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Inverse problems for a nonlocal wave equation with an involution perturbation

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Abstract

Two inverse problems for the wave equation with involution are considered. Results on existence and uniqueness of solutions of these problems are presented.©2016 All rights reserved.

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1. Introduction and Preliminaries

As it is well documented, differential evolution equations with time delays have been the subject of a huge number of articles and a sizeable number of books either from the theoretical point of view or for practical considerations, see for example [12] and [34]. Also, differential equations with operations on the space variable received great attention starting with Carleman [6] (equations with shift (involution)) and followed with great attention by Przewoerska-Rolewicz [17, 18, 19, 20, 21, 22, 23], Aftabizadeh et al. [1], Andreev [3], [4], Burlutskayaa et al. [5], Gupta [7], [8], [9], Watkins [31], Viner [29], [30], and Wiener [32], [33]; for spectral problems and inverse problems for equations with involutions we may cite the recent works of Kaliev et al. [10], [11], Sadybekov et al. [15], [16], [25], Sarsenbi et al. [13], [26] and [27]. The papers of Aliev [2] and Rus [24] concern the maximum principle for equations with a delay in the space variable. For general facts about partial functional-differential equations (equations with transformations in the arguments), we refer to the books of Skubachevskii [28] and Wu [34]. In this paper, we consider inverse problems for the wave equation with involution.

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2. Statement of the problems

Let $\Omega = \{-\pi \leq x \leq \pi, 0 \leq t \leq T\}$ be a rectangular domain. In this domain, we consider the equation

$$u_{tt}(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x), \ (x,t) \in \Omega, \tag{2.1}$$

where, ε is a nonzero real number such that $|\varepsilon| < 1$.

We specifically consider the following two problems.

Problem P1. Find a pair of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (2.1) and the following conditions:

$$u(x,0) = \phi(x), \ \ u_t(x,0) = \rho(x), \ \ u(x,T) = \psi(x), \ x \in [-\pi, \pi], \tag{2.2}$$

$$u(-\pi, t) = 0, \ \ u(\pi, t) = 0, \ t \in [0,T], \tag{2.3}$$

where $\phi(x)$ and $\psi(x)$ are given sufficiently smooth functions.

Problem P2. Find a pair of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (2.1), conditions (2.2), and

$$u_x(-\pi, t) = 0, \ \ u_x(\pi, t) = 0, \ t \in [0,T]. \tag{2.4}$$

By a regular solution to Problem P1 or P2, we mean a pair of functions $(u(x,t), f(x))$, where $u(x,t) \in C^{2,2}_{x,t}(\Omega)$, $f(x) \in C[-\pi, \pi]$.

3. The Spectral Problem

Using the method of separation of variables for solving the homogeneous partial differential equation in problems P1 and P2 leads to the spectral problem consisting of the equation

$$X''(x) - \varepsilon X''(-x) + \lambda X(x) = 0, \quad -\pi \leq x \leq \pi, \tag{3.1}$$

and one of the following boundary conditions

$$X(-\pi) = X(\pi) = 0, \tag{3.2}$$

$$X'(-\pi) = X'(\pi) = 0. \tag{3.3}$$

The Sturm–Liouville problem for the equation (3.1) with one of the boundary conditions (3.2)–(3.3) is self-adjoint; it has real eigenvalues and their corresponding eigenfunctions form a complete orthonormal basis in $L_2(-\pi, \pi)$ [14]. To further investigate the problems under consideration, we need to calculate the explicit form of the eigenvalues and eigenfunctions.

By expressing the function X as a sum of even and odd functions [13], it can be shown that the problem (3.1), (3.2) has the following eigenvalues

$$\lambda_{k,1} = (1+\varepsilon)k^2, \ k \in \mathbb{N}, \quad \lambda_{k,2} = (1-\varepsilon)\left(k+\frac{1}{2}\right)^2, \ k \in \mathbb{N} \cup \{0\},$$

and the corresponding normalized eigenfunctions are given by

$$X_{k,1} = \frac{1}{\sqrt{\pi}}\sin kx, \ k \in \mathbb{N}, \quad X_{k,2} = \frac{1}{\sqrt{\pi}}\cos\left(k+\frac{1}{2}\right)x, \ k \in \mathbb{N} \cup \{0\}. \tag{3.4}$$

