$$g(y_0) \geq g(y_1) \geq g(y_2) \geq \cdots \geq g(y_n) \geq g(y_{n+1}) \leq \cdots. \tag{3.6}$$

In view of the fact that $\sup_{t\in\mathbb{R}} F_{g(x_1),g(x_0)}(t) = 1$ and $\sup_{t\in\mathbb{R}} F_{g(y_1),g(y_0)}(t) = 1$, and by (ii) of Definition 2.3, one can find $t > 0$, such that

$$F_{g(x_1),g(x_0)}(\phi(t)) > 0 \text{ and } F_{g(y_1),g(y_0)}(\phi(t)) > 0$$

for $g(x_0) \leq g(x_1)$ and $g(y_0) \geq g(y_1)$, which implies that $F_{g(x_1),g(x_0)}(\phi(\frac{t}{c})) > 0$ and $F_{g(y_1),g(y_0)}(\phi(\frac{t}{c})) > 0$, then (3.1) gives that

$$\begin{aligned}\psi(\frac{1}{F_{g(x_2),g(x_1)}(\phi(t))} - 1) &= \psi(\frac{1}{F_{T(x_1,y_1),T(x_0,y_0)}(\phi(t))} - 1)\\ &\leq \frac{1}{2}\psi(\frac{1}{F_{g(x_1),g(x_0)}(\phi(\frac{t}{c}))} - 1 + \frac{1}{F_{g(y_1),g(y_0)}(\phi(\frac{t}{c}))} - 1),\end{aligned} \tag{3.7}$$

$$\psi(\frac{1}{F_{g(y_2),g(y_1)}(\phi(t))} - 1) \leq \frac{1}{2}\psi(\frac{1}{F_{g(y_1),g(y_0)}(\phi(\frac{t}{c}))} - 1 + \frac{1}{F_{g(x_1),g(x_0)}(\phi(\frac{t}{c}))} - 1). \tag{3.8}$$

From (3.7) and (3.8), we have

$$\psi(\frac{1}{F_{g(x_2),g(x_1)}(\phi(t))} - 1) + \psi(\frac{1}{F_{g(y_2),g(y_1)}(\phi(t))} - 1) \leq \psi(\frac{1}{F_{g(x_1),g(x_0)}(\phi(\frac{t}{c}))} - 1 + \frac{1}{F_{g(y_1),g(y_0)}(\phi(\frac{t}{c}))} - 1).$$

By (ii) of Definition 2.5, we have

$$\psi(\frac{1}{F_{g(x_2),g(x_1)}(\phi(t))} - 1 + \frac{1}{F_{g(y_2),g(y_1)}(\phi(t))} - 1) \leq \psi(\frac{1}{F_{g(x_2),g(x_1)}(\phi(t))} - 1) + \psi(\frac{1}{F_{g(y_2),g(y_1)}(\phi(t))} - 1),$$

which implies that

$$\psi(\frac{1}{F_{g(x_2),g(x_1)}(\phi(t))}-1+\frac{1}{F_{g(y_2),g(y_1)}(\phi(t))}-1)\leq\psi(\frac{1}{F_{g(x_1),g(x_0)}(\phi(\frac{t}{c}))}-1+\frac{1}{F_{g(y_1),g(y_0)}(\phi(\frac{t}{c}))}-1).$$

Using the fact that ψ is non-decreasing, we get

$$\frac{1}{F_{g(x_2),g(x_1)}(\phi(t))}-1+\frac{1}{F_{g(y_2),g(y_1)}(\phi(t))}-1\leq\frac{1}{F_{g(x_1),g(x_0)}(\phi(\frac{t}{c}))}-1+\frac{1}{F_{g(y_1),g(y_0)}(\phi(\frac{t}{c}))}-1.$$

From the above inequality we deduce that

$$F_{g(x_2),g(x_1)}(\phi(t))>0, F_{g(y_2),g(y_1)}(\phi(t))>0, \text{and so} F_{g(x_2),g(x_1)}(\phi(\frac{t}{c}))>0, F_{g(y_2),g(y_1)}(\phi(\frac{t}{c}))>0.$$

Again, by using (3.1), we have

$$\begin{aligned}\frac{1}{F_{g(x_3),g(x_2)}(\phi(t))}-1+\frac{1}{F_{g(y_3),g(y_2)}(\phi(t))}-1 &\leq \frac{1}{F_{g(x_2),g(x_1)}(\phi(\frac{t}{c}))}-1+\frac{1}{F_{g(y_2),g(y_1)}(\phi(\frac{t}{c}))}-1\\ &\leq \frac{1}{F_{g(x_1),g(x_0)}(\phi(\frac{t}{c^2}))}-1+\frac{1}{F_{g(y_1),g(y_0)}(\phi(\frac{t}{c^2}))}-1.\end{aligned}$$

Repeating the above procedure successively, we obtain

$$\frac{1}{F_{g(x_{n+1}),g(x_n)}(\phi(t))}-1+\frac{1}{F_{g(y_{n+1}),g(y_n)}(\phi(t))}-1\leq\frac{1}{F_{g(x_1),g(x_0)}(\phi(\frac{t}{c^n}))}-1+\frac{1}{F_{g(y_1),g(y_0)}(\phi(\frac{t}{c^n}))}-1.$$

