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L_p-dual mixed geominimal surface areas

Yan Li[a], Wang Weidong[a,*], Si Lin[b]

[a] *Department of Mathematics, China Three Gorges University, Yichang, 443002, P. R. China.*

[b] *Department of Mathematics, Beijing Forestry University, Beijing, 100083, P. R. China.*

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Abstract

Zhu, Zhou and Xu showed an integral formula of L_p-mixed geominimal surface area by the p-Petty body. In this paper, we give an integral representation of L_p-dual mixed geominimal surface area and establish several related inequalities.

Keywords: L_p-mixed geominimal surface area, L_p-dual mixed geominimal surface area, integral representation.
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1. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of convex bodies whose centroid lie at the origin in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_c^n$, respectively. Let $\mathcal{S}_o^n$ and $\mathcal{S}_c^n$ respectively denote the set of star bodies (about the origin) and the set of star bodies whose centroid lie at the origin in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ denote the n-dimensional volume of the body K. For the standard unit ball B in $\mathbb{R}^n$, its volume is written by $\omega_n = V(B)$.

The notion of L_p-geominimal surface area was given by Lutwak (see [7]). For $K \in \mathcal{K}_o^n$, $p \geq 1$, the L_p-geominimal surface area, $G_p(K)$, of K is defined

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf\{nV_p(K, L)V(L^*)^{\frac{p}{n}} : L \in \mathcal{K}_o^n\}.$$

Here $V_p(K, L)$ denotes L_p-mixed volume of $K, L \in \mathcal{K}_o^n$ (see [6, 7]) and L^* denotes the polar of L. For the case $p = 1$, $G_p(K)$ is just classical geominimal surface area which is introduced by Petty ([8]). Some

*Corresponding author
Email address: wangwd722@163.com (Wang Weidong)

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affine isoperimetric inequalities related to the classical and L_p geominimal surface areas can be found in [1, 4, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17].

In [7], Lutwak gave the infimum in the definition of L_p-geominimal surface area.

Proposition 1.1. *For $K \in \mathcal{K}_o^n$ and $p \geq 1$, there exists a unique body $T_pK \in \mathcal{T}^n$ with*

$$G_p(K) = nV_p(K, T_pK).$$

Here $\mathcal{T}^n = \{T \in \mathcal{K}^n : s(T) = o, V(T^*) = \omega_n\}$, $s(T)$ denotes the Santaló point of T, the body T_pK is called the p-Petty body of K and T_p^*K denotes the polar of T_pK. When $p = 1$, the subscript will often be suppressed and defined by Petty (see [8]).

From Proposition 1.1, Zhu, Zhou and Xu (see [18]) obtained the following fact.

Proposition 1.2. *For $K \in \mathcal{F}_o^n$ and $p \geq 1$, there exists a unique convex body $T_pK \in \mathcal{T}^n$ with*

$$G_p(K) = \int_{S^{n-1}} h_{T_pK}^p(u) f_p(K, u) dS(u).$$

Here $h_M(\cdot)$ denotes the support function of $M \in \mathcal{K}^n$, $f_p(K, \cdot)$ denotes the L_p-curvature function of $K \in \mathcal{K}_o^n$ and $\mathcal{F}_o^n$ denotes the set of $\mathcal{K}_o^n$ that have a positive continuous L_p-curvature function.

Moreover, Zhu, Zhou and Xu (see [18]) studied the L_p-mixed geominimal surface area. They defined the L_p-mixed geominimal surface areas as follows:

Definition 1.3. Let $K_i \in \mathcal{F}_o^n$ and $p \geq 1$, for each i $(i = 1, \cdots, n)$, there exists a unique body (Petty body of K_i) $T_i = T_pK_i \in \mathcal{T}^n$ with

$$G_p(K_1, \cdots, K_n) = \int_{S^{n-1}} [g_p(K_1, u) \cdots g_p(K_n, u)]^{\frac{1}{n}} dS(u).$$

Here $g_p(K_i, \cdot) = h_{T_i}^p(\cdot) f_p(K_i, \cdot)$.

For the L_p-mixed geominimal surface area, they (see [18]) proved the following results.

