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Fixed points and functional equation problems via cyclic admissible generalized contractive type mappings

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Abstract

In this paper, we present some fixed point theorems for cyclic admissible generalized contractions involving C-class functions and admissible mappings in metric-like spaces. We obtain some new results which extend and improve many recent results in the literature. In order to illustrate the effectiveness of the obtained results, several examples and applications to functional equations arising in dynamic programming are also given.

Keywords: Common fixed point, C-class functions, admissible mappings, partial metric space, metric-like space.
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1. Introduction and Preliminaries

The notion of metric-like (dislocated) metric spaces was introduced by Hitzler and Seda [12] as a generalization of a metric space. They generalized the Banach Contraction Principle [7] in such spaces. Metric-like spaces were discovered by Amini-Harandi [11] who established some fixed point results. Then, this interesting topic has been considered by several authors, see for example [6], [16], [17], [24].

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Here, we present some useful notions and facts.

In the sequel, the letters $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ will denote the set of all real numbers, the set of all non negative real numbers and the set of all natural numbers, respectively.

Definition 1.1 ([11])**.** A mapping $\sigma : X \times X \to \mathbb{R}^+$, where X is a nonempty set, is said to be metric-like on X if for any $x, y, z \in X$, the following three conditions hold true:

(σ1) $\sigma(x, y) = 0 \Rightarrow x = y$;

(σ2) $\sigma(x, y) = \sigma(y, x)$;

(σ3) $\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z)$.

The pair (X, σ) is then called a metric-like space. Note that a metric-like on X satisfies all of the conditions of a metric except that $\sigma(x, x)$ may be positive for $x \in X$. Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls

$$B_\sigma(x, \varepsilon) = \{y \in X : |\sigma(x, y) - \sigma(x, x)| < \varepsilon\}, \quad \text{for all } x \in X \text{ and } \varepsilon > 0.$$

A sequence $\{x_n\}$ in the metric-like space (X, σ) converges to a point $x \in X$ if and only if $lim_{n\to\infty}\sigma(x_n, x) = \sigma(x, x)$.

Let (X, σ) and (Y, τ) be metric-like spaces, and let $F : X \to Y$ be a continuous mapping. Then

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) \Rightarrow \lim_{n\to\infty} \sigma(Fx_n, Fx) = \sigma(Fx, Fx).$$

A sequence $\{x_n\}_{n=0}^{\infty}$ of elements of X is called σ-Cauchy if $lim_{m,n\to\infty}\sigma(x_m, x_n)$ exists and is finite. The metric-like space (X, σ) is called complete if for each σ-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there is some $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{m,n\to\infty} \sigma(x_m, x_n).$$

Lemma 1.2 ([17])**.** *Let (X, σ) be a metric-like space. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ where $x \in X$ and $\sigma(x, x) = 0$. Then, for all $y \in X$, we have $\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y)$.*

In 2012, Samet et al. [23] presented the notion of α-admissible mappings and proved the existence theorems of fixed point for such mappings satisfy some generalized contractive condition. The results obtained by Samet et al. [23] extend and generalize the existing fixed point results in the literature, in particular the Banach contraction principle. After that, several authors considered the generalizations of this new approach (see [15], [19], [22]). Recently, Alizadeh et al. [3] introduced the concept of cyclic (α, β)-admissible mapping and proved some new fixed point results which generalize and modify some recent results in the literature.

Definition 1.3 ([3])**.** Let $f : X \to X$ and $\alpha, \beta : X \to \mathbb{R}^+$. We say that f is a cyclic (α, β)-admissible mapping if the following conditions hold:

(*i*) $\alpha(x) \geq 1$ for some $x \in X$ implies $\beta(fx) \geq 1$,

(*ii*) $\beta(x) \geq 1$ for some $x \in X$ implies $\alpha(fx) \geq 1$.

Definition 1.4. An element $x \in X$ is called a common point of mappings $f, g : X \to X$ if $fx = gx = x$. Therewithal, the set of all common points of f and g is shown by $\mathcal{F}(f, g)$.

Very recently, Ansari [4] defined the concept of C-class functions and presented new fixed point results which improve and extend some results in the literature. For more details, also see [10, 14].

Definition 1.5 ([4])**.** A mapping $\boldsymbol{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called *C-class* function if it is continuous and satisfies following conditions:

(1) $\mathbf{F}(s, t) \leq s$;

(2) $\mathbf{F}(s, t) = s$ implies that either $s = 0$ or $t = 0$ for all $s, t \in \mathbb{R}^+$.

Note that $\mathbf{F}(0,0)=0$.
We denote the set of C-class functions as $\mathcal{C}$.

Example 1.6 ([4])**.** The following functions $\mathbf{F}:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t\in\mathbb{R}^+$:
(1) $\mathbf{F}(s,t)=s-t$, if $\mathbf{F}(s,t)=s$ then $t=0$;
(2) $\mathbf{F}(s,t)=ks$ for $0<k<1$, if $\mathbf{F}(s,t)=s$ then $s=0$;
(3) $\mathbf{F}(s,t)=\dfrac{s}{(1+t)^r}$ for $r\in(0,+\infty)$ if $\mathbf{F}(s,t)=s$ then $s=0$ or $t=0$;
(4) $\mathbf{F}(s,t)=\log_a\left(\frac{t+a^s}{1+t}\right)$ for $a>1$, if $\mathbf{F}(s,t)=s$ then $s=0$ or $t=0$;
(5) $\mathbf{F}(s,1)=\ln\left(\frac{1+a^s}{2}\right)$ for $a>e$, if $\mathbf{F}(s,1)=s$ then $s=0$;
(6) $\mathbf{F}(s,t)=slog_{t+a}a$ for $a>1$, if $\mathbf{F}(s,t)=s$ then $s=0$ or $t=0$;
(7) $\mathbf{F}(s,t)=(s+l)^{\frac{1}{(1+t)^r}}-l$ for $l>1$, $r\in(0,+\infty)$, if $\mathbf{F}(s,t)=s$ then $t=0$.

