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New higher order multi-step methods for solving scalar equations

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Abstract

We introduce and analyze two new multi-step iterative methods with convergence order four and five based on modified homotopy perturbation methods, using the system of coupled equations involving an auxiliary function. We also present the convergence analysis and various numerical examples to demonstrate the validity and efficiency of our methods. These methods are a good addition and also a generalization of the existing methods for solving nonlinear equations.

Keywords: Iterative methods, nonlinear equations, order of convergence, auxiliary function, modified homotopy method.
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1. Introduction

Most of the problems in diverse fields of mathematics and engineering lead to the nonlinear equations whose exact solutions are quite difficult or even impossible to find. Accordingly, the development of numerical techniques to solve the nonlinear equations has gained huge devotion of scientists and engineers. Various iterative methods involving different techniques including Taylor series, quadrature formulas, decomposition, homotopy, etc., have been introduced for the purpose, see [1, 2, 3, 4, 5, 6, 8, 9, 10, 15, 18, 19] and references

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therein. These include the methods with quadratic, cubic and higher order convergence. Chun [4] introduced an iterative method with fourth order convergence, using the technique of decomposition, in 2005.

The homotopy perturbation method (HPM) was first introduced by He [7] in 1999. He further modified his method in different ways [10, 11, 12, 13, 14]. Afterward, HPM has been used extensively by researchers in order to solve linear and nonlinear equations. In 2009, Javidi [16] developed fourth and fifth order methods using modified HPM for solving nonlinear equations.

Shah and Noor [18] have proposed some algorithms, using auxiliary functions and decomposition technique to find solutions of nonlinear equations. They write the original nonlinear equation in the form of a coupled system to achieve the results.

In the present paper, we construct two new methods, following the technique of Shah and Noor [18] based on HPM with convergence order four and five, for solving nonlinear equations. The methods of Javidi [16] are particular cases of our iterative schemes. The performance of our proposed methods has revealed through a comparative study with some known methods, by considering some test examples.

2. Iterative Methods

Consider the nonlinear equation

$$f(x) = 0. \tag{2.1}$$

Let α be a simple zero of Equation (2.1) and γ be an initial guess sufficiently close to α. Assume that $g(x)$ is an auxiliary function such that

$$f(x)g(x) = 0. \tag{2.2}$$

Using Taylor series, we write nonlinear Equation (2.2) in the form of the coupled system as follows:

$$f(\gamma)g(\gamma) + [f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)](x-\gamma) + h(x) = 0. \tag{2.3}$$

Equation (2.3) can be written in the from

$$h(x) = f(x)g(\gamma) - f(\gamma)g(\gamma) - [f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)](x-\gamma). \tag{2.4}$$

Also, Equation (2.3) can be rewritten as

$$x = \gamma - \frac{f(\gamma)g(\gamma)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]} - \frac{h(x)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]}. \tag{2.5}$$

We write Equation (2.5) in the following way

$$x = c + N(x), \tag{2.6}$$

where

$$c = \gamma - \frac{f(\gamma)g(\gamma)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]}, \tag{2.7}$$

and

$$N(x) = -\frac{h(x)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]}. \tag{2.8}$$

Now, we construct a homotopy [7] $\Psi(x,\eta,\xi) : (\mathbb{R}\times[0,1])\times\mathbb{R} \to \mathbb{R}$ for Equation (2.6), which satisfies

$$\Psi(x,\eta,\xi) = x - c - \eta N(x) - \eta(1-\eta)\xi = 0, \quad \xi, x \in \mathbb{R}, \quad \eta \in [0,1], \tag{2.9}$$

where η is an embedding parameter and ξ is an unknown real number. The trivial problem

$$\Psi(x,0,\xi) = x - c = 0, \tag{2.10}$$

is incessantly deformed to the original problem

$$\Psi(x, 1, \xi) = x - c - N(x) = 0, \tag{2.11}$$

as the embedding parameter η monotonically increases from 0 to 1. The modified HMP uses the embedding parameter η, as an expanding parameter, to obtain [7].

$$x = x_0 + \eta x_1 + \eta^2 x_2 + \cdots . \tag{2.12}$$

Therefore, the approximate solution of Equation (2.1) can readily be obtained as

$$x^* = \lim_{\eta \to 1} x = x_0 + x_1 + x_2 + \cdots . \tag{2.13}$$

It has been proved that the Series (2.13) is convergent [7].