If we change x_0 with x_r in the previous inequalities, then for all $n>r$ we get

$$\frac{1}{F_{g(x_{n+1}),g(x_n)}(\phi(c^rt))}-1+\frac{1}{F_{g(y_{n+1}),g(y_n)}(\phi(c^rt))}-1\leq\frac{1}{F_{g(x_1),g(x_0)}(\phi(\frac{c^rt}{c^{n-r}t}))}-1+\frac{1}{F_{g(y_1),g(y_0)}(\phi(\frac{c^rt}{c^{n-r}t}))}-1.$$

Since $\phi(\frac{c^rt}{c^{n-r}t})\to\infty$ as $n\to\infty$ for all $0<r<n$, we have

$$\lim_{n\to\infty}F_{g(x_1),g(x_0)}(\phi(\frac{c^rt}{c^{n-r}t}))=1 \quad\text{and}\quad \lim_{n\to\infty}F_{g(y_1),g(y_0)}(\phi(\frac{c^rt}{c^{n-r}t}))=1.$$

Thus,

$$\lim_{n\to\infty}(\frac{1}{F_{g(x_{n+1}),g(x_n)}(\phi(c^rt))}-1)\leq\lim_{n\to\infty}(\frac{1}{F_{g(x_{n+1}),g(x_n)}(\phi(c^rt))}-1+\frac{1}{F_{g(y_{n+1}),g(y_n)}(\phi(c^rt))}-1)\leq 0,$$
$$\lim_{n\to\infty}(\frac{1}{F_{g(y_{n+1}),g(y_n)}(\phi(c^rt))}-1)\leq\lim_{n\to\infty}(\frac{1}{F_{g(x_{n+1}),g(x_n)}(\phi(c^rt))}-1+\frac{1}{F_{g(y_{n+1}),g(y_n)}(\phi(c^rt))}-1)\leq 0,$$

which imply that

$$\lim_{n\to\infty}F_{g(x_{n+1}),g(x_n)}(\phi(c^rt))=1 \quad\text{and}\quad \lim_{n\to\infty}F_{g(y_{n+1}),g(y_n)}(\phi(c^rt)=1. \tag{3.9}$$

Now, let $\epsilon>0$ be given, by (i) and (iv) of Definition 2.3, we can find $r\in\mathbb{Z}^+$ such that $\phi(c^rt)<\epsilon$. It follows from (3.9) that

$$\lim_{n\to\infty}F_{g(x_{n+1}),g(x_n)}(\epsilon)\geq\lim_{n\to\infty}F_{g(x_{n+1}),g(x_n)}(\phi(c^rt))=1 \quad\text{and}\quad \lim_{n\to\infty}F_{g(y_{n+1}),g(y_n)}(\epsilon)\geq 1.$$

By using a triangle inequality, we obtain

$$F_{g(x_{n+p}),g(x_n)}(\epsilon)\geq\Delta(F_{g(x_{n+p}),g(x_{n+p-1})}(\frac{\epsilon}{p}),\Delta(F_{g(x_{n+p-1}),g(x_{n+p-2})}(\frac{\epsilon}{p}),\cdots,F_{g(x_{n+1}),g(x_n)}(\frac{\epsilon}{p}))).$$

Thus, letting $n\to\infty$ and making use of (3.9), for any integer, we get

$$\lim_{n\to\infty}F_{g(x_{n+p}),g(x_n)}(\epsilon)=1 \quad\text{for every } \epsilon>0.$$

Hence $\{g(x_n)\}$ is a *Cauchy* sequence.

Similarly, we can also prove that $\{g(y_n)\}$ is also a Caucgy sequence. Since $(X, \le, F, \Delta)$ is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x \quad \text{and} \quad \lim_{n\to\infty} g(y_n) = y. \tag{3.10}$$

From (3.10) and continuity of g, we have

$$\lim_{n\to\infty} g(g(x_n)) = g(x) \quad \text{and} \quad \lim_{n\to\infty} g(g(y_n)) = g(y).$$

From (3.2) and commutativity of T and g, we have

$$g(g(x_{n+1})) = g(T(x_n, y_n)) = T(g(x_n), g(y_n)) \text{ and } g(g(y_{n+1})) = g(T(y_n, x_n)) = T(g(y_n), g(x_n)). \tag{3.11}$$

We now show that $g(x) = T(x, y)$ and $g(y) = T(y, x)$. Suppose that the assumption (a) holds. Taking the limit as $n \to \infty$ in (3.11), by (3.10) and continuity of T we get

$$g(x) = \lim_{n\to\infty} g(g(x_{n+1})) = \lim_{n\to\infty} T(g(x_n), g(y_n)) = T(\lim_{n\to\infty} g(x_n), \lim_{n\to\infty} g(y_n)) = T(x, y),$$
$$g(y) = \lim_{n\to\infty} g(g(y_{n+1})) = \lim_{n\to\infty} T(g(y_n), g(x_n)) = T(\lim_{n\to\infty} g(y_n), \lim_{n\to\infty} g(x_n)) = T(y, x).$$