Similarly, the eigenvalues for the problem (3.1), (3.3) are given by

$$\lambda_{k,1} = (1+\varepsilon)\left(k+\frac{1}{2}\right)^2, \quad \lambda_{k,2} = (1-\varepsilon)k^2, \ k \in \mathbb{N} \cup \{0\},$$

and the corresponding normalized eigenfunctions are

$$X_{k,1} = \frac{1}{\sqrt{\pi}}\sin\left(k+\frac{1}{2}\right)x, \quad X_{k,2} = \frac{1}{\sqrt{\pi}}\cos kx, \ k \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

The systems of functions (3.4) and (3.5) are complete in $L_2(-\pi, \pi)$.

4. Main results

Here we present the existence and uniqueness results for Problems P1 and P2.

Theorem 4.1. *Let* $\phi(x), \rho(x), \psi(x) \in C^4[-\pi, \pi]$ *and* $\phi^{(i)}(\pm\pi) = \rho^{(i)}(\pm\pi) = \psi^{(i)}(\pm\pi) = 0$, $i = 0, 1, 2, 3$. *Then, for a nonzero real number* ε *such that* $|\varepsilon| < 1$, *Problem P1 has a unique solution which can be written in the form*

$$u(x,t) = \phi(x) + \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\left(C^{(4)}\right)_{1k} \cos \sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t}{\left(k+\frac{1}{2}\right)^4} \cos\left(k+\frac{1}{2}\right)x$$
$$+ \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\left(C^{(4)}\right)_{3k} \sin \sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t - \left(C^{(4)}\right)_{1k}}{\left(k+\frac{1}{2}\right)^4} \cos\left(k+\frac{1}{2}\right)x$$
$$+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\left(C^{(4)}\right)_{2k} \cos \sqrt{1+\varepsilon}\,kt + \left(C^{(4)}\right)_{4k} \sin \sqrt{1+\varepsilon}\,kt - \left(C^{(4)}\right)_{2k}}{k^4} \sin kx,$$

and

$$f(x) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\left(\left(\phi^{(4)}\right)_{1k} - \left(C^{(4)}\right)_{1k}\right)}{\left(k+\frac{1}{2}\right)^2} \cos\left(k+\frac{1}{2}\right)x$$
$$+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\left(\left(\phi^{(4)}\right)_{2k} - \left(C^{(4)}\right)_{2k}\right)}{k^2} \sin kx,$$

where,

$$\left(C^{(4)}\right)_{1k} = \frac{\left(\phi^{(4)}\right)_{1k} - \left(\psi^{(4)}\right)_{1k} + \left(C^{(4)}\right)_{3k} \sin \sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T}{1 - \cos \sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T},$$
$$\left(C^{(4)}\right)_{3k} = \frac{\left(\rho^{(4)}\right)_{1k}}{\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)},$$
$$\left(C^{(4)}\right)_{2k} = \frac{\left(\phi^{(4)}\right)_{2k} - \left(\psi^{(4)}\right)_{2k} + \left(C^{(4)}\right)_{4k} \sin \sqrt{1+\varepsilon}\,kT}{1 - \cos \sqrt{1+\varepsilon}\,kT},$$
$$\left(C^{(4)}\right)_{4k} = \frac{\left(\rho^{(4)}\right)_{2k}}{\sqrt{1+\varepsilon}\,k},$$

and

$$\left(g^{(4)}\right)_{1k} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} g^{(4)}(x) \cos\left(k+\frac{1}{2}\right)x \, dx, \quad \left(g^{(4)}\right)_{2k} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} g^{(4)}(x) \sin kx \, dx, \quad for \quad g = \phi, \psi, \rho.$$

Theorem 4.2. *Let* $\phi(x), \rho(x), \psi(x) \in C^4[-\pi, \pi]$ *and* $\phi^{(i)}(\pm\pi) = \rho^{(i)}(\pm\pi) = \psi^{(i)}(\pm\pi) = 0$, $i = 0, 1, 2, 3$. *Then, for a nonzero real number* ε *such that* $|\varepsilon| < 1$, *Problem P2 has a unique solution which can be written in the form*