Using the Taylor series expansion of $N(x)$ about x_0 and applying the modified HPM to the Equation (2.1), we can write Equation (2.6) as

$$x - c - \eta \left\{ N(x_0) + (x - x_0)\frac{N'(x_0)}{1!} + (x - x_0)^2 \frac{N''(x_0)}{2!} + \cdots \right\} - \eta(1-\eta)\xi = 0. \tag{2.14}$$

Substitution x from Equation (2.12) into Equation (2.14) gives

$$\begin{aligned} & x_0 + \eta x_1 + \eta^2 x_2 + \cdots \\ & - c - \eta \left\{ N(x_0) + (x_0 + \eta x_1 + \eta^2 x_2 + \cdots - x_0)\frac{N'(x_0)}{1!} \right. \\ & \left. + (x_0 + \eta x_1 + \eta^2 x_2 + \cdots - x_0)^2 \frac{N''(x_0)}{2!} + \cdots \right\} - \eta(1-\eta)\xi \\ & = 0. \end{aligned} \tag{2.15}$$

Comparing the alike powers of η on both sides, we obtain

$$\eta^0 : x_0 - c = 0, \tag{2.16}$$

$$\eta^1 : x_1 - N(x_0) - \xi = 0, \tag{2.17}$$

$$\eta^2 : x_2 - x_1 N'(x_0) + \xi = 0, \tag{2.18}$$

$$\eta^3 : x_3 - x_2 N'(x_0) - \frac{1}{2} x_1^2 N''(x_0) = 0. \tag{2.19}$$

From Equation (2.16), we get

$$x_0 = c = \gamma - \frac{f(\gamma)g(\gamma)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]}. \tag{2.20}$$

Thus,

$$x \approx x_0 = \gamma - \frac{f(\gamma)g(\gamma)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]}. \tag{2.21}$$

The above relation enable us to suggest the following iteration process.

Algorithm 2.1. *For a given x_0, compute the approximate solution x_{n+1}, by the following iterative scheme:*

$$x_{n+1} = x_n - \frac{f(x_n)g(x_n)}{[f'(x_n)g(x_n) + f(x_n)g'(x_n)]}, \quad n = 0, 1, 2, \ldots . \tag{2.22}$$

The iterative method defined in Equation (2.22) has also been introduced by He [14] and Noor [17] for generating various iterative methods for solving nonlinear equations.

Now we find the value of parameter ξ by setting $x_2 = 0$ which provides methods with better convergence order and high efficiency index.

Hence by substituting $x_1 = N(x_0) + \xi$ from Equation (2.17) into Equation (2.18), we infer

$$-(N(x_0) + \xi)N'(x_0) + \xi = 0, \tag{2.23}$$

and

$$\xi = \frac{N(x_0)N'(x_0)}{1 - N'(x_0)}. \tag{2.24}$$

From Equations (2.4) and (2.20), it can be easily obtained that

$$h(x_0) = f(x_0)g(\gamma). \tag{2.25}$$

From Equations (2.4), (2.8) and (2.20), we get

$$N(x_0) = -\frac{f(x_0)g(\gamma)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]}, \tag{2.26}$$

and

$$N'(x_0) = 1 - \frac{f'(x_0)g(\gamma)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]}. \tag{2.27}$$

Using Equations (2.17) and (2.24), we obtain

$$x_1 = N(x_0) + \frac{N(x_0)N'(x_0)}{1 - N'(x_0)} = \frac{N(x_0)}{1 - N'(x_0)}. \tag{2.28}$$