Thus we proved that

$$g(x) = T(x, y) \quad \text{and} \quad g(y) = T(y, x).$$

Suppose now that (b) holds. Since

$$F_{g(x),T(x,y)}(\epsilon) \ge \Delta(F_{g(x),g(g(x_{n+1}))}(\frac{\epsilon}{2}), F_{g(g(x_{n+1})),T(x,y)}(\frac{\epsilon}{2})). \tag{3.12}$$

By using (i) of Definition 2.3, we can find $s > 0$ such that $\phi(s) < \frac{\epsilon}{2}$. Since $\lim_{n\to\infty} g(g(x_n)) = g(x)$ and $\lim_{n\to\infty} g(g(y_n)) = g(y)$, then there exists $n_0 \in \mathbb{Z}^+$, such that

$$F_{g(g(x_n)),g(x)}(\phi(s)) > 0 \text{ and } F_{g(g(y_n)),g(y)}(\phi(s)) > 0$$

for all $n > n_0$. Since $\{g(x_n)\}$ is non-decreasing and $\{g(x_n)\} \to x$, and as $\{g(y_n)\}$ is non-increasing and $\{g(y_n)\} \to y$, by (3.1) and (3.11) we get

$$\begin{aligned}\psi(\frac{1}{F_{g(g(x_{n+1})),T(x,y)}(\phi(s))} - 1) &= \psi(\frac{1}{F_{g(T(x_n,y_n)),T(x,y)}(\phi(s))} - 1) = \psi(\frac{1}{F_{T(g(x_n),g(y_n)),T(x,y)}(\phi(s))} - 1)\\ &\le \frac{1}{2}\psi(\frac{1}{F_{g(g(x_n)),g(x)}(\phi(\frac{s}{c}))} - 1 + \frac{1}{F_{g(g(y_n)),g(y)}(\phi(\frac{s}{c}))} - 1),\end{aligned}$$

$$\begin{aligned}\psi(\frac{1}{F_{g(g(y_{n+1})),T(y,x)}(\phi(s))} - 1) &= \psi(\frac{1}{F_{g(T(y_n,x_n)),T(y,x)}(\phi(s))} - 1) = \psi(\frac{1}{F_{T(g(y_n),g(x_n)),T(y,x)}(\phi(s))} - 1)\\ &\le \frac{1}{2}\psi(\frac{1}{F_{g(g(x_n)),g(x)}(\phi(\frac{s}{c}))} - 1 + \frac{1}{F_{g(g(y_n)),g(y)}(\phi(\frac{s}{c}))} - 1).\end{aligned}$$

So, by the above inequalities and (ii) of Definition 2.5, we have

$$\begin{aligned}\frac{1}{F_{g(g(x_{n+1})),T(x,y)}(\frac{\epsilon}{2})} - 1 &\le \frac{1}{F_{g(g(x_{n+1})),T(x,y)}(\phi(s))} - 1\\ &\le \frac{1}{F_{g(g(x_{n+1})),T(x,y)}(\phi(s))} - 1 + \frac{1}{F_{g(g(y_{n+1})),T(y,x)}(\phi(s))} - 1\\ &\le \frac{1}{F_{g(g(x_n)),g(x)}(\phi(\frac{s}{c}))} - 1 + \frac{1}{F_{g(g(y_n)),g(y)}(\phi(\frac{s}{c}))} - 1,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\frac{1}{F_{g(g(y_{n+1})),T(y,x)}(\frac{\epsilon}{2})}-1 &\leq \frac{1}{F_{g(g(y_{n+1})),T(y,x)}(\phi(s))}-1\\ &\leq \frac{1}{F_{g(g(x_{n+1})),T(x,y)}(\phi(s))}-1+\frac{1}{F_{g(g(y_{n+1})),T(y,x)}(\phi(s))}-1\\ &\leq \frac{1}{F_{g(g(x_n)),g(x)}(\phi(\frac{s}{c}))}-1+\frac{1}{F_{g(g(y_n)),g(y)}(\phi(\frac{s}{c}))}-1.\end{aligned}$$

Letting $n \to \infty$ in (3.13), we obtain

$$\lim_{n\to\infty} F_{g(g(x_{n+1})),T(x,y)}(\frac{\epsilon}{2}))=1. \tag{3.14}$$

From (3.12) and (3.14), we get $F_{g(x),T(x,y)}(\epsilon)=1$ for every $\epsilon>0$, which implies that $g(x)=T(x,y)$.

Similarly, one can show that $g(y)=T(y,x)$. Thus we proved that g and T have a coupled coincidence point. □

Taking $g=I_X$ (the identity mapping on X), $c=\frac{1}{2}$ and $\phi(t)=\varphi(t)=t$ for all $t\geq 0$ in Theorem 3.1, we get the following result.