In this paper, we introduce the notion of cyclic admissible generalized contractions involving C-class functions and establish several common fixed point theorems this type mappings in the context of metric-like spaces. We also give several examples and results which generalize, extend and improve many recent results in the literature. As a consequence of our results, we discuss the existence and uniqueness of the common bounded solution of a functional equation arising in dynamic programming.

2. Main Results

Before proceeding to our results, let us give some lemma and definitions which will be used efficiently in the proof of main results.

Lemma 2.1. *Let (X,σ) be a metric-like space and let $\{z_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty}\sigma\left(z_n,z_{n+1}\right)=0. \tag{2.1}$$

If $\lim_{n,m\to\infty}\sigma\left(z_{2n},z_{2m}\right)\neq 0$, then there exist $\varepsilon>0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k>m_k>k$ and the following four sequences tend to ε^+ when $k\to\infty$:

$$\sigma\left(z_{2n_k},z_{2m_k}\right),\quad \sigma\left(z_{2m_k},z_{2n_k+1}\right),\quad \sigma\left(z_{2m_k-1},z_{2n_k}\right),\quad \sigma\left(z_{2m_k-1},z_{2n_k+1}\right).$$

Proof. Suppose that $\{z_{2n}\}$ is a sequence in (X,σ) satisfying (2.1) such that $\lim_{n,m\to\infty}\sigma\left(z_{2n},z_{2m}\right)\neq 0$. Then there exist $\varepsilon>0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $n_k>m_k>k$ such that

$$\sigma\left(z_{2n_k},z_{2m_k}\right)\geq\varepsilon \quad\text{and}\quad \sigma\left(z_{2n_k-2},z_{2m_k}\right)<\varepsilon.$$

Then, using the triangular inequality we have

$$\begin{aligned}\varepsilon\leq\sigma\left(z_{2n_k},z_{2m_k}\right)&\leq\sigma\left(z_{2n_k},z_{2n_k-1}\right)+\sigma\left(z_{2n_k-1},z_{2n_k-2}\right)+\sigma\left(z_{2n_k-2},z_{2m_k}\right)\\&<\sigma\left(z_{2n_k},z_{2n_k-1}\right)+\sigma\left(z_{2n_k-1},z_{2n_k-2}\right)+\varepsilon.\end{aligned}$$

Taking $k\to\infty$ on both sides of above inequality and using (2.1), we obtain

$$\lim_{k\to\infty}\sigma\left(z_{2n_k},z_{2m_k}\right)=\varepsilon^+. \tag{2.2}$$

Again, using the triangular inequality, we get

$$\left|\sigma\left(z_{2m_k},z_{2n_k+1}\right)-\sigma\left(z_{2n_k},z_{2m_k}\right)\right|\leq\sigma\left(z_{2n_k},z_{2n_k+1}\right).$$

Passing to the limit when $k\to\infty$ and using (2.1) and (2.2), we deduce

$$\lim_{k\to\infty}\sigma\left(z_{2m_k},z_{2n_k+1}\right)=\varepsilon^+.$$

Similarly, one can easily show that

$$\lim_{k\to\infty} \sigma\left(z_{2m_k-1}, z_{2n_k}\right) = \lim_{k\to\infty} \sigma\left(z_{2m_k-1}, z_{2n_k+1}\right) = \varepsilon^+.$$

□

Definition 2.2. Let $f, g : X \to X$ and $\alpha, \beta : X \to \mathbb{R}^+$. We say that (f, g) is a cyclic (α, β)-admissible pair if

(i) $\alpha(x) \geq 1$ for some $x \in X$ implies $\beta(fx) \geq 1$,
(ii) $\beta(x) \geq 1$ for some $x \in X$ implies $\alpha(gx) \geq 1$.

Example 2.3. Let $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $fx = -\left(x + x^3\right)$ and $gx = -\left(x^3 + x^5\right)$. Suppose that $\alpha, \beta : \mathbb{R} \to \mathbb{R}^+$ are given by $\alpha(x) = e^x$ for all $x \in \mathbb{R}$ and $\beta(y) = e^{-y}$ for all $y \in \mathbb{R}$. Then (f, g) is a cyclic (α, β)-admissible. Indeed, if $\alpha(x) = e^x \geq 1$, then $x \geq 0$ and so, $\beta(fx) = e^{x+x^3} \geq 1$. Also, if $\beta(y) = e^{-y} \geq 1$, then $y \leq 0$ and so, $\alpha(gy) = e^{-\left(y^3+y^5\right)} \geq 1$.

We denote by Ψ the set of all functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:
$(i)_\psi$ ψ is a strictly increasing and continuous,
$(ii)_\psi$ $\psi(t) = 0$ if and only if $t = 0$,
and Φ the set of all continuous functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$.

Definition 2.4. Let (X, σ) be a metric-like space and (f, g) be a cyclic (α, β)-admissible pair. We say that (f, g) is a cyclic admissible generalized contraction pair if

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi\left(\sigma\left(fx, gy\right)\right) \leq \mathbf{F}(\psi\left(M\left(x, y\right)\right), \phi\left(M\left(x, y\right)\right)), \tag{2.3}$$

for all $x, y \in X$, where $\mathbf{F} \in \mathcal{C}$, $\psi \in \Psi$ and $\phi \in \Phi$, and

$$M(x, y) = \max\{\sigma(x, y), \sigma(x, fx), \sigma(y, gy), \left[\sigma(x, gy) + \sigma(y, fx)\right]/4\}.$$

If we take $f = g$ in the above definition, we get following definition.