Thus, by combiming Equations (2.26) and (2.27) with Equation (2.28), we get

$$x_1 = -\frac{f(x_0)}{f'(x_0)}. \tag{2.29}$$

Using Equations (2.20) and (2.29), we obtain

$$x \approx x_0 + x_1 = \gamma - \frac{f(\gamma)g(\gamma)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]} - \frac{f(x_0)}{f'(x_0)}. \tag{2.30}$$

This formulation allows us the following recurrence relation for solving nonlinear equations.

Algorithm 2.2. *For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:*

$$\begin{aligned} x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)}, \quad n = 0, 1, 2, \ldots, \\ y_n &= x_n - \frac{f(x_n)g(x_n)}{[f'(x_n)g(x_n) + f(x_n)g'(x_n)]}. \end{aligned} \tag{2.31}$$

Now, from Equation (2.19), it can easily be obtained that

$$\begin{aligned} x_3 &= \frac{1}{2}x_1^2 N''(x_0) \\ &= -\frac{1}{2}\left(\frac{f(x_0)}{f'(x_0)}\right)^2 \frac{f''(x_0)g(\gamma)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]}. \end{aligned} \tag{2.32}$$

Thus using Equations (2.30) and (2.32), we get

$$\begin{aligned} x &\approx x_0 + x_1 + x_2 + x_3 \\ &= \gamma - \frac{f(\gamma)g(\gamma)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]} - \frac{f(x_0)}{f'(x_0)} \\ &\quad - \frac{1}{2}\left(\frac{f(x_0)}{f'(x_0)}\right)^2 \frac{f''(x_0)g(\gamma)}{[f'(\gamma)g(\gamma) + f(\gamma)g'(\gamma)]}. \end{aligned} \tag{2.33}$$

This formulation allows us the following recurrence relation for solving nonlinear equations.

Algorithm 2.3. *For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:*

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)g(x_n)}{[f'(x_n)g(x_n) + f(x_n)g'(x_n)]} - \frac{f(y_n)}{f'(y_n)} \\ &\quad - \frac{1}{2}\left(\frac{f(y_n)}{f'(y_n)}\right)^2 \frac{f''(y_n)g(x_n)}{[f'(x_n)g(x_n) + f(x_n)g'(x_n)]}, \\ y_n &= x_n - \frac{f(x_n)g(x_n)}{[f'(x_n)g(x_n) + f(x_n)g'(x_n)]}. \end{aligned} \tag{2.34}$$

It is to note that for different values of auxiliary function $g(x)$, several iterative methods with higher order convergence can be developed from the main iterative scheme, established in this paper, that is, Algorithms 2.2 and 2.3. Implementation of these schemes in an effective manner, the proper selection of the auxiliary function plays a vital role. We take $g(x) = e^{-\alpha x_n}$ for an illustration. Thus Algorithms 2.2 and 2.3 take the following form:

Algorithm 2.4. *For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:*

$$\begin{aligned} x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)}, \quad n = 0, 1, 2, \ldots, \\ y_n &= x_n - \frac{f(x_n)}{[f'(x_n) - \alpha f(x_n)]}. \end{aligned}$$

Algorithm 2.4 is a two-step predictor-corrector method.

For $\alpha = 0$, Algorithm 2.4 is the same as obtained in Javidi [16, Algorithm 2.1].

Algorithm 2.5. *For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:*

$$\begin{aligned} x_{n+1} &= y_n - \frac{f(y_n)}{f'(y_n)} - \frac{1}{2}\left(\frac{f(y_n)}{f'(y_n)}\right)^2 \frac{f''(y_n)}{[f'(x_n) - \alpha f(x_n)]}, \quad n = 0, 1, 2, \ldots, \\ y_n &= x_n - \frac{f(x_n)}{[f'(x_n) - \alpha f(x_n)]}. \end{aligned}$$

Algorithm 2.5 is also a two-step predictor-corrector method.