Corollary 3.2. *Let $(X,\leq)$ be a partially ordered set and (X,F,Δ) be a complete Menger PM-space with a continuous t-norm. Suppose $T: X\times X\to X$ is such that T has the mixed monotone property and*

$$\frac{1}{F_{T(x,y),T(u,v)}(\frac{1}{2}t)}-1\leq \frac{1}{2}(\frac{1}{F_{x,u}(t)}-1+\frac{1}{F_{y,v}(t)}-1)$$

for all $t>0$ such that $F_{x,u}(t)>0$ and $F_{y,v}(t)>0$, $x,y,u,v\in X$ for which $x\leq u$ and $y\geq v$. Suppose either

(a) T is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\}\to x$, then $x_n\leq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\}\to y$, then $y_n\leq y$ for all n.

If there exist $x_0,y_0\in X$ such that

$$x_0\leq T(x_0,y_0) \quad \text{and} \quad y_0\geq T(y_0,x_0),$$

then there exist $x,y\in X$ such that

$$x=T(x,y) \quad \text{and} \quad y=T(y,x),$$

that is, T has a coupled fixed point.

Now we shall prove the existence and uniqueness theorem of a coupled fixed point. Note that if $(X,\leq)$ is a partially ordered set, then we endow the product $X\times X$ with the following partial order:

$$\text{for all } (x,y),(u,v)\in X\times X, \quad (x,y)\leq(u,v)\Leftrightarrow x\leq u, y\geq v.$$

Theorem 3.3. *In addition to the hypotheses of* Theorem 3.1*, suppose that for every $(x,y),(x^*,y^*)\in X\times X$ there exists a $(u,v)\in X\times X$ such that $(T(u,v),T(v,u))$ is comparable to $(T(x,y),T(y,x))$ and $(T(x^*,y^*),T(y^*,x^*))$. Then T and g have a unique coupled common fixed point, that is, there exists a unique $(x,y)\in X\times X$ such that*

$$x=g(x)=T(x,y) \quad \text{and} \quad y=g(y)=T(y,x).$$

Proof. From Theorem 3.1, the set of coupled coincidences is non-empty. We shall first show that if (x,y) and (x^*,y^*) are coupled coincidence points, that is, if $g(x)=T(x,y)$, $g(y)=T(y,x)$ and $g(x^*)=T(x^*,y^*)$, $g(y^*)=T(y^*,x^*)$, then

$$g(x) = g(x^*) \quad \text{and} \quad g(y) = g(y^*). \tag{3.15}$$

By assumption there exists a $(u, v) \in X$ such that $(T(u, v), T(v, u))$ is comparable to $(T(x, y), T(y, x))$ and $(T(x^*, y^*), T(y^*, x^*))$. Putting $u_0 = u$ and $v_0 = v$ and choose $u_1, v_1 \in X$ such that $g(u_1) = T(u_0, v_0)$ and $g(v_1) = T(v_0, u_0)$. Then, similarly as the proof of Theorem 3.1, we can inductively define sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ such that

$$g(u_{n+1}) = T(u_n, v_n) \quad \text{and} \quad g(v_{n+1}) = T(v_n, u_n).$$

Similarly, setting $x_0 = x$, $y_0 = y$, $x_0^* = x^*$ and $y_0^* = y^*$, we can define $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(x_n^*)\}$, $\{g(y_n^*)\}$. Then it is easy to show that

$$g(x_n) = T(x, y),\ g(y_n) = T(y, x) \quad \text{and} \quad g(x_n^*) = T(x^*, y^*),\ g(y_n^*) = T(y^*, x^*) \quad \text{for all } n \geq 1.$$

Since $(T(x, y), T(y, x)) = (g(x_1), g(y_1)) = (g(x), g(y))$ and $(T(u, v), T(v, u)) = (g(u_1), g(v_1))$ are comparable, then $g(x) \leq g(u_1)$ and $g(y) \geq g(v_1)$. It is easy to show that $((g(x), g(y)))$ and $(g(u_n), g(v_n))$ are comparable, that is $g(x) \leq g(x_n)$ and $g(y) \geq g(y_n)$ for all $n \geq 1$. Following the proof of Theorem 3.1, we can choose a $t > 0$ such that $F_{g(x),g(u_n)}(\phi(\frac{t}{c})) > 0$ and $F_{g(y),g(v_n)}(\phi(\frac{t}{c})) > 0$ for all $n \geq 0$. Thus from (3.1)

$$\begin{aligned}
\psi(\frac{1}{F_{g(x),g(u_{n+1})}(\phi(t))} - 1) &= \psi(\frac{1}{F_{T(x,y),T(u_n,v_n)}(\phi(t))} - 1) \\
&\leq \frac{1}{2}\psi(\frac{1}{F_{g(x),g(u_n)}(\phi(\frac{t}{c}))} - 1 + \frac{1}{F_{g(y),g(v_n)}(\phi(\frac{t}{c}))} - 1),
\end{aligned}$$

$$\begin{aligned}
\psi(\frac{1}{F_{g(y),g(v_{n+1})}(\phi(t))} - 1) &= \psi(\frac{1}{F_{T(y,x),T(v_n,u_n)}(\phi(t))} - 1) \\
&\leq \frac{1}{2}\psi(\frac{1}{F_{g(x),g(u_n)}(\phi(\frac{t}{c}))} - 1 + \frac{1}{F_{g(y),g(v_n)}(\phi(\frac{t}{c}))} - 1).
\end{aligned}$$