Theorem 1.4. *If $n \neq p > 1$, and $K_1, \ldots, K_n \in \mathcal{F}_o^n$, then for $1 \leq m \leq n$,*

$$\Big[G_p(K_1, \ldots, K_n)\Big]^m \leq \prod_{i=0}^{m-1} G_p(K_1, \cdots, K_{n-m}, \underbrace{K_{n-i}, \cdots, K_{n-i}}_{m}).$$

Equality holds if and only if the K_j are dilates of each other for $j = n - m + 1, \cdots, n$. If $m = 1$ equality holds trivially.

Theorem 1.5. *If $K, L \in \mathcal{F}_o^n$, $n \neq p \geq 1$, $i, j, k \in \mathbb{R}$ and $i < j < k$, then*

$$G_{p,j}(K, L)^{k-i} \leq G_{p,i}(K, L)^{k-j} G_{p,k}(K, L)^{j-i}$$

with equality if and only if K and L are dilates.

Theorem 1.6. *If $K, L \in \mathcal{F}_c^n$, $n \neq p \geq 1$, $i \in \mathbb{R}$ and $0 < i < n$, then*

$$G_{p,i}(K, L) G_{p,i}(K^*, L^*) \leq (n\omega_n)^2$$

with equality if and only if K and L are dilated ellipsoids.

Here $\mathcal{F}_c^n$ denotes the set of $\mathcal{K}_c^n$ that have a positive continuous L_p-curvature function.

Based on the L_p-dual mixed volume, Wang and Qi (see [13]) gave the notion of L_p-dual geominimal surface area as follows:

Definition 1.7. For $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-dual geominimal surface area, $\widetilde{G}_{-p}(K)$, of K is defined by

$$\omega_n^{-\frac{p}{n}} \widetilde{G}_{-p}(K) = \inf\{n\widetilde{V}_{-p}(K, L)V(L^*)^{-\frac{p}{n}}; L \in \mathcal{K}_o^n\}. \tag{1.1}$$

Here $\widetilde{V}_{-p}(K, L)$ denotes L_p-dual mixed volume of K and L, and

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u)\rho_L^{-p}(u)dS(u), \tag{1.2}$$

where $\rho_K(\cdot)$ denotes the radial function of $K \in \mathcal{S}_o^n$.

In this paper, we firstly obtain the infimum in the above definition as follows:

Proposition 1.8. *If $K \in \mathcal{K}_o^n$ and $p \geq 1$, then there exists a unique body $\widetilde{K} \in \widetilde{\mathcal{T}}^n$, such that*

$$\widetilde{G}_{-p}(K) = n\widetilde{V}_{-p}(K, \widetilde{K}). \tag{1.3}$$

Here $\widetilde{\mathcal{T}}^n = \{\widetilde{T} \in \mathcal{K}_o^n : V(\widetilde{T}^*) = \omega_n\}$.

By Proposition 1.8 and (1.2), we have the following integral representation of $\widetilde{G}_{-p}(K)$.

Proposition 1.9. *For $K \in \mathcal{K}_o^n$ and $p \geq 1$, there exists a unique body $\widetilde{K} \in \widetilde{\mathcal{T}}^n$ with*

$$\widetilde{G}_{-p}(K) = \int_{S^{n-1}} \rho(K, u)^{n+p}\rho(\widetilde{K}, u)^{-p}dS(u). \tag{1.4}$$

Next, corresponding to Definition 1.3, we define integral form of L_p-dual mixed geominimal surface area by Proposition 1.9.

Definition 1.10. For each $K_i \in \mathcal{K}_o^n$ and $p \geq 1$, there exists a unique body $\widetilde{K}_i \in \widetilde{\mathcal{T}}^n$ $(i = 1, \cdots, n)$ with

$$\widetilde{G}_{-p}(K_1, \ldots, K_n) = \int_{S^{n-1}} [\rho(K_1, u)^{n+p}\rho(\widetilde{K}_1, u)^{-p} \cdots \rho(K_n, u)^{n+p}\rho(\widetilde{K}_n, u)^{-p}]^{\frac{1}{n}} dS(u). \tag{1.5}$$