$$u(x,t) = \phi(x) + \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\left(C^{(4)}\right)_{1k} \cos \sqrt{1-\varepsilon}\,kt + \left(C^{(4)}\right)_{3k} \sin \sqrt{1-\varepsilon}\,kt - \left(C^{(4)}\right)_{1k}}{k^4} \cos kx$$
$$+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\left(C^{(4)}\right)_{2k} \cos \sqrt{1+\varepsilon}\left(k+\frac{1}{2}\right)t}{\left(k+\frac{1}{2}\right)^4} \sin\left(k+\frac{1}{2}\right)x,$$
$$+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\left(C^{(4)}\right)_{4k} \sin \sqrt{1+\varepsilon}\left(k+\frac{1}{2}\right)t - \left(C^{(4)}\right)_{2k}}{\left(k+\frac{1}{2}\right)^4} \sin\left(k+\frac{1}{2}\right)x,$$

and

$$f(x) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\left(\left(\phi^{(4)}\right)_{1k} - \left(C^{(4)}\right)_{1k}\right)}{k^2} \cos kx$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\left(\left(\phi^{(4)}\right)_{2k} - \left(C^{(4)}\right)_{2k}\right)}{\left(k+\frac{1}{2}\right)^2} \sin\left(k+\frac{1}{2}\right) x,$$

where,

$$\left(C^{(4)}\right)_{1k} = \frac{\left(\phi^{(4)}\right)_{1k} - \left(\psi^{(4)}\right)_{1k} + \left(C^{(4)}\right)_{3k} \sin\sqrt{1-\varepsilon}\,kT}{1 - \cos\sqrt{1-\varepsilon}\,kT},$$

$$\left(C^{(4)}\right)_{3k} = \frac{\left(\rho^{(4)}\right)_{1k}}{\sqrt{1-\varepsilon}\,k},$$

$$\left(C^{(4)}\right)_{2k} = \frac{\left(\phi^{(4)}\right)_{2k} - \left(\psi^{(4)}\right)_{2k} + \left(C^{(4)}\right)_{4k} \sin\sqrt{1+\varepsilon}\left(k+\frac{1}{2}\right) T}{1 - \cos\sqrt{1+\varepsilon}\left(k+\frac{1}{2}\right) T},$$

$$\left(C^{(4)}\right)_{4k} = \frac{\left(\rho^{(4)}\right)_{2k}}{\sqrt{1+\varepsilon}\left(k+\frac{1}{2}\right)},$$

and

$$\left(g^{(4)}\right)_{1k} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} g^{(4)}(x) \cos kx \ dx, \quad \left(g^{(4)}\right)_{2k} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} g^{(4)}(x) \sin\left(k+\frac{1}{2}\right) x \ dx, \quad for \quad g = \phi, \psi, \rho.$$

5. Proofs of Results

5.1. Existence

Here, we give a full proof of the existence of a solution to Problem P1 as given in Theorem 4.1.

As the eigenfunctions system (3.4), corresponding to Problem P1, forms an orthonormal basis in $L_2(-\pi, \pi)$ (this follows from the self-adjoint problem (3.1), (3.2)), the functions $u(x,t)$ and $f(x)$ can be represented as follows

$$u(x,t) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} u_k(t) \cos\left(k+\frac{1}{2}\right) x + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} v_k(t) \sin kx, \tag{5.1}$$

and

$$f(x) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} f_{1k} \cos\left(k+\frac{1}{2}\right) x + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} f_{2k} \sin kx, \tag{5.2}$$

where $u_k(t)$, $v_k(t)$, f_{1k} and f_{2k} are unknown. Substituting (5.1) and (5.2) into equation (2.1), we obtain the following equations for the functions $u_k(t)$, $v_k(t)$ and the constants f_{1k}, f_{2k}:

$$u_k''(t) + (1-\varepsilon)\left(k+\frac{1}{2}\right)^2 u_k(t) = f_{1k},$$

and

$$v_k''(t) + (1+\varepsilon)k^2 v_k(t) = f_{2k}.$$

Solving these two equations, we get

$$u_k(t) = C_{1k} \cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right) t + C_{3k} \sin\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right) t + \frac{f_{1k}}{(1-\varepsilon)\left(k+\frac{1}{2}\right)^2}$$

and

$$v_k(t) = C_{2k} \cos \sqrt{1+\varepsilon} \, kt + C_{4k} \sin \sqrt{1+\varepsilon} \, kt + \frac{f_{2k}}{(1+\varepsilon) \, k^2},$$