Definition 2.5. Let (X, σ) be a metric-like space and let f is a cyclic (α, β)-admissible mapping. We say that f is a cyclic admissible generalized contraction if

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi\left(\sigma\left(fx, fy\right)\right) \leq \mathbf{F}(\psi\left(M_f\left(x, y\right)\right), \phi\left(M_f\left(x, y\right)\right))$$

for all $x, y \in X$, where $\mathbf{F} \in \mathcal{C}$, $\psi \in \Psi$ and $\phi \in \Phi$, and

$$M_f(x, y) = \max\{\sigma(x, y), \sigma(x, fx), \sigma(y, fy), \left[\sigma(x, fy) + \sigma(y, fx)\right]/4\}.$$

Our main result is as follows:

Theorem 2.6. *Let (X, σ) be a complete metric-like space, $f, g : X \to X$ be two mappings and (f, g) be a cyclic admissible generalized contraction pair. Assume that the following conditions are satisfied:*

(a) *there exists $x_0 \in X \times X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;*
(b) *f and g are continuous;*
(c) *If $\sigma(x, x) = 0$ for some $x \in X$, then $\alpha(x) \geq 1$ and $\beta(x) \geq 1$.*

Then f and g have a common fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ whenever $x, y \in \mathcal{F}(f, g)$, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Define the sequence $\{x_n\}$ in X as follows: $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$, for all $n \geq 0$.

Since (f, g) is a cyclic (α, β)-admissible mapping and $\alpha(x_0) \geq 1$, then $\beta(fx_0) = \beta(x_1) \geq 1$ which implies $\alpha(gx_1) = \alpha(x_2) \geq 1$. By continuing this process, we get $\alpha(x_{2n}) \geq 1$ and $\beta(x_{2n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Similarly, since (f, g) is a cyclic (α, β)-admissible and $\beta(x_0) \geq 1$, we have $\beta(x_{2n}) \geq 1$ and $\alpha(x_{2n+1}) \geq 1$ for all $n \in \mathbb{N}_0$. Then, we deduce

$$\alpha(x_n) \geq 1 \quad \text{and} \quad \beta(x_n) \geq 1 \quad \text{for all } n \in \mathbb{N}_0. \tag{2.4}$$

Suppose that $x_{2n} \neq x_{2n+1}$ for all $n \in \mathbb{N}_0$. Then $\sigma(x_{2n}, x_{2n+1}) > 0$ for all $n \in \mathbb{N}_0$. Indeed, if $x_{2n} \neq x_{2n+1}$ and $\sigma(x_{2n}, x_{2n+1}) = 0$, then by $(\sigma 1)$ we have $x_{2n} = x_{2n+1}$, which is a contradiction.

Since $\alpha(x_{2n}) \beta(x_{2n+1}) \geq 1$ from (2.4), applying (2.3), we obtain

$$\begin{aligned} \psi(\sigma(x_{2n+1}, x_{2n+2})) &= \psi(\sigma(fx_{2n}, gx_{2n+1})) \\ &\leq \mathbf{F}(\psi(M(x_{2n}, x_{2n+1})), \phi(M(x_{2n}, x_{2n+1}))) \\ &< \psi(M(x_{2n}, x_{2n+1})), \end{aligned} \tag{2.5}$$

which implies $\psi(\sigma(x_{2n+1}, x_{2n+2})) < \psi(M(x_{2n}, x_{2n+1}))$, and so

$$\sigma(x_{2n+1}, x_{2n+2}) < M(x_{2n}, x_{2n+1}). \tag{2.6}$$

Now from the triangle inequality for σ, we have

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{\sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n}, fx_{2n}), \sigma(x_{2n+1}, gx_{2n+1}), \\ &\qquad [\sigma(x_{2n}, gx_{2n+1}) + \sigma(x_{2n+1}, fx_{2n})]/4\} \\ &= \max\{\sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n+1}, x_{2n+2}), \\ &\qquad [\sigma(x_{2n}, x_{2n+2}) + \sigma(x_{2n+1}, x_{2n+1}]/4\} \\ &\leq \max\{\sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n+1}, x_{2n+2}), \\ &\qquad [\sigma(x_{2n}, x_{2n+1}) + \sigma(x_{2n+1}, x_{2n+2})]/2\} \\ &= \max\{\sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

Thus, from (2.6), we get

$$\sigma(x_{2n+1}, x_{2n+2}) < \max\{\sigma(x_{2n}, x_{2n+1}), \sigma(x_{2n+1}, x_{2n+2})\}. \tag{2.7}$$

By a similar proof, one can also show that

$$\sigma(x_{2n}, x_{2n+1}) < \max\{\sigma(x_{2n-1}, x_{2n}), \sigma(x_{2n}, x_{2n+1})\}. \tag{2.8}$$

Therefore, from (2.7) and (2.8),

$$\sigma(x_n, x_{n+1}) < \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\} \quad \text{for all } n \geq 1. \tag{2.9}$$

If there exists $n_0 \in \mathbb{N}$ such that $\sigma(x_{n_0}, x_{n_0+1}) > \sigma(x_{n_0-1}, x_{n_0})$, then by (2.9), we have

$$\sigma(x_{n_0}, x_{n_0+1}) < \sigma(x_{n_0}, x_{n_0+1}),$$

which is a contradiction. Hence, $\sigma(x_n, x_{n+1}) \leq \sigma(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$ and so, by (2.5), we obtain

$$\psi(\sigma(x_n, x_{n+1})) \leq \mathbf{F}(\psi(\sigma(x_{n-1}, x_n)), \phi(\sigma(x_{n-1}, x_n))). \tag{2.10}$$