Adding and by (ii) of Definition 2.5, we get

$$\begin{aligned}
\frac{1}{F_{g(x),g(u_{n+1})}(\phi(t))} - 1 + \frac{1}{F_{g(y),g(v_{n+1})}(\phi(t))} - 1 &\leq \frac{1}{F_{g(x),g(u_n)}(\phi(\frac{t}{c}))} - 1 + \frac{1}{F_{g(y),g(v_n)}(\phi(\frac{t}{c}))} - 1 \\
&\ \ \vdots \\
&\leq \frac{1}{F_{g(x),g(u_0)}(\phi(\frac{t}{c^n}))} - 1 + \frac{1}{F_{g(y),g(v_0)}(\phi(\frac{t}{c^n}))} - 1.
\end{aligned}$$

If we change u_r with u_0 in the previous, then for all $n > r$ we get

$$\frac{1}{F_{g(x),g(u_{n+1})}(\phi(c^r t))} - 1 + \frac{1}{F_{g(y),g(v_{n+1})}(\phi(c^r t))} - 1 \leq \frac{1}{F_{g(x),g(u_r)}(\phi(\frac{c^r t}{c^{n-r}}))} - 1 + \frac{1}{F_{g(y),g(v_r)}(\phi(\frac{c^r t}{c^{n-r}}))} - 1.$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} F_{g(x),g(u_{n+1})}(\phi(c^r t)) = 1 \quad \text{and} \quad \lim_{n\to\infty} F_{g(y),g(v_{n+1})}(\phi(c^r t)) = 1.$$

Now, let $\epsilon > 0$ be given, by (i) and (iv) of Definition 2.3, we can find a $r \in \mathbb{Z}^+$ such that $\phi(c^r t) < \frac{\epsilon}{2}$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} F_{g(x),g(u_{n+1})}(\frac{\epsilon}{2}) \geq \lim_{n\to\infty} F_{g(x),g(u_{n+1})}(\phi(c^r t)) = 1, \\
\lim_{n\to\infty} F_{g(y),g(v_{n+1})}(\frac{\epsilon}{2}) \geq \lim_{n\to\infty} F_{g(y),g(v_{n+1})}(\phi(c^r t)) = 1.
\end{aligned} \tag{3.16}$$

Similarly, we can prove that

$$\lim_{n\to\infty} F_{g(x^*),g(u_{n+1})}(\frac{\epsilon}{2}) = 1 \quad \text{and} \quad \lim_{n\to\infty} F_{g(y^*),g(v_{n+1})}(\frac{\epsilon}{2}) = 1. \tag{3.17}$$

By using a triangle inequality,(3.16) and (3.17),

$$F_{g(x),g(x^*)}(\epsilon) \geq \Delta(F_{g(x),g(u_{n+1})}(\frac{\epsilon}{2}), F_{g(x^*),g(u_{n+1})}(\frac{\epsilon}{2})) \to 1 \quad \text{as } n \to \infty,$$

$$F_{g(y),g(y^*)}(\epsilon) \geq \Delta(F_{g(y),g(v_{n+1})}(\frac{\epsilon}{2}), F_{g(y^*),g(v_{n+1})}(\frac{\epsilon}{2})) \to 1 \quad \text{as } n \to \infty.$$

Hence $g(x) = g(x^*)$ and $g(y) = g(y^*)$. Thus we proved (3.15).

Since $g(x) = T(x, y)$ and $g(y) = T(y, x)$, by commutativity of T and g, we have

$$g(g(x)) = g(T(x, y)) = T(g(x), g(y)) \quad \text{and} \quad g(g(y)) = g(T(y, x)) = T(g(y), g(x)). \tag{3.18}$$

Denote $g(x) = z$ and $g(y) = w$. Then from (3.18), we obtain

$$g(z) = T(z, w) \quad \text{and} \quad g(w) = T(w, z). \tag{3.19}$$

Thus (z, w) is a coupled coincidence point. Then from (3.15) with $x^* = z$ and $y^* = w$ it follows $g(z) = g(x)$ and $g(w) = g(y)$, that is

$$g(z) = z \quad \text{and} \quad g(w) = w. \tag{3.20}$$

From (3.19) and (3.20), we have

$$z = g(z) = T(z, w) \quad \text{and} \quad w = g(w) = T(w, z).$$

Therefore, (z, w) is a coupled common fixed point of T and g. To prove the uniqueness, assume that (p, q) is another coupled common fixed point. Then by (3.15) we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. This completes the proof. □

Corollary 3.4. *In addition to the hypotheses of* Corollary 3.2*, suppose that for every* $(x, y), (x^*, y^*) \in X \times X$ *there exist a* $(u, v) \in X \times X$ *such that* $(T(u, v), T(v, u))$ *is comparable to* $(T(x, y), T(y, x))$ *and* $(T(x^*, y^*), T(y^*, x^*))$*. Then* T *has a unique coupled fixed point, that is, there exists a unique* $(x, y) \in X \times X$ *such that*

$$x = T(x, y) \quad \text{and} \quad y = T(y, x).$$

Corollary 3.5. *In addition to the hypotheses of* Corollary 3.2*, if* x_0 *and* y_0 *are comparable then* T *has a fixed point, that is* $x = T(x, x)$*.*