where the constants C_{1k}, C_{2k}, C_{3k}, C_{4k}, f_{1k}, and f_{2k} are to be determined using the conditions in (2.2); expanding the functions $\phi(x), \rho(x)$ and $\psi(x)$ using the eigenfunctions system (3.4), we obtain

$$C_{1k} + \frac{f_{1k}}{(1-\varepsilon)\left(k+\frac{1}{2}\right)^2} = \phi_{1k}, \quad C_{2k} + \frac{f_{2k}}{(1+\varepsilon) \, k^2} = \phi_{2k}$$

$$\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)C_{3k} = \rho_{1k}, \qquad \sqrt{1+\varepsilon} \, kC_{4k} = \rho_{2k}$$

$$C_{1k} \cos \sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T + C_{3k} \sin \sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T + \frac{f_{1k}}{(1-\varepsilon)\left(k+\frac{1}{2}\right)^2} = \psi_{1k},$$

and

$$C_{2k} \cos \sqrt{1+\varepsilon} \, kT + C_{4k} \sin \sqrt{1+\varepsilon} \, kT + \frac{f_{2k}}{(1+\varepsilon) \, k^2} = \psi_{2k},$$

where, $\phi_{ik}, \rho_{ik}, \psi_{ik}, \, i = 1, 2$ are the coefficients of the expansions of the functions $\phi(x), \rho(x), \psi(x)$ that are given by

$$g_{1k} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} g(x) \cos \left(k+\frac{1}{2}\right) x \, dx, \quad g_{2k} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} g(x) \sin kx \, dx, \quad \text{for} \quad g = \phi, \rho, \psi.$$

Solving the above set of equations for C_{1k}, C_{2k}, C_{3k}, C_{4k}, f_{1k}, and f_{2k}, we get

$$C_{1k} = \frac{\phi_{1k} - \psi_{1k} + C_{3k} \sin \sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T}{1 - \cos \sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T},$$

$$C_{3k} = \frac{\rho_{1k}}{\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)},$$

$$C_{2k} = \frac{\phi_{2k} - \psi_{2k} + C_{4k} \sin \sqrt{1+\varepsilon} \, kT}{1 - \cos \sqrt{1+\varepsilon} \, kT},$$

$$C_{4k} = \frac{\rho_{2k}}{\sqrt{1+\varepsilon} \, k},$$

and

$$f_{1k} = (1-\varepsilon)\left(k+\frac{1}{2}\right)^2 (\phi_{1k} - C_{1k}), \qquad f_{2k} = (1+\varepsilon) \, k^2 (\phi_{2k} - C_{2k}). \tag{5.3}$$

Now, substituting $u_k(t)$, $v_k(t)$, f_{1k}, f_{2k} into (5.1) and (5.2), we obtain

$$u(x,t) = \phi(x) + \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \left(C_{1k} \cos \sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t + C_{3k} \sin \sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)t - C_{1k}\right) \cos \left(k+\frac{1}{2}\right)x$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \left(C_{2k} \cos \sqrt{1+\varepsilon} \, kt + C_{4k} \sin \sqrt{1+\varepsilon} \, kt - C_{2k}\right) \sin kx,$$

and

$$f(x) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} (1-\varepsilon)\left(k+\frac{1}{2}\right)^2 (\phi_{1k} - C_{1k}) \cos \left(k+\frac{1}{2}\right)x + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} (1+\varepsilon) \, k^2 (\phi_{2k} - C_{2k}) \sin kx.$$

Moreover, if $\phi^{(i)}(\pm\pi) = \rho^{(i)}(\pm\pi) = \psi^{(i)}(\pm\pi) = 0$, $i = 0, 1, 2, 3$, then on integration by parts, $\phi_{ik}, \rho_{ik}, \psi_{ik}$, $i = 1, 2$, can be written as

$$g_{1k} = \frac{\left(g^{(4)}\right)_{1k}}{\left(k + \frac{1}{2}\right)^4} \quad \text{and} \quad g_{2k} = \frac{\left(g^{(4)}\right)_{2k}}{k^4}$$

for $g = \phi, \rho, \psi$, where, $\left(\phi^{(4)}\right)_{ik}, \left(\rho^{(4)}\right)_{ik}, \left(\psi^{(4)}\right)_{ik}, i = 1, 2$ are the coefficients of the expansions of the functions $\phi^{(4)}(x)$, $\rho^{(4)}(x)$, $\psi^{(4)}(x)$ that are given by

$$\left(g^{(4)}\right)_{1k} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} g^{(4)}(x) \cos\left(k + \frac{1}{2}\right) x \, dx, \quad \left(g^{(4)}\right)_{2k} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} g^{(4)}(x) \sin kx \, dx,$$

for $g = \phi, \psi, \rho$.

