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Well-posedness and general decay of solution for a transmission problem with viscoelastic term and delay

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Abstract

In this paper, we consider a transmission problem in a bounded domain with a viscoelastic term and a delay term. Under appropriate hypotheses on the relaxation function and the relationship between the weight of the damping and the weight of the delay, we prove the well-posedness result by using Faedo-Galerkin method. By introducing suitable Lyapunov functionals, we establish a general decay result, from which the exponential and polynomial types of decay are only special cases.

Keywords: Wave equation, transmission problem, general decay, viscoelastic term, delay.
2010 MSC: 35B37, 35L55, 93D15, 93D20.

1. Introduction

In this paper, we study the transmission system with a viscoelastic term and a delay term

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \displaystyle\int_0^t g(t-s)u_{xx}(x,s)\mathrm{d}s \\ \qquad\qquad +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \qquad\qquad (x,t) \in (L_1, L_2) \times (0,+\infty), \end{cases} \tag{1.1}$$

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under the boundary and transmission conditions

$$\begin{cases} u(0,t) = u(L_3,t) = 0, & \\ u(L_i,t) = v(L_i,t), & i = 1,2, \\ \left(a - \displaystyle\int_0^t g(s)\mathrm{d}s\right) u_x(L_i,t) = bv_x(L_i,t), & i = 1,2, \end{cases} \tag{1.2}$$

and the initial conditions

$$\begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega, \quad t \in [0,\tau], \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in (L_1,L_2), \end{cases} \tag{1.3}$$

where $0 < L_1 < L_2 < L_3$, $\Omega = (0,L_1) \cup (L_2,L_3)$, a, b, μ_1, μ_2 are positive constants, and $\tau > 0$ is the delay.

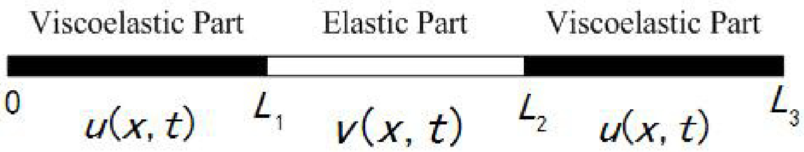

Figure 1: The configuration.

The transmission problems like (1.1)-(1.3) related to the wave propagation over a body consists of two different types of materials: the elastic part and the viscoelastic part.

In recent years, many authors have investigated wave equations with viscoelastic damping and showed that the dissipation produced by the viscoelastic part can produce the decay of the solution, see [5, 6, 7, 8, 11, 16, 18, 20, 22, 26, 27, 28] and the references therein. For example, Cavalcanti et al. [8] studied the following equation:

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau + a(x)u_t + |u|^\gamma u = 0, \quad \text{in} \quad \Omega \times (0,\infty),$$

where $a : \Omega \to \mathbb{R}_+$. Under the conditions that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions and

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0,$$

the authors showed the exponential decay. Then Berrimi and Messaoudi [5] proved the same result under weaker conditions on both a and g. Berrimi and Messaoudi [6] considered the equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau = |u|^\gamma u, \quad \text{in} \quad \Omega \times (0,\infty),$$

with only the viscoelastic dissipation and proved that the solution energy decays exponentially or polynomially depending on the rate of the decay of the relaxation function g. In all previous works, the rates of decay of relaxation functions were either exponential or polynomial type. For a wider class of relaxation functions, Messaoudi [22] investigated the following viscoelastic equation:

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\mathrm{d}\tau = 0, \quad \text{in} \quad \Omega \times (0,\infty),$$

in a bounded domain, and established a more general decay result, from which the usual exponential and polynomial decay rates are only special cases.

It is well known that delay effects, which arise in many practical problems, may be sources of instability. Hence, the control of PDEs with time delay effects has become an active area of research in recent years. For example, it was proved in [10, 15, 17, 19, 24] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms were used. A boundary stabilization problem for the wave equation with interior delay was studied in [1]. The authors proved an exponential stability result under some Lions geometric condition. Kirane and Said-Houari [12] considered the viscoelastic wave equation with a delay

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,t-s)\mathrm{d}s + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \quad \text{in} \quad \Omega \times (0,\infty),$$

where μ_1 and μ_2 are positive constants. They established a general energy decay result under the condition that $0 \le \mu_2 \le \mu_1$. Later, Liu [14] improved this result by considering the equation with a time-varying delay term, with not necessarily positive coefficient μ_2 of the delay term.