Let $\widetilde{g}_{-p}(K_i, \cdot) = \rho(K_i, \cdot)^{n+p}\rho(\widetilde{K}_i, \cdot)^{-p}$, then $\widetilde{G}_{-p}(K_1, \ldots, K_n)$ can be written as follows:

$$\widetilde{G}_{-p}(K_1, \ldots, K_n) = \int_{S^{n-1}} [\widetilde{g}_{-p}(K_1, u) \cdots \widetilde{g}_{-p}(K_n, u)]^{\frac{1}{n}} dS(u). \tag{1.6}$$

Let $\underbrace{K_1 = \cdots = K_{n-i}}_{n-i} = K$ and $\underbrace{K_{n-i+1} = \cdots = K_n}_{i} = L$ $(i = 0, 1, \cdots, n)$ in (1.6), we denote $\widetilde{G}_{-p,i}(K, L) = \widetilde{G}_{-p}(\underbrace{K, \cdots, K}_{n-i}, \underbrace{L, \cdots, L}_{i})$. More general, if i is any real, we define that: for $K, L \in \mathcal{K}_o^n$, $p \geq 1$, the ith L_p-dual mixed geominimal surface area, $\widetilde{G}_{-p,i}(K, L)$, of K and L by

$$\widetilde{G}_{-p,i}(K, L) = \int_{S^{n-1}} \widetilde{g}_{-p}(K, u)^{\frac{n-i}{n}} \widetilde{g}_{-p}(L, u)^{\frac{i}{n}} dS(u). \tag{1.7}$$

From Proposition 1.8, we easily know $\widetilde{B} = B$. Thus, let $L = B$ in (1.7) and write

$$\widetilde{G}_{-p,i}(K, B) = \widetilde{G}_{-p,i}(K), \tag{1.8}$$

then (1.7), (1.8) and $\rho(B,\cdot)=1$ yield

$$\widetilde{G}_{-p,i}(K)=\int_{S^{n-1}}\widetilde{g}_{-p}(K,u)^{\frac{n-i}{n}}dS(u). \tag{1.9}$$

Obviously, from (1.6), (1.7) and (1.9), we have

$$\widetilde{G}_{-p,0}(K)=\widetilde{G}_{-p}(K), \tag{1.10}$$

$$\widetilde{G}_{-p,0}(K,L)=\widetilde{G}_{-p}(K),\qquad \widetilde{G}_{-p,n}(K,L)=\widetilde{G}_{-p}(L). \tag{1.11}$$

Further, associated with the L_p-dual mixed geominimal surface areas, we give the following dual results of Theorems 1.4,1.5 and 1.6, respectively.

Theorem 1.11. *If $K_1,\cdots,K_n\in\mathcal{K}_o^n$, $1\le m\le n$, then for $p\ge 1$,*

$$\left[\widetilde{G}_{-p}(K_1,\cdots,K_n)\right]^m\le\prod_{i=0}^{m-1}\widetilde{G}_{-p}(K_1,\cdots,K_{n-m},\underbrace{K_{n-i},\cdots,K_{n-i}}_{m}) \tag{1.12}$$

with equality if and only if there exist positive constants $c_1,c_2,\cdots,c_m$ such that for all $u\in S^{n-1}$,

$$c_1\rho^{n+p}_{K_n}(u)\rho^{-p}_{\widetilde{K}_n}(u)=c_2\rho^{n+p}_{K_{n-1}}(u)\rho^{-p}_{\widetilde{K}_{n-1}}(u)=\cdots=c_m\rho^{n+p}_{K_{n-m+1}}(u)\rho^{-p}_{\widetilde{K}_{n-m+1}}(u).$$

Theorem 1.12. *For $K,L\in\mathcal{K}_o^n$, $p\ge 1$, $i,j,k\in\mathbb{R}$. If $i<j<k$, then*

$$\widetilde{G}_{-p,j}(K,L)^{k-i}\le\widetilde{G}_{-p,i}(K,L)^{k-j}\widetilde{G}_{-p,k}(K,L)^{j-i} \tag{1.13}$$

with equality if and only if K and L are dilates of each other.

Theorem 1.13. *If $K,L\in\mathcal{K}_c^n$, $p\ge 1$, $i\in\mathbb{R}$ and $0\le i\le n$, then*

$$\widetilde{G}_{-p,i}(K,L)\widetilde{G}_{-p,i}(K^*,L^*)\le(n\omega_n)^2 \tag{1.14}$$

with equality if and only if K and L are dilated ellipsoids of each other.