Then the constants C_{1k}, C_{2k}, C_{3k}, C_{4k}, f_{1k}, and f_{2k} can be written as

$$C_{1k} = \frac{\left(C^{(4)}\right)_{1k}}{\left(k + \frac{1}{2}\right)^4}, \qquad C_{3k} = \frac{\left(C^{(4)}\right)_{3k}}{\left(k + \frac{1}{2}\right)^4},$$

$$C_{2k} = \frac{\left(C^{(4)}\right)_{2k}}{k^4}, \qquad C_{4k} = \frac{\left(C^{(4)}\right)_{4k}}{k^4},$$

and

$$f_{1k} = \frac{(1-\varepsilon)}{\left(k + \frac{1}{2}\right)^2} \left(\left(\phi^{(4)}\right)_{1k} - \left(C^{(4)}\right)_{1k}\right), \qquad f_{2k} = \frac{(1+\varepsilon)}{k^2} \left(\left(\phi^{(4)}\right)_{2k} - \left(C^{(4)}\right)_{2k}\right), \tag{5.4}$$

where,

$$\left(C^{(4)}\right)_{1k} = \frac{\left(\phi^{(4)}\right)_{1k} - \left(\psi^{(4)}\right)_{1k} + \left(C^{(4)}\right)_{3k} \sin\sqrt{1-\varepsilon}\left(k + \frac{1}{2}\right) T}{1 - \cos\sqrt{1-\varepsilon}\left(k + \frac{1}{2}\right) T}, \quad \left(C^{(4)}\right)_{3k} = \frac{\left(\rho^{(4)}\right)_{1k}}{\sqrt{1-\varepsilon}\left(k + \frac{1}{2}\right)},$$

$$\left(C^{(4)}\right)_{2k} = \frac{\left(\phi^{(4)}\right)_{2k} - \left(\psi^{(4)}\right)_{2k} + \left(C^{(4)}\right)_{4k} \sin\sqrt{1+\varepsilon}\, kT}{1 - \cos\sqrt{1+\varepsilon}\, kT}, \qquad \left(C^{(4)}\right)_{4k} = \frac{\left(\rho^{(4)}\right)_{2k}}{\sqrt{1+\varepsilon}\, k}.$$

Hence, the solution to Problem P1 can be written as

$$u(x, t) = \phi(x) + \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\left(C^{(4)}\right)_{1k} \cos\sqrt{1-\varepsilon}\left(k + \frac{1}{2}\right) t}{\cos} \left(k + \frac{1}{2}\right) x$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\left(C^{(4)}\right)_{3k} \sin\sqrt{1-\varepsilon}\left(k + \frac{1}{2}\right) t - \left(C^{(4)}\right)_{1k}}{\left(k + \frac{1}{2}\right)^4} \cos\left(k + \frac{1}{2}\right) x$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\left(C^{(4)}\right)_{2k} \cos\sqrt{1+\varepsilon}\, kt + \left(C^{(4)}\right)_{4k} \sin\sqrt{1+\varepsilon}\, kt - \left(C^{(4)}\right)_{2k}}{k^4} \sin kx, \tag{5.5}$$

and

$$f(x) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(1-\varepsilon)\left(\left(\phi^{(4)}\right)_{1k} - \left(C^{(4)}\right)_{1k}\right)}{\left(k + \frac{1}{2}\right)^2} \cos\left(k + \frac{1}{2}\right) x$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{(1+\varepsilon)\left(\left(\phi^{(4)}\right)_{2k} - \left(C^{(4)}\right)_{2k}\right)}{k^2} \sin kx. \tag{5.6}$$

This ends the proof of Theorem 4.1.

Similarly, one can prove the existence of a formal solution to Problem P2 as given in Theorem 4.2.

5.2. Convergence of the series

In order to justify that the formal solution is indeed a true solution, we will prove that all the operations performed before are valid.