Transmission problems related to (1.1)-(1.3) have also been extensively studied. Bastos and Raposo [4] investigated the transmission problem with frictional damping and showed the well-posedness and exponential stability of the total energy. Muñoz Rivera and Portillo Oquendo [23] considered the transmission problem of viscoelastic waves and proved that the dissipation produced by the viscoelastic part can produce exponential decay of the solution, no matter how small its size is. Bae [3] studied the transmission problem, in which one component is clamped and the other is in a viscoelastic fluid producing a dissipative mechanism on the boundary, and established a decay result which depends on the rate of the decay of the relaxation function.

Motivated by the above results, we intend to consider the well-posedness and the general decay result of problem (1.1)-(1.3) under some hypotheses in this paper. The main difficulty we encounter here arises from the simultaneous appearance of the viscoelastic term and the delay term. Our first intention is to study the well-posedness of problem (1.1)-(1.3) by making use of Faedo-Galerkin procedure, that is Faedo-Galerkin approximation together with energy estimates. For asymptotic behavior, we prove a general decay result from which the exponential and polynomial types of decay are only special cases by introducing suitable Lyapunov functionals.

The paper is organized as follows. In Section 2, we give some materials needed for our work and state our main results. In Section 3, we prove the well-posedness of the problem. The general decay result is proved in Section 4.

2. Preliminaries and main results

In this section, we present some materials that shall be used in order to prove our main results. Let us first introduce the following notations:

$$\begin{aligned}
(g * h)(t) &:= \int_0^t g(t-s)h(s)\mathrm{d}s, \\
(g \diamond h)(t) &:= \int_0^t g(t-s)|h(t)-h(s)|\mathrm{d}s, \\
(g\Box h)(t) &:= \int_0^t g(t-s)|h(t)-h(s)|^2\mathrm{d}s.
\end{aligned}$$

We easily see that the above operators satisfy

$$\begin{aligned}
(g * h)(t) &= \left(\int_0^t g(s)\mathrm{d}s\right) h(t) - (g \diamond h)(t), \\
|(g \diamond h)(t)|^2 &\le \left(\int_0^t |g(s)|\mathrm{d}s\right) (|g|\Box h)(t).
\end{aligned}$$

Lemma 2.1 ([9])**.** *For any $g, h \in C^1(\mathbb{R})$, the following equation holds*

$$2[g * h]h' = g'\Box h - g(t)|h|^2 - \frac{d}{dt}\left\{g\Box h - \left(\int_0^t g(s)\mathrm{d}s\right)|h|^2\right\}.$$

Proof. Differentiating the expression

$$g\Box h - \left(\int_0^t g(s)\mathrm{d}s\right)|h|^2,$$

we get the result. □

For the relaxation function g, we assume the conditions
(G1) g: $\mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g \in L^1(0,\infty), \quad g(0) > 0, \quad 0 < \beta(t) := a - \int_0^t g(s)\mathrm{d}s \quad \text{and} \quad 0 < \beta_0 := a - \int_0^\infty g(s)\mathrm{d}s.$$

(G2) There exists a nonincreasing differentiable function $\xi(t)$: $\mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \le -\xi(t)g(t), \quad \forall t \ge 0 \quad \text{and} \quad \int_0^\infty \xi(t)\mathrm{d}t = +\infty.$$

These hypotheses imply that

$$\beta_0 \le \beta(t) \le a. \tag{2.1}$$

As in [24], we introduce the following variable:

$$z(x,\rho,t) = u_t(x, t - \tau\rho), \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,\infty).$$

Then the above variable z satisfies

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,\infty). \tag{2.2}$$

Thus, system (1.1) becomes

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + g * u_{xx} \\ \qquad\qquad + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) = 0, & (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, & (x,t) \in (L_1, L_2) \times (0,+\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & (x,\rho,t) \in \Omega \times (0,1) \times (0,+\infty), \end{cases} \tag{2.3}$$

and the boundary and transmission conditions (1.2) becomes

$$\begin{cases} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), & i = 1,2, \quad t \in (0,+\infty), \\ \left(a - \int_0^t g(s)\mathrm{d}s\right) u_x(L_i,t) = bv_x(L_i,t), & i = 1,2, \quad t \in (0,+\infty), \\ z(x,0,t) = u_t(x,t), & (x,t) \in \Omega \times (0,+\infty), \\ z(x,1,t) = f_0(x,t-\tau), & (x,t) \in \Omega \times (0,\tau). \end{cases} \tag{2.4}$$