2. Notation and Background Materials

If $K\in\mathcal{K}^n$, then its support function, $h_K=h(K,\cdot):\mathbb{R}^n\longrightarrow(-\infty,+\infty)$, is defined by (see [3])

$$h(K,x)=\max\{x\cdot y:y\in K\},\qquad x\in\mathbb{R}^n, \tag{2.1}$$

where $x\cdot y$ denotes the standard inner product of x and y.

If K is a compact star-shaped (with respect to the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K(\cdot)=\rho(K,\cdot):\mathbb{R}^n\setminus\{0\}\to[0,\infty)$, is defined by (see [3, 10])

$$\rho(K,u)=\max\{\lambda\ge 0:\lambda u\in K\},\quad u\in S^{n-1}. \tag{2.2}$$

If ρ_K is positive and continuous, K will be called a star body (respect to the origin). Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u\in S^{n-1}$.

If E is a nonempty subset in $\mathbb{R}^n$, the polar set, E^*, of E is defined by (see [3, 10])

$$E^*=\{x\in\mathbb{R}^n:x\cdot y\le 1,y\in E\}.$$

If $K\in\mathcal{K}_o^n$, it follows that $K^{**}=K$ and

$$\rho(K,u)^{-1}=h(K^*,u),\quad \rho(K^*,u)^{-1}=h(K,u) \tag{2.3}$$

for all $u \in S^{n-1}$.

For $K \in \mathcal{K}_o^n$ and its polar body, the well-known Blaschke-Stantaló inequality can be stated that (see [3]): *If $K \in \mathcal{K}_c^n$, then*

$$V(K)V(K^*) \leq \omega_n^2 \tag{2.4}$$

with equality if and only if K is an ellipsoid centered at the origin.

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \star K \widetilde{+}_{-p} \mu \star L \in \mathcal{S}_o^n$, of K and L is defined by (see [2, 7])

$$\rho(\lambda \star K \widetilde{+}_{-p} \mu \star L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}, \tag{2.5}$$

where the operation '$\widetilde{+}_{-p}$' is called L_p-harmonic radial addition and $\lambda \star K$ denotes the L_p-harmonic radial scalar multiplication. From (2.5), we can obtain $\lambda \star K = \lambda^{-\frac{1}{p}} K$.

If $K, L \in \mathcal{K}_o^n$ (rather than being in $\mathcal{S}_o^n$), then

$$(\lambda \star K \widetilde{+}_{-p} \mu \star L)^* = \lambda \cdot K^* +_p \mu \cdot L^*. \tag{2.6}$$

For the L_p-harmonic radial combinations, Lutwak (see [7]) proved the following dual L_p-Brunn-Minkowski inequality.

Lemma 2.1. *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), then*

$$V(\lambda \star K \widetilde{+}_{-p} \mu \star L)^{\frac{-p}{n}} \geq \lambda V(K)^{\frac{-p}{n}} + \mu V(L)^{\frac{-p}{n}} \tag{2.7}$$

with equality if and only if K and L are dilates of each other.

3. Proof of Proposition 1.8

In order to prove Proposition 1.8, we require the following lemmas.

Lemma 3.1 ([7]). *Let $\mathcal{C}^n$ denote the set of compact convex subsets of Euclidean n-space $\mathbb{R}^n$. Suppose $K_i \in \mathcal{K}_o^n$ and $K_i \to L \in \mathcal{C}^n$. If the sequence $V(K_i^*)$ is bounded, then $L \in \mathcal{K}_o^n$.*

Lemma 3.2. *Suppose $K_i \to K \in \mathcal{K}_o^n$ and $L_i \to L \in \mathcal{K}_o^n$. If $p \geq 1$, then $\widetilde{V}_{-p}(K_i, L_i) \to \widetilde{V}_{-p}(K, L)$.*