It follows that the sequence $\{\sigma(x_n, x_{n+1})\}$ is decreasing and bounded below. Thus, there exists $r \geq 0$ such that $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = r$. Next, we claim that $r = 0$. Assume on the contrary that $r > 0$. Consider the properties of ϕ, $\mathbf{F}$ and ψ, letting $n \to \infty$ in (2.10), we obtain

$$\psi(r) \leq \mathbf{F}(\psi(r), \phi(r)),$$

which implies $r = 0$, that is,

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0. \tag{2.11}$$

Now, we prove that $\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0$. It is sufficient to show that $\lim_{n,m\to\infty} \sigma(x_{2n}, x_{2m}) = 0$. Suppose, to the contrary, that $\lim_{n,m\to\infty} \sigma(x_{2n}, x_{2m}) \neq 0$. Then, using Lemma 2.1 we get that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the following four sequences tend to ε^+ when $k \to \infty$:

$$\sigma(x_{2n_k}, x_{2m_k}), \quad \sigma(x_{2n_k+1}, x_{2m_k}), \quad \sigma(x_{2n_k}, x_{2m_k-1}), \quad \sigma(x_{2n_k+1}, x_{2m_k-1}). \tag{2.12}$$

Since $\alpha(x_{2n_k})\beta(x_{2m_k-1}) \geq 1$ from (2.4), by (2.3), we have

$$\begin{aligned} \psi(\sigma(x_{2n_k+1}, x_{2m_k})) &= \psi(\sigma(fx_{2n_k}, gx_{2m_k-1})) \\ &\leq \mathbf{F}(\psi(M(x_{2n_k}, x_{2m_k-1})), \phi(M(x_{2n_k}, x_{2m_k-1}))), \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} M(x_{2n_k}, x_{2m_k-1}) &= \max\{\sigma(x_{2n_k}, x_{2m_k-1}), \sigma(x_{2n_k}, fx_{2n_k}), \sigma(x_{2m_k-1}, gx_{2m_k-1}), \\ &\qquad [\sigma(x_{2n_k}, gx_{2m_k-1}) + \sigma(x_{2m_k-1}, fx_{2n_k})]/4\} \\ &= \max\{\sigma(x_{2n_k}, x_{2m_k-1}), \sigma(x_{2n_k}, x_{2n_k+1}), \sigma(x_{2m_k-1}, x_{2m_k}), \\ &\qquad [\sigma(x_{2n_k}, x_{2m_k}) + \sigma(x_{2m_k-1}, x_{2n_k+1})]/4\}. \end{aligned}$$

Now, from the properties of ϕ, ψ and $\mathbf{F}$ and using (2.11) and (2.12) as $k \to \infty$ in (2.13), we have

$$\psi(\varepsilon) \leq \mathbf{F}(\psi(\varepsilon), \phi(\varepsilon)),$$

which implies that $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Thus, $\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0$, that is, $\{x_n\}$ is a σ-Cauchy sequence in X. From the completeness of (X, σ), there exists $z \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z, z) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{2.14}$$

Since f and g are continuous, we have

$$\lim_{n\to\infty} \sigma(x_{2n+1}, gz) = \lim_{n\to\infty} \sigma(fx_{2n}, gz) = \sigma(fz, gz), \tag{2.15}$$

and

$$\lim_{n\to\infty} \sigma(fz, x_{2n+2}) = \lim_{n\to\infty} \sigma(fz, gx_{2n+1}) = \sigma(fz, gz). \tag{2.16}$$

On the other hand, by Lemma 1.2 and (2.14), we get

$$\lim_{n\to\infty} \sigma(x_{2n+1}, gz) = \sigma(z, gz), \tag{2.17}$$

and

$$\lim_{n\to\infty} \sigma(fz, x_{2n+2}) = \sigma(fz, z). \tag{2.18}$$

Combining (2.15) and (2.17), we deduce that $\sigma(z, gz) = \sigma(fz, gz)$. Also, by (2.16) and (2.18), we have $\sigma(fz, z) = \sigma(fz, gz)$, and so

$$\sigma(z, gz) = \sigma(fz, z) = \sigma(fz, gz) \tag{2.19}$$

From (2.14) and the condition (c), we get $\alpha(z)\beta(z) \geq 1$. Then, using (2.19), by (2.3), we obtain

$$\begin{aligned} \psi(\sigma(z, gz)) &= \psi(\sigma(fz, gz)) \\ &\leq \mathbf{F}(\psi(M(z, z)), \phi(M(z, z))), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} M(z,z) &= \max\left\{\sigma(z,z), \sigma(z,fz), \sigma(z,gz), \frac{\sigma(z,gz)+\sigma(z,fz)}{4}\right\} \\ &= \max\left\{0, \sigma(z,gz), \sigma(z,gz), \frac{\sigma(z,gz)}{2}\right\} = \sigma(z,gz). \end{aligned}$$

Therefore, from (2.20), we get

$$\psi(\sigma(z,gz)) \leq \mathbf{F}(\psi(\sigma(z,gz)), \phi(\sigma(z,gz))),$$

which implies $\sigma(z,gz)=0$, that is, $z=gz$. Again applying (2.3), we have

$$\begin{aligned} \psi(\sigma(fz,z)) &= \psi(\sigma(fz,gz)) \\ &\leq \mathbf{F}(\psi(M(z,z)), \phi(M(z,z))) \\ &= \mathbf{F}(\psi(\sigma(z,gz)), \phi(\sigma(z,gz))) \\ &= \mathbf{F}(\psi(\sigma(fz,z)), \phi(\sigma(fz,z))), \end{aligned}$$

which implies $\sigma(fz,z)=0$, that is, $z=fz$. Hence, we obtain that $z=fz=gz$. To prove the uniqueness, suppose that w is another common fixed point of f and g. Then, by (d), we have $\alpha(z)\beta(w)\geq 1$. Thus, applying (2.3), we deduce

$$\psi(\sigma(z,w)) = \psi(\sigma(fz,gw)) \leq \mathbf{F}(\psi(M(z,w)), \phi(M(z,w))), \tag{2.21}$$

where

$$\begin{aligned} M(z,w) &= \max\left\{\sigma(z,w), \sigma(z,fz), \sigma(w,gw), \frac{\sigma(z,gw)+\sigma(w,fz)}{4}\right\} \\ &= \max\left\{\sigma(z,w), 0, 0, \frac{\sigma(z,w)}{2}\right\} = \sigma(z,w). \end{aligned}$$

Therefore, from (2.21), we have

$$\psi(\sigma(z,w)) \leq \mathbf{F}(\psi(\sigma(z,w)), \phi(\sigma(z,w))),$$

which implies $\sigma(z,w)=0$, that is, $z=w$. This completes the proof. □

In the following results we replace the continuity assumption of f and g in Theorem 2.6 by a suitable condition on sequences.