Similar to [25], we denote the Hilbert spaces

$$\mathcal{V} = \left\{(u,v) \in H^1(\Omega) \cap H^1(L_1,L_2) : u(0,t) = u(L_3,0) = 0, u(L_i,t) = v(L_i,t), \right.$$
$$\left. \left(a - \int_0^t g(s)\mathrm{d}s\right) u_x(L_i,t) = bv_x(L_i,t), i = 1,2\right\}$$

and

$$\mathcal{L}^2 = L^2(\Omega) \times L^2(L_1, L_2).$$

Then the existence result reads as follows:

Theorem 2.1. *Assume that $\mu_2 \le \mu_1$, $(G1)$ and $(G2)$ hold. Then given $(u_0, v_0) \in \mathcal{V}$, $(u_1, v_1) \in \mathcal{L}^2$, and $f_0 \in L^2((0,1),\Omega)$, there exists a unique weak solution (u, v, z) of problem* (2.3)-(2.4) *such that*

$$(u, v) \in C(0, \infty; \mathcal{V}) \cap C^1(0, \infty; \mathcal{L}^2),$$
$$z \in C(0, \infty; L^2((0, 1), \Omega)).$$

For any regular solution of (1.1)-(1.3), we define the energy as

$$\begin{aligned} E(t) =& \frac{1}{2}\int_\Omega u_t^2(x,t)\mathrm{d}x + \frac{1}{2}\beta(t)\int_\Omega u_x^2(x,t)\mathrm{d}x + \frac{1}{2}\int_\Omega (g\Box u_x)\mathrm{d}x \\ &+ \frac{1}{2}\int_{L_1}^{L_2}\left[v_t^2(x,t) + bv_x^2(x,t)\right]\mathrm{d}x + \frac{\zeta}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x, \end{aligned} \tag{2.5}$$

where ζ is a positive constant such that

$$\tau\mu_2 < \zeta < \tau(2\mu_1 - \mu_2). \tag{2.6}$$

Our decay result reads as follows:

Theorem 2.2. *Let (u, v, z) be the solution of problem* (1.1)-(1.3)*. Assume that $\mu_2 < \mu_1$, $(G1)$, $(G2)$ and*

$$b > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2}\beta_0, \quad a > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2}\beta_0 \tag{2.7}$$

hold, then there exist constants $K, k > 0$ such that, for all $t \in \mathbb{R}_+$,

$$E(t) \le Ke^{-k\int_0^t \xi(s)\mathrm{d}s}. \tag{2.8}$$

3. Well-posedness of the problem

In this section, we will prove the existence and uniqueness of problem (1.1)-(1.3) by using Faedo-Galerkin method.

Proof of Theorem 2.1. We divide the proof of Theorem 2.1 into two steps: the Faedo-Galerkin approximation and the energy estimates.

Step 1: Faedo-Galerkin approximation.

We construct approximations of the solution (u, v, z) by the Faedo-Galerkin method as follows. For $n \ge 1$, let $W_n = \mathrm{span}\{w_1, \dots, w_i\}$ be a Hilbertian basis of the space $H^1(\Omega)$ and $Y_n = \mathrm{span}\{\psi_1, \dots, \psi_i\}$ be a Hilbertian basis of the space $H^1(L_1, L_2)$.

Now, we define for $1 \le j \le n$ the sequence $\varphi_j(x, \rho)$ as follows:

$$\varphi_j(x, 0) = w_j(x).$$

Then we may extend $\varphi_j(x, 0)$ by $\varphi_j(x, \rho)$ over $L^2((0, 1), \Omega)$ and denote $V_n = \mathrm{span}\{\varphi_1, \dots, \varphi_n\}$.

We choose sequences $\left(u_0^{(n)}\right)$, $\left(u_1^{(n)}\right)$ in W_n, $\left(v_0^{(n)}\right)$, $\left(v_1^{(n)}\right)$ in Y_n and a sequence $\left(z_0^{(n)}\right)$ in V_n such that $(u_0^{(n)}, u_1^{(n)}, v_0^{(n)}, v_1^{(n)}) \to (u_0, u_1, v_0, v_1)$ strongly in $\mathcal{V}$ and $z_0^{(n)} \to f_0$ strongly in $L^2((0, 1), \Omega)$.