Proof. Following Corollary 3.2, T has a coupled fixed point (x, y). We only have to show that $x = y$. Since x_0 and y_0 are comparable, we may assume that $x_0 \geq y_0$. By using the mathematical induction, one can show that

$$x_n \geq y_n \quad \text{for all } n \geq 0,$$

where $x_{n+1} = T(x_n, y_n)$ and $y_{n+1} = T(y_n, x_n)$, $n = 0, 1, 2\cdots$. Following the proof of Theorem 3.1, we can choose a $t > 0$ such that $F_{x_n,y_n}(2t) > 0$ and $F_{y_n,x_n}(2t) > 0$ for all $n \geq 0$. Thus we have

$$\begin{aligned}
\frac{1}{F_{x_{n+1},y_{n+1}}(t)} - 1 &= \frac{1}{F_{T(x_n,y_n),T(y_n,x_n)}(t)} - 1 \\
&\leq \frac{1}{2}(\frac{1}{F_{x_n,y_n}(2t)} - 1 + \frac{1}{F_{y_n,x_n}(2t)} - 1), \\
\frac{1}{F_{y_{n+1},x_{n+1}}(t)} - 1 &= \frac{1}{F_{T(y_n,x_n),T(x_n,y_n)}(t)} - 1 \\
&\leq \frac{1}{2}(\frac{1}{F_{x_n,y_n}(2t)} - 1 + \frac{1}{F_{y_n,x_n}(2t)} - 1).
\end{aligned}$$

Adding, we get

$$\begin{aligned}\frac{1}{F_{x_{n+1},y_{n+1}}(t)}-1+\frac{1}{F_{y_{n+1},x_{n+1}}(t)}-1 &\le \frac{1}{F_{x_n,y_n}(2t)}-1+\frac{1}{F_{y_n,x_n}(2t)}-1\\ &\vdots\\ &\le \frac{1}{F_{x_0,y_0}(2^nt)}-1+\frac{1}{F_{y_0,x_0}(2^nt)}-1\end{aligned}$$

If we change u_r with u_0 in the previous, then for all $n > r$ we get

$$\frac{1}{F_{x_{n+1},y_{n+1}}(\frac{1}{2^r}t)}-1+\frac{1}{F_{y_{n+1},x_{n+1}}(\frac{1}{2^r}t)}-1 \le \frac{1}{F_{x_r,y_r}(2^{n-2r}t)}-1+\frac{1}{F_{y_r,x_r}(2^{n-2r}t)}-1.$$

Letting $n\to\infty$, we have

$$\lim_{n\to\infty} F_{x_{n+1},y_{n+1}}(\frac{1}{2^r}t)=1 \quad \text{and} \quad \lim_{n\to\infty} F_{y_{n+1},x_{n+1}}(\frac{1}{2^r}t)=1.$$

Now, let $\epsilon > 0$ be given, we can find a $r\in\mathbb{Z}^+$ such that $\frac{1}{2^r} < \frac{\epsilon}{3}$. Then we have

$$\lim_{n\to\infty} F_{x_{n+1},y_{n+1}}(\frac{\epsilon}{3}) \ge \lim_{n\to\infty} F_{x_{n+1},y_{n+1}}(\frac{1}{2^r}t)=1. \tag{3.21}$$

By using a triangle inequality, we have

$$F_{x,y}(\epsilon) \ge \Delta(F_{x,x_{n+1}}(\frac{\epsilon}{3}),\Delta(F_{x_{n+1},y_{n+1}}(\frac{\epsilon}{3}),F_{y_{n+1},y}(\frac{\epsilon}{3}))).$$

Letting $n\to\infty$ and using (3.21) we have

$$F_{x,y}(\epsilon)=1 \quad \text{for any } \epsilon>0.$$

Hence $x = y$. □

4. An application

In this section, we study the existence of a unique solution to a nonlinear integral equation, as an application to the fixed point theorem proved in section 3.

Consider the integral equation

$$x(r)=\int_a^b (K_1(r,s)+K_2(r,s))[f(s,x(s))+g(s,x(s))+(\mu_1-\mu_2)x(s)]ds \quad r\in I=[a,b],\mu_1,\mu_2>0. \tag{4.1}$$

We assume that K_1, K_2, f and g satisfy the following conditions

Assumption 4.1. (i) $K_1(r,s)\ge 0$ and $K_2(r,s)\le 0$ for all $r,s\in[a,b]$.
(ii) There exist $\lambda_1,\lambda_2,\mu_1,\mu_2>0$ such that for all $x,y\in\mathbb{R}$, $x\ge y$,

$$0\le f(r,x)-f(r,y)+\mu_1(x-y)\le\lambda_1(x-y)$$

and

$$-\lambda_2(x-y)\le g(r,x)-g(r,y)+\mu_2(y-x)\le 0.$$

(iii) $\max\{\lambda_1,\lambda_2\}\sup\limits_{r\in I}\int_a^b(K_1(r,s)-K_2(r,s))ds\le\frac{1}{4}$.