Proof. Since $K_i \to K \in \mathcal{K}_o^n$ and $L_i \to L \in \mathcal{K}_o^n$, $\rho_{K_i} \to \rho_K$ and $\rho_{L_i} \to \rho_L$ uniformly on S^{n-1}. Notice that ρ_K, ρ_L is continuous, the ρ_{K_i} and ρ_{L_i} are uniformly bounded on S^{n-1}. Hence,

$$\rho_{K_i}^{n+p} \to \rho_K^{n+p},\ \rho_{L_i}^{-p} \to \rho_L^{-p}.$$

This yields

$$\int_{S^{n-1}} \rho(K_i, u)^{n+p} \rho(L_i, u)^{-p} du \to \int_{S^{n-1}} [\rho(K, u)^{n+p} \rho(L, u)^{-p}] dS(u),$$

i.e.

$$\widetilde{V}_{-p}(K_i, L_i) \to \widetilde{V}_{-p}(K, L).$$

□

Proof of Proposition 1.8. From the definition of $\widetilde{G}_{-p}(K)$, there exists a sequence $M_i \in \mathcal{K}_o^n$ such that $V(M_i^*) = \omega_n$, with $V_p(K, B) \geq V_p(K, M_i)$, for all i, and

$$\widetilde{V}_{-p}(K, M_i) \to \widetilde{G}_{-p}(K)$$

Since $M_i \in \mathcal{K}_o^n$, thus M_i are uniformly bounded for all i (see [7]). From this, the Blaschke selection theorem guarantees the existence of a subsequence of the M_i, which will also be denoted by M_i, and a compact convex $L \in \mathcal{C}^n$, such that $M_i \to L$. Since $V(M_i^*) = \omega_n$, Lemma 3.1 gives $L \in \mathcal{K}_o^n$. Now, $M_i \to L$ implies that $M_i^* \to L^*$, and since $V(M_i^*) = \omega_n$, it follows that $V(L^*) = \omega_n$. Lemma 3.2 can now be used to conclude that L will serve as the desired body $\widetilde{K}$.

The uniqueness of the minimizing body is easily demonstrated as follows. Suppose $L_1, L_2 \in \mathcal{K}_o^n$, such that $V(L_1^*) = \omega_n = V(L_2^*)$, and

$$\widetilde{V}_{-p}(K, L_1) = \widetilde{V}_{-p}(K, L_2).$$

Define $L \in \mathcal{K}_o^n$, by

$$L = \frac{1}{2} \star L_1 \widetilde{+}_{-p} \frac{1}{2} \star L_2,$$

$$\widetilde{V}_{-p}(K, L) = \widetilde{V}_{-p}(K, L_1) = \widetilde{V}_{-p}(K, L_2).$$

Since obviously,

$$L^* = \frac{1}{2} \cdot L_1^* +_p \frac{1}{2} \cdot L_2^*,$$

and $V(L_1^*) = \omega_n = V(L_2^*)$, it follows from Lemma 2.1 that $V(L^*) \geq \omega_n$, with equality if and only if $L_1 = L_2$. Thus,

$$\widetilde{V}_{-p}(K, L) V(L^*)^{\frac{p}{n}} < \widetilde{V}_{-p}(K, L_1) V(L_1^*)^{\frac{p}{n}}$$

is the contradiction that would arise if it were the case that $L_1 \neq L_2$. This completes the proof. □

4. Proofs of Results

In order to prove Theorem 1.11, we require the following lemma (see [5]).

Lemma 4.1. *If $f_0, f_1, \cdots f_m$ are (strictly) positive continuous functions defined on S^{n-1} and $\lambda_1, \cdots \lambda_m$ are positive constants the sum of whose reciprocals is unity, then*

$$\int_{S^{n-1}} f_0(u) \cdots f_m(u) dS(u) \leq \prod_{i=1}^{m} \left(\int_{S^{n-1}} f_0(u) f_i^{\lambda_i}(u) dS(u) \right)^{\frac{1}{\lambda_i}} \tag{4.1}$$

with equality if and only if there exist positive constants $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that $\alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u)$ for all $u \in S^{n-1}$.