Theorem 2.7. *Let (X,σ) be a complete metric-like space, f, $g: X \to X$ be two mappings and (f,g) be a cyclic admissible generalized contraction. Assume that the following conditions are satisfied:*

(a) there exists $x_0 \in X \times X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$;

(b) if $\{x_n\}$ a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$ for all n, then $\beta(x) \geq 1$.

Then f and g have a common fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ whenever $x,y \in \mathcal{F}(f,g)$, then f and g have a unique common fixed point.

Proof. We define the sequence $\{x_n\}$ as in Theorem 2.6. Proceeding along the same lines as in the proof of Theorem 2.6, we know that $\lim_{n\to\infty}\sigma(x_n,z) = \sigma(z,z) = 0$. Then, by the condition (b) and (2.4), we deduce that $\alpha(x_{2n})\beta(z) \geq 1$ for all $n \in \mathbb{N}_0$.

Now, applying inequality (2.3), we obtain

$$\begin{aligned} \psi(\sigma(x_{2n+1},gz)) &= \psi(\sigma(fx_{2n},gz)) \\ &\leq \mathbf{F}(\psi(M(x_{2n},z)), \phi(M(x_{2n},z))), \end{aligned} \tag{2.22}$$

where

$$M\left(x_{2n},z\right)=\max\{\sigma(x_{2n},z),\sigma(x_{2n},x_{2n+1}),\sigma(z,gz),\frac{\sigma(x_{2n},gz)+\sigma(z,x_{2n+1})}{4}\}.$$

Taking $n\to\infty$ in the inequality (2.22) and using Lemma 2.1, we have

$$\psi\left(\sigma\left(z,gz\right)\right)\leq\mathbf{F}\left(\psi\left(\sigma\left(z,gz\right)\right),\phi\left(\sigma\left(z,gz\right)\right)\right),$$

which implies $\sigma\left(z,gz\right)=0$, that is, $z=gz$. Similarly, we may show that $z=fz$.

The uniqueness of common fixed point follows from Theorem 2.6. □

Now, we present an example to support the useability of our results.

Example 2.8. Let $X=\{0,1,2\}$ and define $\sigma:X\times X\to\mathbb{R}^+$ as follows:

$$\sigma\left(0,0\right)=3,\qquad\sigma\left(1,1\right)=1,\qquad\sigma\left(2,2\right)=2,\qquad\sigma\left(1,0\right)=\sigma\left(0,1\right)=9$$
$$\sigma\left(2,0\right)=\sigma\left(0,2\right)=5,\qquad\sigma\left(2,1\right)=\sigma\left(1,2\right)=4.$$

Then (X,σ) *is a* complete metric-like space. Let $f,g:X\to X$ be defined by

$$f0=0,\quad f1=2,\quad f2=1,$$
$$g0=0,\quad g1=1,\quad g2=1,$$

and $\alpha,\beta:X\to\mathbb{R}^+$ be defined by

$$\alpha\left(x\right)=\begin{cases}1, & \text{if } x=1\\ \frac{1}{2}, & \text{otherwise}\end{cases}\qquad\text{and}\qquad\beta\left(x\right)=\begin{cases}1, & \text{if } x=2\\ \frac{1}{4}, & \text{otherwise}\end{cases}.$$

Let $x\in X$ such that $\alpha\left(x\right)\geq1$. This implies that $x=1$ and $f1=2$, and so $\beta\left(fx\right)\geq1$. Again, let $x\in X$ such that $\beta\left(x\right)\geq1$. This implies that $x=2$ and $g2=1$, and so $\alpha\left(gx\right)\geq1$. This means that (f,g) is a cyclic (α,β) admissible mapping.

Also, we define the mappings $\mathbf{F}:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ by $\mathbf{F}(s,t)=\frac{s}{1+t}$ and $\psi,\phi:\mathbb{R}^+\to\mathbb{R}^+$ by $\psi\left(t\right)=t$ and $\phi\left(t\right)=\frac{t}{1000}$. Then, (f,g) is a cyclic admissible generalized contraction pair. Indeed, let $\alpha\left(x\right)\beta\left(y\right)\geq1$ for some $x,y\in X$. Then, by the definitions of α and β, we get $x=1$ and $y=2$, and so

$$\begin{aligned}\psi\left(\sigma\left(fx,gy\right)\right)&=\sigma\left(f1,g2\right)=4\\ &\leq\frac{9}{1+\frac{9}{1000}}=\frac{\psi\left(M\left(0,1\right)\right)}{1+\phi\left(M\left(0,1\right)\right)}\\ &=\mathbf{F}(\psi\left(M\left(x,y\right)\right),\phi\left(M\left(x,y\right)\right)),\end{aligned}$$

where

$$\begin{aligned}M\left(0,1\right)&=\max\{\sigma\left(0,1\right),\sigma(0,f0),\sigma(1,g1),\left[\sigma(0,g1)+\sigma(1,f0)\right]/4\}\\ &=\max\{9,3,1,\frac{9+9}{4}\}=9.\end{aligned}$$

It is also obvious that the condition (b) of Theorem 2.7 is satisfied. Therefore, f and g have a unique common fixed point which is $u=0$.