We define the approximations

$$\left(u^{(n)}(x,t), v^{(n)}(x,t)\right) = \sum_{i=1}^n h_i^{(n)}(t)(w_i(x), \psi_i(x)) \quad \text{and} \quad z^{(n)}(x,\rho,t) = \sum_{i=1}^n f_i^{(n)}(t)\varphi_i(x),$$

where $\left(u^{(n)}(t), v^{(n)}(t), z^{(n)}(t)\right)$ is a solution to the following Cauchy problem:

$$\begin{cases} \displaystyle\int_\Omega u_{tt}^{(n)} w_i \mathrm{d}x - \left[\left(au_x^{(n)} - g * u_x^{(n)}\right) w_i\right]_{\partial\Omega} + \int_\Omega a u_x^{(n)} w_{ix} \mathrm{d}x - \int_\Omega \left(g * u_x^{(n)}\right) w_{ix} \mathrm{d}x \\ \displaystyle + \int_\Omega \mu_1 u_t^{(n)} w_i \mathrm{d}x + \int_\Omega \mu_2 z^{(n)}(x,1,t) w_i \mathrm{d}x = 0, \\ \displaystyle\int_{L_1}^{L_2} v_{tt}^{(n)} \psi_i \mathrm{d}x + \int_{L_1}^{L_2} b v_x^{(n)} \psi_{ix} \mathrm{d}x - \left[b v_x^{(n)} \psi_i\right]_{L_1}^{L_2} = 0, \\ z^{(n)}(x,0,t) = u_t^{(n)}(x,t), \\ \left(u^{(n)}(0), u_t^{(n)}(0)\right) = \left(u_0^{(n)}, u_1^{(n)}\right) \end{cases} \tag{3.1}$$

and

$$\begin{cases} \displaystyle\int_\Omega \left(\tau z_t^{(n)}(x,\rho,t) + z_\rho^{(n)}(x,\rho,t)\right) \varphi_i \mathrm{d}x = 0, \\ z^{(n)}(\rho,0) = z_0^{(n)}. \end{cases} \tag{3.2}$$

Similar to [21], according to the standard theory of ordinary differential equations, the finite dimensional problem (3.1)-(3.2) have a solution $\left(h_i^{(n)}(t), f_i^{(n)}(t)\right)_{i=1,\dots,n}$ defined on $[0, t_n)$.

Step 2: Energy estimates.

Multiplying the first and the second equation of (3.1) by $\left(h_i^{(n)}\right)'(t)$, we have

$$\begin{aligned} &\int_\Omega u_{tt}^{(n)} u_t^{(n)} \mathrm{d}x - \left[\left(au_x^{(n)} - g * u_x^{(n)}\right) w_i\right]_{\partial\Omega} \times \left(h_i^{(n)}\right)'(t) + \int_\Omega a u_x^{(n)} u_{xt}^{(n)} \mathrm{d}x \\ &- \int_\Omega \left(g * u_x^{(n)}\right) u_{xt}^{(n)} \mathrm{d}x + \int_\Omega \mu_1 u_t^{(n)} u_t^{(n)} \mathrm{d}x + \int_\Omega \mu_2 z^{(n)}(x,1,t) u_t^{(n)} \mathrm{d}x = 0 \end{aligned} \tag{3.3}$$

and

$$\int_{L_1}^{L_2} v_{tt}^{(n)} v_t^{(n)} \mathrm{d}x + \int_{L_1}^{L_2} b v_x^{(n)} v_{xt}^{(n)} \mathrm{d}x - \left[b v_x^{(n)} \psi_i\right]_{L_1}^{L_2} \times \left(h_i^{(n)}\right)'(t) = 0. \tag{3.4}$$

Multiplying the first equation of (3.2) by $\dfrac{\zeta}{\tau} f_i^{(n)}(t)$ and integrating over $(0,t) \times (0,1)$, we get

$$\frac{\zeta}{2} \int_\Omega \int_0^1 \left(z^{(n)}\right)^2 (x,\rho,t) \mathrm{d}\rho \mathrm{d}x + \frac{\zeta}{\tau} \int_0^t \int_\Omega \int_0^1 z_\rho^{(n)} z^{(n)}(x,\rho,s) \mathrm{d}\rho \mathrm{d}x \mathrm{d}s = \frac{\zeta}{2} \int_\Omega \int_0^1 \left(z_0^{(n)}\right)^2 \mathrm{d}\rho \mathrm{d}x. \tag{3.5}$$