Proof of Theorem 1.11. Let $\lambda_i = m$ $(1 \leq i \leq n)$,

$$f_0(u) = [\widetilde{g}_{-p}(K_1, u) \ldots \widetilde{g}_{-p}(K_{n-m}, u)]^{\frac{1}{n}},$$

$$f_{i+1}(u) = [\widetilde{g}_{-p}(K_{n-i}, u)]^{\frac{1}{n}}, \qquad (0 \leq i \leq m-1),$$

by (1.6) and Lemma 4.1, we get

$$\begin{aligned}
\widetilde{G}_{-p}(K_1, \ldots, K_n) &= \int_{S^{n-1}} [\widetilde{g}_{-p}(K_1, u) \cdots \widetilde{g}_{-p}(K_n, u)]^{\frac{1}{n}} dS(u) \\
&\leq \prod_{i=0}^{m-1} \left(\int_{S^{n-1}} f_0(u) f_{i+1}(u)^m dS(u) \right)^{\frac{1}{m}} \\
&= \prod_{i=0}^{m-1} \left(\int_{S^{n-1}} [\widetilde{g}_{-p}(K_1, u) \cdots \widetilde{g}_{-p}(K_{n-m}, u) \widetilde{g}_{-p}(K_{n-i}, u)^m]^{\frac{1}{n}} dS(u) \right)^{\frac{1}{m}} \\
&= \prod_{i=0}^{m-1} \widetilde{G}_{-p}(K_1, \ldots, K_{n-m}, \underbrace{K_{n-i}, \ldots, K_{n-i}})^{\frac{1}{m}}.
\end{aligned}$$

According to the equality condition of Lemma 4.1, we see that equality holds in inequality (1.12) if and only if there exist positive constants $c_1, c_2, \cdots, c_m$ such that

$$c_1\rho_{K_n}^{n+p}(u)\rho_{\widetilde{K}_n}^{-p}(u) = c_2\rho_{K_{n-1}}^{n+p}(u)\rho_{\widetilde{K}_{n-1}}^{-p}(u) = \cdots = c_m\rho_{K_{n-m+1}}^{n+p}(u)\rho_{\widetilde{K}_{n-m+1}}^{-p}(u)$$

for all $u \in S^{n-1}$. $\square$

Corollary 4.2. *If $K_1, \cdots, K_n \in \mathcal{K}_o^n$, then for $p \geq 1$,*

$$\left[\widetilde{G}_{-p}(K_1, \cdots, K_n)\right]^n \leq \widetilde{G}_{-p}(K_1)\cdots\widetilde{G}_{-p}(K_n) \tag{4.2}$$

with equality if and only if there exist constants $c_1, c_2, \cdots, c_n$ (not all zero) such that for all $u \in S^{n-1}$,

$$c_1\rho_{K_n}^{n+p}(u)\rho_{\widetilde{K}_n}^{-p}(u) = c_2\rho_{K_{n-1}}^{n+p}(u)\rho_{\widetilde{K}_{n-1}(u)}^{-p} = \cdots = c_n\rho_{K_1}^{n+p}(u)\rho_{\widetilde{K}_1}^{-p}(u).$$

Proof. Let $m = n$ in Theorem 1.11 and by (1.1), we easily obtain Corollary 4.2.

According to the equality condition of (1.12), we see that equality holds in (4.2) if and only if there exist constants $c_1, c_2, \cdots, c_n$ (not all zero) such that for all $u \in S^{n-1}$,

$$c_1\rho_{K_n}^{n+p}(u)\rho_{\widetilde{K}_n}^{-p}(u) = c_2\rho_{K_{n-1}}^{n+p}(u)\rho_{\widetilde{K}_{n-1}}^{-p}(u) = \cdots = c_n\rho_{K_1}^{n+p}(u)\rho_{\widetilde{K}_1}^{-p}(u).$$

$\square$

Proof of Theorem 1.12. Since $i < j < k$, thus $\frac{k-i}{k-j} > 1$. From (1.6) and Hölder inequality, we obtain