For $\alpha = 0$, Algorithm 2.5 is the same as obtained in Javidi [16, Algorithm 2.2].

Obviously, our proposed methods are a generalization of Javidi's method [16].

3. Convergence analysis

In this section, convergence criteria of new proposed algorithms is studied in the form of the following theorem.

Theorem 3.1. *Assume that the function $f : I \subset \mathbb{R} \to \mathbb{R}$ for an open interval I has a simple root $\alpha \in I$. Let $f(x)$ be sufficiently differentiable in a neighborhood of α. Then the convergence orders of the methods defined by Algorithms 2.2 and 2.3 are four and five, respectively.*

Proof. Let α be a simple zero of $f(x)$. Since f is sufficiently differentiable, the Taylor series expressions of $f(x_n)$ and $f'(x_n)$ about α are given by

$$f(x_n) = f'(\alpha)\{e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O\left(e_n^6\right)\} \tag{3.1}$$

and

$$f'(x_n) = f'(\alpha)\{1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_4^3 + 5c_5 e_n^4 + O\left(e_n^5\right)\}, \tag{3.2}$$

where $e_n = x_n - \alpha$ and $c_j = \frac{1}{j!}\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 2, 3, \ldots$.

Now, expanding $f(x_n)g(x_n)$, $f'(x_n)g(x_n)$ and $f(x_n)g'(x_n)$ by Taylor series, we get

$$
\begin{aligned}
f(x_n)g(x_n) = f'(\alpha)\Big\{ & g(\alpha)e_n + (c_2 g(\alpha) + g'(\alpha))e_n^2 \\
& + \left(\frac{1}{2}g''(\alpha) + c_2 g'(\alpha) + c_3 g(\alpha)\right)e_n^3 + O\left(e_n^4\right)\Big\},
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
f'(x_n)g(x_n) = f'(\alpha)\Big\{ & g(\alpha) + (2c_2 g(\alpha) + g'(\alpha))e_n \\
& + \left(\frac{1}{2}g''(\alpha) + 2c_2 g'(\alpha) + 3c_3 g(\alpha)\right)e_n^2 \\
& + \left(\frac{1}{6}g'''(\alpha) + c_2 g''(\alpha) + 3c_3 g'(\alpha) + 4c_4 g(\alpha)\right)e_n^3 + O\left(e_n^4\right)\Big\},
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
f(x_n)g'(x_n) = f'(\alpha)\Big\{ & g'(\alpha)e_n + (c_2 g'(\alpha) + g''(\alpha))e_n^2 \\
& + \left(\frac{1}{2}g'''(\alpha) + c_2 g''(\alpha) + c_3 g'(\alpha)\right)e_n^3 + O\left(e_n^4\right)\Big\}.
\end{aligned} \tag{3.5}
$$

From Equations (3.3), (3.4) and (3.5), we get

$$
\begin{aligned}
\frac{f(x_n)g(x_n)}{f'(x_n)g(x_n) + f(x_n)g'(x_n)} = {} & e_n - \left(c_2 + \frac{g'(\alpha)}{g(\alpha)}\right)e_n^2 \\
& + \left(2c_2\frac{g'(\alpha)}{g(\alpha)} - 2c_3 - \frac{g''(\alpha)}{g(\alpha)} + 2c_2^2 + 2\left(\frac{g'(\alpha)}{g(\alpha)}\right)^2\right)e_n^3 + O\left(e_n^4\right).
\end{aligned} \tag{3.6}
$$

Using Equation (3.6), we get

$$
\begin{aligned}
y_n = \alpha + {} & \left(c_2 + \frac{g'(\alpha)}{g(\alpha)}\right)e_n^2 \\
& + \left(2c3 - 2c_2\frac{g'(\alpha)}{g(\alpha)} + \frac{g''(\alpha)}{g(\alpha)} - 2c_2^2 - 2\left(\frac{g'(\alpha)}{g(\alpha)}\right)^2\right)e_n^3 + O\left(e_n^4\right).
\end{aligned} \tag{3.7}
$$