To handle the last term in the left-hand side of (3.5), we remark that

$$\begin{aligned} \int_0^t \int_\Omega \int_0^1 z_\rho^{(n)} z^{(n)}(x,\rho,s) \mathrm{d}\rho \mathrm{d}x \mathrm{d}s &= \frac{1}{2} \int_0^t \int_\Omega \int_0^1 \frac{\partial}{\partial \rho} \left(z^{(n)}\right)^2 (x,\rho,s) \mathrm{d}\rho \mathrm{d}x \mathrm{d}s \\ &= \frac{1}{2} \int_0^t \int_\Omega \left(\left(z^{(n)}\right)^2 (x,1,s) - \left(z^{(n)}\right)^2 (x,0,s)\right) \mathrm{d}x \mathrm{d}s. \end{aligned} \tag{3.6}$$

Integrating (3.3) and (3.4) over $(0,t)$, counting them and (3.5) up, taking into account (3.6) and using Lemma 2.1, we obtain

$$\begin{aligned} &\mathscr{E}_n(t) + \left(\mu_1 - \frac{\zeta}{2\tau}\right) \int_0^t \int_\Omega \left(u_t^{(n)}\right)^2 (x,s) \mathrm{d}x \mathrm{d}s + \frac{\zeta}{2\tau} \int_0^t \int_\Omega \left(z^{(n)}\right)^2 (x,1,s) \mathrm{d}x \mathrm{d}s \\ &+ \mu_2 \int_0^t \int_\Omega z^{(n)}(x,1,s) u_t^{(n)}(x,s) \mathrm{d}x \mathrm{d}s + \frac{1}{2} \int_0^t \int_\Omega g(t) \left|u_x^{(n)}\right|^2 \mathrm{d}x \mathrm{d}s - \frac{1}{2} \int_0^t \int_\Omega \left(g' \Box u_x^{(n)}\right) \mathrm{d}x \mathrm{d}s \\ =&\mathscr{E}_n(0), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned}\mathscr{E}_n(t) =&\frac{1}{2}\int_\Omega \left(u_t^{(n)}\right)^2(x,t)\mathrm{d}x + \frac{1}{2}\beta(t)\int_\Omega \left(u_x^{(n)}\right)^2(x,t)\mathrm{d}x + \frac{1}{2}\int_\Omega \left(g\Box u_x^{(n)}\right)\mathrm{d}x \\ &+ \frac{1}{2}\int_{L_1}^{L_2}\left[\left(v_t^{(n)}\right)^2(x,t) + b\left(v_x^{(n)}\right)^2(x,t)\right]\mathrm{d}x + \frac{\zeta}{2}\int_\Omega\int_0^1 \left(z^{(n)}\right)^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x. \end{aligned}\tag{3.8}$$

At this point, we have to distinguish the following two cases:

Case 1: We suppose that $\mu_2 < \mu_1$ and choose ζ satisfying (2.6). Young's inequality gives us that

$$\begin{aligned}&\mathscr{E}_n(t) + \left(\mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_2}{2}\right)\int_0^t\int_\Omega \left(u_t^{(n)}\right)^2(x,s)\mathrm{d}x\mathrm{d}s + \left(\frac{\zeta}{2\tau} - \frac{\mu_2}{2}\right)\int_0^t\int_\Omega \left(z^{(n)}\right)^2(x,1,s)\mathrm{d}x\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t\int_\Omega g(t)\left|u_x^{(n)}\right|^2\mathrm{d}x\mathrm{d}s - \frac{1}{2}\int_0^t\int_\Omega \left(g'\Box u_x^{(n)}\right)\mathrm{d}x\mathrm{d}s \\ \le&\mathscr{E}_n(0).\end{aligned}$$

Consequently, using (2.6), we have

$$\begin{aligned}&\mathscr{E}_n(t) + c_1\int_0^t\int_\Omega \left(u_t^{(n)}\right)^2(x,s)\mathrm{d}x\mathrm{d}s + c_2\int_0^t\int_\Omega \left(z^{(n)}\right)^2(x,1,s)\mathrm{d}x\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t\int_\Omega g(t)\left|u_x^{(n)}\right|^2\mathrm{d}x\mathrm{d}s - \frac{1}{2}\int_0^t\int_\Omega \left(g'\Box u_x^{(n)}\right)\mathrm{d}x\mathrm{d}s \\ \le&\mathscr{E}_n(0).\end{aligned}\tag{3.9}$$

Case 2: We suppose that $\mu_2 = \mu_1 = \mu$ and choose $\zeta = \tau\mu$. Then (3.9) takes the form

$$\mathscr{E}_n(t) + \frac{1}{2}\int_0^t\int_\Omega g(t)\left|u_x^{(n)}\right|^2\mathrm{d}x\mathrm{d}s - \frac{1}{2}\int_0^t\int_\Omega \left(g'\Box u_x^{(n)}\right)\mathrm{d}x\mathrm{d}s \le \mathscr{E}_n(0).\tag{3.10}$$