If we take $f=g$ in Theorem 2.6 (or Theorem 2.7), we have the following corollary.

Corollary 2.9. *Let (X,σ) be a complete metric-like space, $f:X\to X$ be a mapping such that f is a cyclic admissible generalized contraction. Suppose that the following assertions hold:*

(i) the conditions of (a), (b) and (c) hold in Theorem 2.6 *or*

(ii) the conditions of (a) and (b) hold in Theorem 2.7.

Then f has a fixed point. Moreover, if $\alpha\left(x\right)\geq1$ and $\beta\left(y\right)\geq1$ whenever x and y are two fixed points of f then f has a unique fixed point.

Corollary 2.10. *Let (X,σ) be a complete metric-like space, $f,\ g : X \to X$ be two mappings such that (f,g) is a cyclic (α,β) admissible pair. Assume that, for all $x,y \in X$,*

$$\alpha(x)\beta(y)\psi(\sigma(fx,gy)) \le \mathbf{F}(\psi(M(x,y)),\phi(M(x,y))), \tag{2.23}$$

where $\psi \in \Psi$, $\mathbf{F} \in \mathcal{C}$ and $\phi \in \Phi$. Suppose also that the following assertions hold:
(i) the conditions of (a), (b) and (c) hold in Theorem 2.6 *or*
(ii) the conditions of (a) and (b) hold in Theorem 2.7.
Then f and g have a common fixed point. Moreover, if $\alpha(x) \ge 1$ and $\beta(y) \ge 1$ whenever $x,y \in \mathcal{F}(f,g)$, then f and g have a unique common fixed point.

Proof. Let $\alpha(x)\beta(y) \ge 1$ for all $x,y \in X$. Then from (2.23), we have

$$\psi(\sigma(fx,gy)) \le \mathbf{F}(\psi(M(x,y)),\phi(M(x,y))).$$

This implies that the inequality (2.3) holds. Therefore, the proof follows from Theorem 2.6. □

If we take $\psi(t) = t$ in Corollary 2.10, we have the following corollary.

Corollary 2.11. *Let (X,σ) be a complete metric-like space, $f,\ g : X \to X$ be two mappings such that (f,g) is a cyclic (α,β) admissible pair. Assume that, for all $x,y \in X$,*

$$\alpha(x)\beta(y)\sigma(fx,gy) \le \mathbf{F}(M(x,y),\phi(M(x,y))), \tag{2.24}$$

where $\mathbf{F} \in \mathcal{C}$ and $\phi \in \Phi$. Suppose also that the following assertions hold:
(i) the conditions of (a), (b) and (c) hold in Theorem 2.6 *or*
(ii) the conditions of (a) and (b) hold in Theorem 2.7.
Then f and g have a common fixed point. Moreover, if $\alpha(x) \ge 1$ and $\beta(y) \ge 1$ whenever $x,y \in \mathcal{F}(f,g)$, then f and g have a unique common fixed point.

If we take $\mathbf{F}(s,t) = ks$ for $k \in (0,1)$ in Corollary 2.11, we have the following corollary.

Corollary 2.12. *Let (X,σ) be a complete metric-like space, $f,\ g : X \to X$ be two mappings such that (f,g) is a cyclic (α,β) admissible pair. Assume that, for all $x,y \in X$,*

$$\alpha(x)\beta(y)\sigma(fx,gy) \le kM(x,y), \tag{2.25}$$

where $k \in (0,1)$. Suppose also that the following assertions hold:
(i) the conditions of (a), (b) and (c) hold in Theorem 2.6 *or*
(ii) the conditions of (a) and (b) hold in Theorem 2.7.
Then f and g have a common fixed point. Moreover, if $\alpha(x) \ge 1$ and $\beta(y) \ge 1$ whenever $x,y \in \mathcal{F}(f,g)$, then f and g have a unique common fixed point.

If we take $\mathbf{F}(s,t) = s - t$ and $\alpha(t) = \beta(t) = 1$ in Theorem 2.6, we have the following corollary.

Corollary 2.13. *Let (X,σ) be a complete metric-like space and $f,\ g : X \to X$ be two mappings. Assume that, for all $x,y \in X$,*

$$\psi(\sigma(fx,gy)) \le \psi(M(x,y)) - \phi(M(x,y)), \tag{2.26}$$

where $\psi \in \Psi$ and $\phi \in \Phi$. Then f and g have a common fixed point.

If we take $\mathbf{F}(s,t) = s/(1+t)^r$ and $\alpha(t) = \beta(t) = 1$ in Theorem 2.6, we have the following corollary.

Corollary 2.14. *Let (X,σ) be a complete metric-like space and $f,\ g : X \to X$ be two mappings. Assume that, for all $x,y \in X$,*

$$\psi(\sigma(fx,gy)) \le \frac{\psi(M(x,y))}{(1+\phi(M(x,y)))^r}, \tag{2.27}$$

where $r > 0$, $\psi \in \Psi$ and $\phi \in \Phi$. Then f and g have a common fixed point.

In 1994, Matthews [20] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks and showed that the Banach contraction principle can be generalized to the partial metric context for applications in program verification.

A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:
($p1$) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
($p2$) $p(x,x) \leq p(x,y)$,
($p3$) $p(x,y) = p(y,x)$,
($p4$) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.
The pair (X, p) is then called a partial metric space.
If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y),$$

is a metric on X. It is well-known that a partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.

Now, we have the following common fixed point result.