Definition 4.2. An element $(\alpha,\beta)\in\mathcal{C}(I,\mathbb{R})\times\mathcal{C}(I,\mathbb{R})$ is called a coupled lower and upper solution of the integral equation (4.1) if $\alpha(r)\le\beta(r)$,

$$\alpha(r) \leq \int_a^b K_1(r,s)(f(s,\alpha(s)) + g(s,\beta(s)) + \mu_1\alpha(s) - \mu_2\beta(s))ds$$
$$+ \int_a^b K_2(r,s)(f(s,\beta(s)) + g(s,\alpha(s)) + \mu_1\beta(s) - \mu_2\alpha(s))ds$$

and

$$\beta(r) \geq \int_a^b K_1(r,s)(f(s,\beta(s)) + g(s,\alpha(s)) + \mu_1 y(s) - \mu_2 x(s))ds$$
$$+ \int_a^b K_2(r,s)(f(s,\alpha(s)) + g(s,\beta(s)) + \mu_1 x(s) - \mu_2 y(s))ds$$

for all $r \in [a,b]$, $\mu_1, \mu_2 > 0$.

Theorem 4.3. *Consider the integral Equation* (4.1) *with* $K_1, K_2 \in \mathcal{C}(I \times I, \mathbb{R})$, $f, g \in \mathcal{C}(I \times \mathbb{R}, \mathbb{R})$ *and* $h \in \mathcal{C}(I, \mathbb{R})$ *and suppose that Assumption 4.1 is satisfied. Then the existence of a coupled lower and upper solution for* (4.1) *provides the existence of a unique solution of* (4.1) *in* $\mathcal{C}(I, \mathbb{R})$.

Proof. Let $X := \mathcal{C}(I, \mathbb{R})$. X is a partially ordered set if we define the following order relation in X:

$$x, y \in \mathcal{C}(I, \mathbb{R}), \qquad x \leq y \Leftrightarrow x(r) \leq y(r) \quad \text{for all } r \in I.$$

And (X, d) is a complete metric space with metric

$$d(x,y) = \sup_{t \in I} |x(r) - y(r)|, \quad x, y \in X.$$

Then (X, F, Δ_m) is a complete Menger PM-pace, where

$$F_{x,y}(t) = \frac{t}{t + d(x,y)} \quad \text{for all } x, y \in X, t > 0.$$

Suppose $\{u_n\}$ is a monotone non-decreasing in X that converges to $u \in X$. Then for every $r \in I$, the sequence of real numbers

$$u_1(r) \leq u_2(r) \leq \cdots \leq u_n(r) \leq \cdots$$

converges to $u(r)$. Therefore, for all $r \in I$, $n \in \mathbb{Z}^+$, $u_n(r) \leq u(r)$. Hence $u_n \leq u$, for all n.

Similarly, we can verify that limit $v(r)$ of a monotone non-increasing sequence $\{v_n\}$ in X is a lower bound for all the elements in the sequence. That is, $v \leq v_n$ for all n. Therefore, condition (b) of Corollary 3.2 holds.

Also, $X \times X = \mathcal{C}(I, \mathbb{R}) \times \mathcal{C}(I, \mathbb{R})$ is a partially ordered set if we define the following order relation in $X \times X$

$$(x,y),(u,v) \in X \times X, (x,y) \leq (u,v) \Leftrightarrow x(r) \leq u(r) \text{ and } y(r) \geq v(r), \quad \text{for all } r \in I.$$

Define $T : X \times X \to X$ by

$$T(x,y)(r) = \int_a^b K_1(r,s)(f(s,x(s)) + g(s,y(s)) + \mu_1 x(s) - \mu_2 y(s))ds$$
$$+ \int_a^b K_2(r,s)(f(s,y(s)) + g(s,x(s)) + \mu_1 y(s) - \mu_2 x(s))ds$$

for all $r \in I$.

For each $r \in I$, $\max\{T(x,y)(r), T(y,x)(r)\}$ and $\min\{T(x,y)(r), T(y,x)(r)\}$ are the upper and lower bounds of $T(x,y)$ and $T(y,x)$, respectively. Therefore, for every $(x,y),(u,v) \in X \times X$, there exists a

$$\big(\max\{T(x,y), T(u,v)\}, \min\{T(y,x), T(v,u)\}\big) \in X \times X$$

that is comparable to $(T(x,y),T(y,x))$ and $(T(u,v),T(v,u))$.