The Taylor series expansions of $f(y_n)$ and $f'(y_n)$ are given by

$$
\begin{aligned}
f(y_n) = f'(\alpha)\Big\{ & \left(c_2 + \frac{g'(\alpha)}{g(\alpha)}\right)e_n^2 \\
& + \left(2c_3 - 2c_2\frac{g'(\alpha)}{g(\alpha)} + \frac{g''(\alpha)}{g(\alpha)} - 2c_2^2 - 2\left(\frac{g'(\alpha)}{g(\alpha)}\right)^2\right)e_n^3 + O\left(e_n^4\right)\Big\}
\end{aligned} \tag{3.8}
$$

and

$$
\begin{aligned}
f'(y_n) = f'(\alpha)\Big\{ & 1 + \left(2c_2^2 + 2c_2\frac{g'(\alpha)}{g(\alpha)}\right)e_n^2 \\
& + \left(4c_3c_2 - 4c_2^2\frac{g'(\alpha)}{g(\alpha)} + 2c_2\frac{g''(\alpha)}{g(\alpha)} - 4c_2^3 - 4c_2\left(\frac{g'(\alpha)}{g(\alpha)}\right)^2\right)e_n^3 + O\left(e_n^4\right)\Big\}.
\end{aligned} \tag{3.9}
$$

Using Equations (3.8) and (3.9), we find

$$\begin{aligned}\frac{f(y_n)}{f'(y_n)} &= \left(c_2 + \frac{g'(\alpha)}{g(\alpha)}\right)e_n^2 \\ &\quad + \left(2c_3 - 2c_2\frac{g'(\alpha)}{g(\alpha)} + \frac{g''(\alpha)}{g(\alpha)} - 2c_2^2 - 2\left(\frac{g'(\alpha)}{g(\alpha)}\right)^2\right)e_n^3 + O\left(e_n^4\right).\end{aligned} \tag{3.10}$$

Thus, using Equations (2.31) and (3.10), the error term for Algorithm 2.2 can be obtained as

$$e_{n+1} = \left(c_2^3 + 2c_2^2\frac{g'(\alpha)}{g(\alpha)} + c_2\left(\frac{g'(\alpha)}{g(\alpha)}\right)^2\right)e_n^4 + O\left(e_n^5\right), \tag{3.11}$$

which shows that Algorithm 2.2 is of convergence order at least four.

Now expanding $f''(y_n)$ and then $f''(y_n)g(x_n)$ by Taylor series, we get

$$\begin{aligned}f''(y_n) &= f'(\alpha)\Bigg\{2c_2 + \left(6c_2c_3 + 6c_3\frac{g'(\alpha)}{g(\alpha)}\right)e_n^2 \\ &\quad + \left(12c_3^2 - 12c_2c_3\frac{g'(\alpha)}{g(\alpha)} + 6c_3\frac{g''(\alpha)}{g(\alpha)} - 12c_2^2c_3 - 12c_3\left(\frac{g'(\alpha)}{g(\alpha)}\right)^2\right)e_n^3 + O\left(e_n^4\right)\Bigg\}\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}f''(y_n)g(x_n) &= f'(\alpha)\Bigg\{2c_2g(\alpha) + 2c_2g'(\alpha)e_n + 6c_2c_3g(\alpha) + c_2g''(\alpha) + 6c_3g'(\alpha)e_n^2 \\ &\quad + \left(\frac{1}{3}c_2g'''(\alpha) - 6c_3c_2g'(\alpha) - \frac{\left(6c_3g'(\alpha)\right)^2}{g(\alpha)}12g(\alpha)c_3^2\right. \\ &\quad \left. + 6c_3g''(\alpha) - 12c_2^2c_3g(\alpha)\right)e_n^3 + O\left(e_n^4\right)\Bigg\}.\end{aligned} \tag{3.13}$$