Corollary 2.15. *Let (X, p) be a complete partial metric space, $f, g : X \to X$ be two mappings such that (f, g) is a cyclic (α, β)-admissible pair and*

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(p(fx,gy)) \leq \mathbf{F}(\psi(N(x,y)), \phi(N(x,y))) \tag{2.28}$$

for all $x, y \in X$, where $\mathbf{F} \in \mathcal{C}$, $\psi \in \Psi$ and $\phi \in \Phi$, and

$$N(x,y) = \max\{p(x,y), p(x,fx), p(y,gy), [p(x,gy) + p(y,fx)]/2\}.$$

Suppose that the following assertions hold:
(i) the conditions of (a), (b) and (c) hold in Theorem 2.6 *or*
(ii) the conditions of (a) and (b) hold in Theorem 2.7.
Then f and g have a common fixed point. Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ whenever $x, y \in \mathcal{F}(f,g)$, then f and g have a unique common fixed point.

3. Cyclic contractions

Let A and B be two nonempty subsets of a set X. A mapping $f : X \to X$ is cyclic (with respect to A and B) if $f(A) \subseteq B$ and $f(B) \subseteq A$.

The fixed point theory of cyclic contractive mappings has a recent origin. Kirk et al. [18] in 2003 initiated this line of research. This work has been followed by works like those in [2], [21]. Cyclic contractive mappings are mappings of which the contraction condition is only satisfied between any two points x and y with $x \in A$ and $y \in B$.

Definition 3.1. Let A and B be two nonempty subsets of a set X and $f, g : X \to X$ be two mappings. The pair (f, g) is called a cyclic with respect to A and B if $fA \subseteq B$ and $gB \subseteq A$.

In this section we give some fixed point results involving cyclic mappings which can be regarded as consequences of the theorems presented in the previous section.

Theorem 3.2. *Let A and B be two closed subsets of complete metric-like space (X, σ) such that $A \cap B \neq \emptyset$ and $f, g : A \cup B \to A \cup B$ be two mappings and (f, g) is a cyclic with respect to A and B. Assume that, for all $x \in A$ and $y \in B$,*

$$\psi(\sigma(fx,gy)) \leq \mathbf{F}(\psi(M(x,y)), \phi(M(x,y))), \tag{3.1}$$

where $\psi \in \Psi$, $\mathbf{F} \in \mathcal{C}$ and $\phi \in \Phi$. Then f and g have a unique common fixed point $z \in A \cap B$.

Proof. Define $\alpha, \beta : X \to [0, \infty)$ by

$$\alpha(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad \beta(x) = \begin{cases} 1, & x \in B, \\ 0, & \text{otherwise} \end{cases}.$$

Let $\alpha(x)\beta(y) \geq 1$. Then $x \in A$ and $y \in B$. Hence, from (3.1) we obtain

$$\psi(\sigma(fx, gy)) \leq \mathbf{F}(\psi(M(x,y)), \phi(M(x,y)))$$

for all $x, y \in A \cup B$. Let $\alpha(x) \geq 1$ for some $x \in X$, so $x \in A$. Hence, $fx \in B$ and so $\beta(fx) \geq 1$. Again, let $\beta(x) \geq 1$ for some $x \in X$, so $x \in B$. Hence, $gx \in A$ and so $\alpha(gx) \geq 1$. Therefore, (f, g) is a cyclic (α, β)-admissible mapping.

There exists an $x_0 \in A \cap B$, as $A \cap B$ is nonempty. This implies that $x_0 \in A$ and $x_0 \in B$ and so $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.

Let $\{x_n\}$ be a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. Then $x_n \in B$ for all $n \in \mathbb{N}$ and so $x \in B$. This implies that $\beta(x) \geq 1$.

Then, the conditions (a) and (b) of Theorem 2.7 hold. So there exist $z \in X$ such that $z = fz = gz$. Since $z \in A$, then $z = fz \in B$ and since $z \in B$, then $z = gz \in A$. Hence $z \in A \cap B$. The uniqueness of the common fixed point follows from (3.1). □

4. Application to functional equations

The existence and uniqueness of solutions of functional equations and system of functional equations arising in dynamic programming have been studied by using different fixed point theorems (see, [1], [5], [13], [15]).

Throughout this section, we assume that U and V are Banach spaces, $W \subseteq U$ is a state space, $D \subseteq V$ is a decision space. Now, we apply our results in order to prove the existence and uniqueness of the common solution of the following functional equations:

$$P = \sup_{y \in D} \{p(x,y) + G(x, y, P(\tau(x,y)))\}, \quad x \in W \tag{4.1}$$

and

$$Q = \sup_{y \in D} \{q(x,y) + K(x, y, Q(\tau(x,y)))\}, \quad x \in W, \tag{4.2}$$

where $\tau : W \times D \to W$, $p, q : W \times D \to \mathbb{R}$ and $G, K : W \times D \times \mathbb{R} \to \mathbb{R}$. It is well known that equations of the type (4.1) and (4.2) provides useful tools for mathematical optimization, computer programming and in dynamic programming (see, [8], [9]).

Let $X := B(W)$ denote the space of all bounded real-valued functions defined on the set W. We consider on X, the metric-like given by

$$\sigma(h, k) = \|h - k\| + \|h\| + \|k\|$$

for all $h, k \in X$, where $\|h\| = \sup_{x \in W} |h(x)|$. Note that σ is also a partial metric on X and since

$$d_\sigma(h, k) = 2\sigma(h,k) - \sigma(h,h) - \sigma(k,k) = 2\|h - k\|,$$

Thus, the space (X, σ) is complete since the metric space $(X, \|\cdot\|)$ is complete.

We consider the operators $f, g : X \to X$ given by

$$fh(x) = \sup_{y \in D} \{p(x,y) + G(x, y, h(\tau(x,y)))\}, \quad x \in W,\ h \in X,$$

$$gh(x) = \sup_{y \in D} \{q(x,y) + K(x, y, h(\tau(x,y)))\}, \quad x \in W,\ h \in X.$$

Suppose that the following conditions hold.