The convergence of the series in (5.5) and (5.6) are based on the following estimates for $u(x,t)$ and $f(x)$,

$$|u(x,t)| \leq |\phi(x)| + \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\sqrt{2}\left|\phi_{1k}^{(4)}\right| + \sqrt{2}\left|\psi_{1k}^{(4)}\right| + 4\left|\rho_{1k}^{(4)}\right|}{\left(\sqrt{1-\varepsilon}\right)\left(1 - \cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)\left(k+\frac{1}{2}\right)^4}$$

$$+ \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\sqrt{2}\left|\phi_{2k}^{(4)}\right| + \sqrt{2}\left|\psi_{2k}^{(4)}\right| + 2\left|\rho_{2k}^{(4)}\right|}{\left(\sqrt{1+\varepsilon}\right)\left(1 - \cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)k^4} \tag{5.7}$$

and

$$|f(x)| \leq \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{6\left|\phi_{1k}^{(4)}\right| + 2\left|\psi_{1k}^{(4)}\right| + 2\sqrt{2}\left|\rho_{1k}^{(4)}\right|}{\left(1 - \cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)\left(k+\frac{1}{2}\right)^2} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{6\left|\phi_{2k}^{(4)}\right| + 2\left|\psi_{2k}^{(4)}\right| + \sqrt{2}\left|\rho_{2k}^{(4)}\right|}{\left(1 - \cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)k^2}. \tag{5.8}$$

Since $\phi(x), \rho(x), \psi(x) \in C^4[-\pi,\pi]$, then by the Bessel inequality for trigonometric series, the following series converge:

$$\sum_{k=0}^{\infty} \left|g_{1k}^{(4)}\right|^2 \leq C \left\|g^{(4)}(x)\right\|_{L_2(-\pi,\pi)}^2, \quad for \quad g = \phi, \rho, \psi \tag{5.9}$$

and

$$\sum_{k=1}^{\infty} \left|g_{2k}^{(4)}\right|^2 \leq C \left\|g^{(4)}(x)\right\|_{L_2(-\pi,\pi)}^2, \quad for \quad g = \phi, \rho, \psi, \tag{5.10}$$

which implies that the set

$$\left\{\varphi_{ik}^{(4)}, \rho_{ik}^{(4)}, \psi_{ik}^{(4)}\right\}, \, k = 1, 2.$$

is bounded. Therefore, by the Weierstrass M-test, the series in (5.7) and (5.8) converge absolutely and uniformly in the region Ω.

Now, using termwise differentiation of the series in (5.5) twice with respect to the variables x and t, we get the following estimates for $u_{xx}(x,t)$ and $u_{tt}(x,t)$,

$$|u_{xx}(x,t)| \leq |\phi''(x)| + \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\sqrt{2}\left|\phi_{1k}^{(4)}\right| + \sqrt{2}\left|\psi_{1k}^{(4)}\right| + 4\left|\rho_{1k}^{(4)}\right|}{\left(\sqrt{1-\varepsilon}\right)\left(1 - \cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)\left(k+\frac{1}{2}\right)^2}$$

$$+ \frac{2}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\sqrt{2}\left|\phi_{2k}^{(4)}\right| + \sqrt{2}\left|\psi_{2k}^{(4)}\right| + 2\left|\rho_{2k}^{(4)}\right|}{\left(\sqrt{1+\varepsilon}\right)\left(1 - \cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)k^2}$$

and

$$|u_{tt}(x,t)| \leq \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{2\left|\phi_{1k}^{(4)}\right| + 2\left|\psi_{1k}^{(4)}\right| + 6\sqrt{2}\left|\rho_{1k}^{(4)}\right|}{\left(1 - \cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)\left(k+\frac{1}{2}\right)^2} + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{2\left|\phi_{2k}^{(4)}\right| + 2\left|\psi_{2k}^{(4)}\right| + 3\sqrt{2}\left|\rho_{2k}^{(4)}\right|}{\left(1 - \cos\sqrt{1-\varepsilon}\left(k+\frac{1}{2}\right)T\right)k^2},$$

which, by using (5.9), (5.10) and the Weierstrass M-test, also converge absolutely and uniformly in Ω.

5.3. Uniqueness

The uniqueness of the solutions to Problems 1 and 2 easily follows from representations of the solutions given in the theorems above, and from the completeness of systems (3.4) and (3.5).

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