From Equations (2.4), (3.5) and (3.13) we get

$$\begin{aligned}&\frac{f''(y_n)g(x_n)}{f'(x_n)g(x_n) + f(x_n)g'(x_n)} \\ &= 2c_2 + \left(-4c_2^2 - 2c_2\frac{g'(\alpha)}{g(\alpha)}\right)e_n \\ &\quad + \left(-2c_2\frac{g''(\alpha)}{g(\alpha)} + 6c_3\frac{g'(\alpha)}{g(\alpha)} + 6c_2^2\frac{g'(\alpha)}{g(\alpha)} + 8c_2^3 + 4c_2\left(\frac{g'(\alpha)}{g(\alpha)}\right)^2\right)e_n^2 \\ &\quad + \left(-c_2\frac{g'''(\alpha)}{g(\alpha)} - 20c_2c_3\frac{g'(\alpha)}{g(\alpha)} - 18c_3\left(\frac{g'(\alpha)}{g(\alpha)}\right)^2 - 8c_2c_4 + 12c_3^2\right. \\ &\quad + 6c_3\frac{g''(\alpha)}{g(\alpha)} + 6c_2^2\frac{g''(\alpha)}{g(\alpha)} + 7c_2g(\alpha)\frac{g'(\alpha)}{\left(g(\alpha)\right)^2} - 16c_2^3\frac{g'(\alpha)}{g(\alpha)} - 14c_2^2\left(\frac{g'(\alpha)}{g(\alpha)}\right)^2 \\ &\quad \left. - 16c_2^4 - 8c_2\left(\frac{g'(\alpha)}{g(\alpha)}\right)^3\right)e_n^3 + O\left(e_n^4\right).\end{aligned} \tag{3.14}$$

Thus the error term for Algorithm 2.3 is given by

$$e_{n+1} = \left(2c_2^4 + 5c_2^3\frac{g'(\alpha)}{g(\alpha)} + 4c_2^2\left(\frac{g'(\alpha)}{g(\alpha)}\right)^2 + c_2\left(\frac{g'(\alpha)}{g(\alpha)}\right)^3\right)e_n^5 + O\left(e_n^6\right). \tag{3.15}$$

This completes the proof. □

4. Numerical examples

In this section, we illustrate the validity and efficiency of our proposed iterative schemes. In Table 1, we present the comparison of our method defined in Algorithm 2.4 (AM1) with Newton's method (NM), and some other methods with convergence order 4, that is, Chun's method (CM1) [4, Equation (10)], Shah and Noor's method (SN) [18, Algorithm 2.3] and Javidi's method (JM1) [16, Algorithm 2.1]. A comparison of our developed method given in Algorithm 2.5 (AM2) with Newton's method (NM) that convergence quadratically, Chun's method (CM2) [4, Equation (11)] and Javidi's method (JM2) [16, Algorithm 2.2] whose order of convergence is 5, is given in Table 2. Here N denotes the number of iterations.

We use $\alpha = 0.5$ and Maple software for numerical computations, $\varepsilon = 10^{-15}$ as tolerance and the following stopping criteria:

(i) $\delta = |x_{n+1} - x_n| < \varepsilon$,

(ii) $|f(x_n)| < \varepsilon$,

(iii) Maximum numbers of iterations $= 500$.