$(A1)$ $p, q : W \times D \to \mathbb{R}$ and $G, K : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded;

$(A2)$ there exist $\xi, \zeta : X \to \mathbb{R}$ such that if $\xi(h) \geq 0$ and $\zeta(k) \geq 0$ for all $h, k \in X$, then for every $(x, y) \in W \times D$, we have

$$|G(x, y, h(x)) - K(x, y, k(x))| \leq \delta |h(x) - k(x)|,$$

where $\delta \in (0, 1/3)$;

$(A3)$ for every $(x, y) \in W \times D$, we have

$$G(x, y, h(x)) \leq h(x) - p(x, y)$$

and

$$K(x, y, k(x)) \leq k(x) - q(x, y),$$

where $p, q : W \times D \to \mathbb{R}$;

$(A4)$ for all $h, k \in X$, $|\sup h(x)| \leq \delta |h(x)|$ and $|\sup k(x)| \leq \delta |k(x)|$ where $\delta \in (0, 1/3)$;

$(A5)$

$$\xi(h) \geq 0 \text{ for some } h \in X \text{ implies } \zeta(fh) \geq 0,$$

and

$$\zeta(h) \geq 0 \text{ for some } h \in X \text{ implies } \xi(gh) \geq 0;$$

$(A6)$ if $\{h_n\}$ is a sequence in X such that $\zeta(h_n) \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$ and $h_n \to h^*$ as $n \to \infty$, then $\zeta(h^*) \geq 0$;

$(A7)$ there exists $h_0 \in X$ such that $\xi(h_0) \geq 0$ and $\zeta(h_0) \geq 0$.

Theorem 4.1. *Assume that conditions* $(A1) - (A7)$ *are satisfied. Then functional equations* (4.1) *and* (4.2) *have a unique common bounded solution in* W.

Proof. Let λ be an arbitrary positive number and $x \in W$ and $h_1, h_2 \in X$ such that $\xi(h_1) \geq 0$ and $\zeta(h_2) \geq 0$. Then there exist $y_1, y_2 \in D$ such that

$$fh_1(x) < p(x, y_1) + G(x, y_1, h_1(\tau(x, y_1))) + \lambda, \tag{4.3}$$

$$gh_2(x) < p(x, y_2) + K(x, y_2, h_2(\tau(x, y_2))) + \lambda, \tag{4.4}$$

$$fh_1(x) \geq p(x, y_2) + G(x, y_2, h_1(\tau(x, y_2))), \tag{4.5}$$

$$gh_2(x) \geq p(x, y_1) + K(x, y_1, h_2(\tau(x, y_1))). \tag{4.6}$$

Next, by using (4.3) and (4.6), we have

$$\begin{aligned} fh_1(x) - gh_2(x) &< G(x, y_1, h_1(\tau(x, y_1))) - K(x, y_1, h_2(\tau(x, y_1))) + \lambda \\ &\leq |G(x, y_1, h_1(\tau(x, y_1))) - K(x, y_1, h_2(\tau(x, y_1)))| + \lambda \\ &\leq \delta |h_1(x) - h_2(x)| + \lambda. \end{aligned} \tag{4.7}$$

Analogously, by using (4.4) and (4.5), we obtain

$$gh_2(x) - fh_1(x) < \delta |h_1(x) - h_2(x)| + \lambda. \tag{4.8}$$

Therefore, from (4.7) and (4.8), we deduce

$$|fh_1(x) - gh_2(x)| < \delta |h_1(x) - h_2(x)| + \lambda$$

or, equivalently,

$$\|fh_1 - gh_2\| \leq \delta \|h_1 - h_2\| + \lambda.$$

Since $\lambda > 0$ is arbitrary, we get

$$\|fh_1 - gh_2\| \leq \delta \|h_1 - h_2\|. \tag{4.9}$$

Again, we have

$$|fh_1(x)| = \left| \sup_{y_1 \in D} \{p(x, y_1) + G(x, y_1, h_1(\tau(x, y_1)))\} \right| \leq \delta |h_1(x)|,$$

and so

$$\|fh_1\| \leq \delta \|h_1\|. \tag{4.10}$$

Proceeding similarly,

$$\|gh_2\| \leq \delta \|h_2\| \tag{4.11}$$

Summing (4.9) to (4.11), we obtain

$$\begin{aligned} \sigma(fh_1, gh_2) &= \|fh_1 - gh_2\| + \|fh_1\| + \|gh_2\| \\ &\leq \delta \|h_1 - h_2\| + \delta \|h_1\| + \delta \|h_2\| \\ &= 3\delta [\|h_1 - h_2\| + \|h_1\| + \|h_2\|] \\ &= \gamma\sigma(h_1, h_2) \leq \gamma M(h_1, h_2), \end{aligned}$$

where $\gamma = 3\delta$ such that $\gamma \in (0, 1)$ and

$$\begin{aligned} M(h_1, h_2) = \max\{ & d(h_1(t), h_2(t)), d(h_1(t), fh_1(t)), d(h_2(t), gh_2(t)), \\ & \frac{1}{4}[d(h_1(t), gh_2(t)) + d(h_2(t), fh_1(t))]\}. \end{aligned}$$

Now, define $\alpha, \beta : B(W) \to [0, \infty)$ by

$$\alpha(h) = \begin{cases} 1, & \text{if } \xi(h) \geq 0 \text{ where } h \in B(W), \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\beta(h) = \begin{cases} 1, & \text{if } \zeta(h) \geq 0 \text{ where } h \in B(W), \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, using the last inequality, we have

$$\alpha(h_1)\beta(h_2)\sigma(fh_1, gh_2) \leq \gamma M(h_1, h_2).$$

It easily shows that all the hypotheses of Corollary 2.12 are satisfied. Therefore f and g have a unique common fixed point, that is, functional equations (4.1) and (4.2) have a unique bounded common solution. □

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