Now we shall show that T has the mixed monotone property. Indeed, for $x_1 \le x_2$, that is, $x_1(r) \le x_2(r)$ for all $r \in I$, by Assumption 4.1, we have

$$\begin{aligned}
T(x_1,y)(r) - T(x_2,y)(r) &= \int_a^b K_1(r,s)(f(s,x_1(s)) + g(s,y(s)) + \mu_1 x_1(s) - \mu_2 y(s))ds \\
&\quad + \int_a^b K_2(r,s)(f(s,y(s)) + g(s,x_1(s)) + \mu_1 y(s) - \mu_2 x_1(s))ds \\
&\quad - \int_a^b K_1(r,s)(f(s,x_2(s)) + g(s,y(s)) + \mu_1 x_2(s) - \mu_2 y(s))ds \\
&\quad - \int_a^b K_2(r,s)(f(s,y(s)) + g(s,x_2(s)) + \mu_1 y(s) - \mu_2 x_2(s))ds \\
&= \int_a^b K_1(r,s)(f(s,x_1(s)) - f(s,x_2(s)) + \mu_1(x_1(s) - x_2(s)))ds \\
&\quad + \int_a^b K_2(r,s)(g(s,x_1(s)) - g(s,x_2(s)) + \mu_2(x_2(s) - x_1(s)))ds \le 0.
\end{aligned}$$

Hence $T(x_1,y)(r) \le T(x_2,y)(r)$ for all $r \in I$, that is $T(x_1,y) \le T(x_2,y)$.

Similarly, if $y_1 \ge y_2$, for all $r \in I$, by Assumption 4.1, we have

$$\begin{aligned}
T(x,y_1)(r) - T(x,y_2)(r) &= \int_a^b K_1(r,s)(g(s,y_1(s)) - g(s,y_2(s)) + \mu_2(y_2(s) - y_1(s)))ds \\
&\quad + \int_a^b K_2(r,s)(f(s,y_1(s)) - f(s,y_2(s)) + \mu_1(y_1(s) - y_2(s)))ds \le 0.
\end{aligned}$$

Hence $T(x,y_1)(r) \le T(x,y_2)(r)$ for all $r \in I$, that is, $T(x,y_1) \le T(x,y_2)$.

Thus, $T(x,y)$ is monotone non-decreasing in x and monotone non-increasing in y.

Now, for $x \ge u$ and $y \le v$, that is, $x(r) \ge u(r)$ and $y(r) \le v(r)$ for all $r \in I$, any $t > 0$, we have

$$\begin{aligned}
\frac{1}{F_{T(x,y),T(u,v)}(\frac{1}{2})t} - 1 &= \frac{2d(T(x,y),T(u,v))}{t} = \frac{2}{t}\sup_{r\in I}|T(x,y)(r) - T(u,v)(r)| \\
&= \frac{2}{t}\sup_{r\in I}\bigg|\int_a^b K_1(r,s)[(f(s,x(s)) - f(s,u(s)) \\
&\quad +\mu_1(x(s) - u(s))) - (g(s,v(s)) - g(s,y(s)) + \mu_2(y(s) - v(s))]ds \\
&\quad - \int_a^b K_2(r,s)[(f(s,v(s)) - f(s,y(s)) \\
&\quad +\mu_1(v(s) - y(s))) - (g(s,x(s)) - g(s,u(s)) + \mu_2(u(s) - x(s))]ds\bigg| \\
&\le \frac{2}{t}\sup_{r\in I}\bigg|\int_a^b K_1(r,s)[\lambda_1(x(s) - u(s)) + \lambda_2(v(s) - y(s))]ds \\
&\quad - \int_a^b K_2(r,s)[\lambda_1(v(s) - y(s)) + \lambda_2(x(s) - u(s))]ds\bigg| \\
&\le \frac{2}{t}\max\{\lambda_1,\lambda_2\}\sup_{r\in I}\int_a^b (K_1(r,s) - K_2(r,s))[(x(s) - u(s)) + (v(s) - y(s))]ds.
\end{aligned}$$

As $x(s) - u(s) \le d(x,u)$, $v(s) - y(s) \le d(v,y)$, for all $s \in [a,b]$, we obtain

$$\frac{1}{F_{T(x,y),T(u,v)}(\frac{1}{2})t} - 1 \le \frac{2}{t}\max\{\lambda_1,\lambda_2\}\sup_{r\in I}\int_a^b (K_1(r,s) - K_2(r,s))[(x(s) - u(s)) + (v(s) - y(s))]ds$$

$$
\begin{aligned}
&\le \frac{2}{t}\max\{\lambda_1,\lambda_2\}[d(x,u)+d(v,y)]\sup_{r\in I}\int_a^b (K_1(r,s)-K_2(r,s))ds\\
&\le \frac{2}{t}\times\frac{1}{4}[d(x,u)+d(v,y)]\\
&\le \frac{1}{2}(\frac{d(x,u)}{t}+\frac{d(v,y)}{t})=\frac{1}{2}(\frac{1}{F_{x,u}(t)}-1+\frac{1}{F_{y,v}(t)}-1).
\end{aligned}
$$

Now, let (α,β) be a coupled lower and upper solution of the integral equation (4.1), then we have $\alpha(r)\le T(\alpha,\beta)(r)$ and $\beta(r)\ge T(\beta,\alpha)(r)$ for all $r\in[a,b]$, that is, $\alpha\le T(\alpha,\beta)$ and $\beta\ge T(\beta,\alpha)$. Finally, Corollary 3.2 gives that T has a unique coupled fixed point (x,y), that is, there exists a unique solution of (4.1) in $\mathcal{C}(\mathcal{I},\mathbb{R})$. □

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