Table 1. Comparison of NM, CM1, SN, JM1 and AM1

$f(x)$	$x_\circ$	*Method*	N	$x[k]$	$f(x_n)$	δ
$\sin^2 x - x^2 + 1$	1.1	NM	5	1.4044916482153412334	$1.828320e-17$	$3.058070e-09$
		CM1	4	1.4044916482153412260	$8.710207e-20$	$1.287627e-13$
		SN	3	[illegible]	$8.710207e-20$	$3.087218e-09$
		JM1	[illegible]	[illegible]	$8.710207e-20$	$3.058070e-09$
		AM1	[illegible]	[illegible]	$8.710207e-20$	$2.018548e-13$
$(x-1)^3 - 1$	1.5	NM	[illegible]	2.0000000000000000000	$0.000000e+00$	$6.031119e-12$
		CM1	10	2.0000000000000000000	$0.000000e+00$	$8.217689e-15$
		SN	4	2.0000000000000000000	$0.000000e+00$	$4.433203e-12$
		JM1	4	2.0000000000000000000	$0.000000e+00$	$6.031119e-12$
		AM1	3	2.0000000000000000000	$0.000000e+00$	$7.355562e-07$
$x^2 - e^x + x + 2$	1.7	NM	7	2.2041177331716202960	$3.305103e-20$	$1.118049e-13$
		CM1	17	2.2041177331716202960	$3.305103e-20$	$4.482317e-08$
		SN	4	2.2041177331716202960	$3.305103e-20$	$6.139530e-13$
		JM1	4	2.2041177331716202960	$3.305103e-20$	$1.118049e-13$
		AM1	3	2.2041177331716202960	$3.305103e-20$	$9.493103e-08$
$e^x - 3x - 3$	3.6	NM	5	3.8695271180759797541	$4.150103e-18$	$5.062765e-10$
		CM1	3	3.8695271180759797669	$3.203428e-16$	$4.536976e-05$
		SN	3	3.8695271180759797539	$7.904086e-19$	$3.146929e-10$
		JM1	3	3.8695271180759797539	$7.904086e-19$	$5.062765e-10$
		AM1	3	3.8695271180759797539	$7.904086e-19$	$3.059515e-14$
$x^3 - 1$	1.5	NM	6	2.1544346900318837218	$5.668276e-19$	$3.486728e-14$
		CM1	4	2.1544346900318837218	$5.668276e-19$	$4.211475e-06$
		SN	3	2.1544346900318837218	$5.668276e-19$	$7.671350e-16$
		JM1	3	2.1544346900318837218	$5.668276e-19$	$2.740790e-07$
		AM1	2	2.1544346900318837218	$5.668276e-19$	$7.225849e-06$
$xe^{x^3} - \sin^3 x$ $+3\cos x - x$	−2.5	NM	9	−2.8543919810023413339	$1.430322e-21$	$4.636514e-13$
		CM1	5	−2.8543919810023413339	$1.430322e-21$	$3.258462e-07$
		SN	5	−2.8543919810023413339	$1.430322e-21$	$1.023313e-08$
		JM1	5	−2.8543919810023413339	$1.430322e-21$	$4.636514e-13$
		AM1	4	−2.8543919810023308714	$8.440064e-16$	$2.719899e-05$
$4x^4 - 4x^2$	0.5	NM	23	0.0000000291733579902	$3.404339e-15$	$2.917336e-08$
		CM1	14	0.0000000210479779344	$1.772070e-15$	$4.630555e-08$
		SN	13	0.0000000201191478099	$1.619120e-15$	$4.591293e-08$
		JM1	12	0.0000000145866789951	$8.510848e-16$	$4.376004e-08$
		AM1	11	0.0000000181863023840	$1.322966e-15$	$5.455891e-08$
$x^3 + 4x^2 - 10$	0.9	NM	5	1.3652300134140968476	$3.037133e-17$	$1.937468e-09$
		CM1	4	1.3652300134140968458	$6.472125e-19$	$2.090766e-15$
		SN	3	1.3652300134140968458	$6.472125e-19$	$3.493290e-15$
		JM1	3	1.3652300134140968458	$6.472125e-19$	$1.937468e-09$
		AM1	2	1.3652300134140968458	$6.472125e-19$	$6.652273e-05$