$$\begin{aligned}
&\widetilde{G}_{-p,i}(K,L)^{\frac{k-j}{k-i}}\widetilde{G}_{-p,k}(K,L)^{\frac{j-i}{k-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}}\widetilde{g}_{-p}(K,u)^{\frac{n-i}{n}}\widetilde{g}_{-p}(L,u)^{\frac{i}{n}}dS(u)\right]^{\frac{k-j}{k-i}} \\
&\quad\times\left[\frac{1}{n}\int_{S^{n-1}}\widetilde{g}_{-p}(K,u)^{\frac{n-k}{n}}\widetilde{g}_{-p}(L,u)^{\frac{k}{n}}dS(u)\right]^{\frac{j-i}{k-i}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}}[\widetilde{g}_{-p}(K,u)^{\frac{(n-i)(k-j)}{n(k-i)}}\widetilde{g}_{-p}(L,u)^{\frac{i(k-j)}{n(k-i)}}]^{\frac{k-i}{k-j}}dS(u)\right]^{\frac{k-j}{k-i}} \\
&\quad\times\left[\frac{1}{n}\int_{S^{n-1}}[\widetilde{g}_{-p}(K,u)^{\frac{(n-k)(j-i)}{n(k-i)}}\widetilde{g}_{-p}(L,u)^{\frac{k(j-i)}{n(k-i)}}]^{\frac{k-i}{j-i}}dS(u)\right]^{\frac{j-i}{k-i}} \\
&\geq \frac{1}{n}\int_{S^{n-1}}\widetilde{g}_{-p}(K,u)^{\frac{n-j}{n}}\widetilde{g}_{-p}(L,u)^{\frac{j}{n}}dS(u) \\
&= \widetilde{G}_{-p,j}(K,L).
\end{aligned}$$

This gives the desired inequality (1.13). According to the equality conditions of the Hölder inequality, we know that equality holds in (1.13) if and only if there exists a constant $\lambda > 0$ such that

$$\left[\widetilde{g}_{-p}(K,u)^{\frac{(n-i)(k-j)}{n(k-i)}}\widetilde{g}_{-p}(L,u)^{\frac{i(k-j)}{n(k-i)}}\right]^{\frac{k-i}{k-j}} = \lambda\left[\widetilde{g}_{-p}(K,u)^{\frac{(n-k)(j-i)}{n(k-i)}}\widetilde{g}_{-p}(L,u)^{\frac{k(j-i)}{n(k-i)}}\right]^{\frac{k-i}{j-i}},$$

i.e. $\widetilde{g}_{-p}(K,u) = \lambda\widetilde{g}_{-p}(L,u)$ for all $u \in S^{n-1}$. Thus equality holds in (1.13) if and only if K and L are dilates of each other. $\square$

Let $L = B$ in Theorem 1.12 and use (1.8), we obtain the following corollary.

Corollary 4.3. *If $K \in \mathcal{K}_o^n$, $p \geq 1$, $i, j, k \in \mathbb{R}$ and $i < j < k$, then*

$$\widetilde{G}_{-p,j}(K)^{k-i} \leq \widetilde{G}_{-p,i}(K)^{k-j} \widetilde{G}_{-p,k}(K)^{j-i} \tag{4.3}$$

with equality if and only if K and L are dilates of each other.

By Theorem 1.12, we also get the Minkowski inequality for the L_p-dual mixed geominimal surface area as follows:

Corollary 4.4. *If $K, L \in \mathcal{K}_o^n$, $n \neq p \geq 1$ and $i \in \mathbb{R}$, then for $i < 0$ or $i > n$,*

$$\widetilde{G}_{-p,i}(K, L)^n \geq \widetilde{G}_{-p}(K)^{n-i} \widetilde{G}_{-p}(L)^i; \tag{4.4}$$

for $0 < i < n$,

$$\widetilde{G}_{-p,i}(K, L)^n \leq \widetilde{G}_{-p}(K)^{n-i} \widetilde{G}_{-p}(L)^i. \tag{4.5}$$

In every case, equality holds if and only if K and L are dilates of each other.

Proof. For $i < 0$, take $(i, j, k) = (i, 0, n)$ in Theorem 1.12, we have

$$\widetilde{G}_{-p,i}(K, L)^n \widetilde{G}_{-p,n}(K, L)^{-i} \geq \widetilde{G}_{-p,0}(K, L)^{n-i}, \tag{4.6}$$

i.e.

$$\widetilde{G}_{-p,i}(K, L)^n \geq \widetilde{G}_{-p}(K)^{n-i} \widetilde{G}_{-p}(L)^i \tag{4.7}$$

with equality if and only if K and L are dilates of each other.