Table 2. Comparison of NM, CM2, JM2 and AM2

$f(x)$	$x_\circ$	*Method*	N	$x\,[k]$	$f(x_n)$	δ
$\sin^2 x - x^2 + 1$	1.1	NM	5	1.4044916482153412334	$1.828320e-17$	$3.058070e-09$
		CM2	4	1.4044916482153412260	$8.710207e-20$	$1.459668e-09$
		JM2	3	1.4044916482153412260	$8.710207e-20$	$1.550744e-11$
		AM2	2	1.4044916482153412185	$1.870565e-17$	$6.449644e-04$
$(x-1)^3 - 1$	1.5	NM	7	2.0000000000000000000	$0.000000e+00$	$6.031119e-12$
		CM2	133	2.0000000000000000000	$0.000000e+00$	$1.416531e-06$
		JM2	30	2.0000000000000000000	$0.000000e+00$	$4.037420e-08$
		AM2	3	2.0000000000000000000	$0.000000e+00$	$1.220248e-06$
$x^2 - e^x + x + 2$	1.7	NM	7	2.2041177331716202960	$3.305103e-20$	$1.118049e-13$
		CM2	8	2.2041177331716202960	$3.305103e-20$	$9.844979e-09$
		JM2	13	2.2041177331716202960	$3.305103e-20$	$4.553875e-15$
		AM2	3	2.2041177331716202960	$3.305103e-20$	$2.541180e-08$
$e^x - 3x - 3$	3.6	NM	5	3.8695271180759797541	$4.150103e-18$	$5.062765e-10$
		CM2	3	3.8695271180759797539	$7.904086e-19$	$4.497358e-05$
		JM2	3	3.8695271180759797539	$7.904086e-19$	$8.123636e-13$
		AM2	2	3.8695271180759797536	$8.201176e-18$	$3.077979e-04$
$x^3 - 1$	1.5	NM	6	2.1544346900318837218	$5.668276e-19$	$3.486728e-14$
		CM2	150	2.1544346900318837218	$5.668276e-19$	$7.135475e-05$
		JM2	3	2.1544346900318837218	$5.668276e-19$	$2.865678e-08$
		AM2	2	2.1544346900318837218	$5.668276e-19$	$2.779608e-06$
$xe^{x^3} - \sin^3 x$ $+3\cos x - x$	-2.5	NM	9	−2.8543919810023413339	$1.430322e-21$	$4.636514e-13$
		CM2	5	−2.8543919810023413339	$1.430322e-21$	$5.547576e-12$
		JM2	4	−2.8543919810023413339	$1.430322e-21$	$2.240722e-06$
		AM2	4	−2.8543919810023413339	$1.430322e-21$	$5.255127e-07$
$4x^4 - 4x^2$	0.5	NM	23	0.0000000291733579902	$3.404339e-15$	$2.917336e-08$
		CM2	12	0.0000000434429282675	$7.549152e-15$	$1.154341e-07$
		JM2	11	[illegible]	[illegible]	$4.835544e-08$
		AM2	[illegible]	[illegible]	[illegible]	$7.447503e-08$
$x^3 + 4x^2 - 10$	0.9	NM	[illegible]	[illegible]	[illegible]	$1.937468e-09$
		CM2	[illegible]	[illegible]	[illegible]	$8.067430e-16$
		JM2	[illegible]	[illegible]	$6.472125e-19$	$5.496336e-12$
		AM2	2	1.3652300134140968458	$6.472125e-19$	$1.506824e-04$

5. Conclusions

The present work comprises a development of two new numerical methods for solving nonlinear equations. The orders of convergence of our proposed methods are four and five. A comparison of these new methods with some known methods with the same convergence order is presented. The comparison given in Tables 1 and 2 obviously indicate the better performance of the newly developed methods. These methods may also be viewed as a generalization of some existing methods. The idea and technique of this paper can be employed to develop and analyze higher order multi-step iterative methods for solving nonlinear equations.

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