In the same way, let $(i, j, k) = (0, n, i)$ for $i > n$ in Theorem 1.12, we obtain

$$\widetilde{G}_{-p,i}(K, L)^n \geq \widetilde{G}_{-p}(K)^{n-i} \widetilde{G}_{-p}(L)^i \tag{4.8}$$

with equality if and only if K and L are dilates of each other.

Similarly, let $(i, j, k) = (0, i, n)$ for $0 < i < n$ in Theorem 1.12, we easily get

$$\widetilde{G}_{-p,i}(K, L)^n \leq \widetilde{G}_{-p}(K)^{n-i} \widetilde{G}_{-p}(L)^i \tag{4.9}$$

with equality if and only if K and L are dilates of each other. □

Let $L = B$ in Corollary 4.4, and notice $G_{-p}(B) = n\omega_n$ by (1.9), we have that:

Corollary 4.5. *If $K \in \mathcal{K}_o^n$, $n \neq p \geq 1$, $i \in \mathbb{R}$, then for $i < 0$ or $i > n$,*

$$\widetilde{G}_{-p,i}(K)^n \geq (n\omega_n)^i \widetilde{G}_{-p}(K)^{n-i}; \tag{4.10}$$

for $0 < i < n$

$$\widetilde{G}_{-p,i}(K)^n \leq (n\omega_n)^i \widetilde{G}_{-p}(K)^{n-i}. \tag{4.11}$$

In each case, equality holds if and only if K is a ball centered at the origin.

In order to prove Theorem 1.13, we require the following result (see [13]).

Lemma 4.6. *If $K \in \mathcal{K}_c^n$, $n \geq p \geq 1$, then*

$$\widetilde{G}_{-p}(K) \widetilde{G}_{-p}(K^*) \leq (n\omega_n)^2$$

with equality if and only if K is an ellipsoid.

Proof of Theorem 1.13. For $0 < i < n$, $n \geq p \geq 1$, and by (4.5) and Lemma 4.6, we have

$$\widetilde{G}_{-p,i}(K, L)^n \widetilde{G}_{-p,i}(K^*, L^*)^n \leq [\widetilde{G}_{-p}(K) \widetilde{G}_{-p}(K^*)]^{n-i} [\widetilde{G}_{-p}(L) \widetilde{G}_{-p}(L^*)]^i \leq (n\omega_n)^{2n}$$

with equality if and only if K and L are dilated ellipsoids of each other. □

For the L_p-dual geominimal surface area, Wang and Qi (see [13]) proved the following result.

Lemma 4.7. *If $K \in \mathcal{K}_c^n$, $p \geq 1$, then*

$$\widetilde{G}_{-p}(K) \geq n(\omega_n)^{-\frac{p}{n}} V(K)^{\frac{n+p}{n}} \tag{4.12}$$

with equality if and only if K is an ellipsoid centered at the origin.

Combining with (4.12), we can prove the following fact.

Theorem 4.8. *If $K, L \in \mathcal{K}_c^n$, $p \geq 1$, $i \in \mathbb{R}$ and $i < 0$, then*

$$\widetilde{G}_{-p,i}(K) \geq n\omega_n^{\frac{(n+p)i-pn}{n^2}} V(K)^{\frac{(n+p)(n-i)}{n^2}} \tag{4.13}$$

with equality if and only if K is a ball centered at the origin.

Proof. For $i < 0$, by (4.10) and (4.12), we get

$$\begin{aligned}\widetilde{G}_{-p,i}(K)^n &\geq (n\omega_n)^i \widetilde{G}_{-p}(K)^{n-i} \\ &\geq (n\omega_n)^i [n(\omega_n)^{-\frac{p}{n}} V(K)^{\frac{n+p}{n}}]^{n-i} \\ &= n^n \omega_n^{\frac{(n+p)i-pn}{n}} V(K)^{\frac{(n+p)(n-i)}{n}},\end{aligned}$$

i.e.

$$\widetilde{G}_{-p,i}(K) \geq n\omega_n^{\frac{(n+p)i-pn}{n^2}} V(K)^{\frac{(n+p)(n-i)}{n^2}}$$

with equality if and only if K is a ball centered at the origin. □

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