Now, since the sequences $\left(u_0^{(n)}\right)_{n\in\mathbb{N}}$, $\left(u_1^{(n)}\right)_{n\in\mathbb{N}}$, $\left(v_0^{(n)}\right)_{n\in\mathbb{N}}$, $\left(v_1^{(n)}\right)_{n\in\mathbb{N}}$, $\left(z_0^{(n)}\right)_{n\in\mathbb{N}}$ converge and using (G2), in the both cases we can find a positive constant c_3 independent of n such that

$$\mathscr{E}_n(t) \le c_3.\tag{3.11}$$

Therefore, using the fact that $\beta(t) \ge \beta_0$, the estimate (3.11) together with (3.8) give us, for all $n \in \mathbb{N}$, $t_n = T$, we deduce

$$\begin{aligned}&\left(u^{(n)}\right)_{n\in\mathbb{N}} \quad \text{is bounded in} \quad L^\infty(0,T;H^1(\Omega)), \\ &\left(v^{(n)}\right)_{n\in\mathbb{N}} \quad \text{is bounded in} \quad L^\infty(0,T;H^1(L_1,L_2)), \\ &\left(u_t^{(n)}\right)_{n\in\mathbb{N}} \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \\ &\left(v_t^{(n)}\right)_{n\in\mathbb{N}} \quad \text{is bounded in} \quad L^\infty(0,T;L^2(L_1,L_2)), \\ &\left(z^{(n)}\right)_{n\in\mathbb{N}} \quad \text{is bounded in} \quad L^\infty(0,T;L^2((0,1),\Omega)).\end{aligned}\tag{3.12}$$

Consequently, we conclude that

$$\begin{aligned}&u^{(n)} \rightharpoonup u \quad \text{weak}^* \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)), \\ &v^{(n)} \rightharpoonup v \quad \text{weak}^* \quad \text{in} \quad L^\infty(0,T;H^1(L_1,L_2)), \\ &u_t^{(n)} \rightharpoonup u_t \quad \text{weak}^* \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)),\end{aligned}$$

$$\begin{aligned} v_t^{(n)} &\rightharpoonup v_t \quad \text{weak}^* \quad \text{in} \quad L^\infty(0,T;L^2(L_1,L_2)),\\ z^{(n)} &\rightharpoonup z \quad \text{weak}^* \quad \text{in} \quad L^\infty(0,T;L^2((0,1),\Omega)). \end{aligned}$$

From (3.12), we have $\left(u^{(n)}\right)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H^1(\Omega))$ and $\left(v^{(n)}\right)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;H^1(L_1,L_2))$. Then, $\left(u^{(n)}\right)_{n\in\mathbb{N}}$ is bounded in$L^2(0,T;H^1(\Omega))$ and $\left(v^{(n)}\right)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^1(L_1,L_2))$. Consequently, $\left(u^{(n)}\right)_{n\in\mathbb{N}}$ is bounded in $H^1(0,T;H^1(\Omega))$ and $\left(v^{(n)}\right)_{n\in\mathbb{N}}$ is bounded in $H^1(0,T;H^1(L_1,L_2))$.

Since the embedding

$$H^1(0,T;H^1(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$$

and

$$H^1(0,T;H^1(L_1,L_2)) \hookrightarrow L^2(0,T;L^2(L_1,L_2))$$

are compact, using Aubin-Lion's theorem [13], we can extract subsequences $\left(u^{(k)}\right)_{k\in\mathbb{N}}$ of $\left(u^{(n)}\right)_{n\in\mathbb{N}}$ and $\left(v^{(k)}\right)_{k\in\mathbb{N}}$ of $\left(v^{(n)}\right)_{n\in\mathbb{N}}$ such that

$$u^{(k)} \to u \quad \text{strongly} \quad \text{in} \quad L^2(0,T;L^2(\Omega))$$

and

$$v^{(k)} \to v \quad \text{strongly} \quad \text{in} \quad L^2(0,T;L^2(L_1,L_2)).$$

Therefore,

$$u^{(k)} \to u \quad \text{strongly} \quad \text{and} \quad \text{a.e.} \quad \text{on} \quad (0,T)\times\Omega$$

and

$$v^{(k)} \to v \quad \text{strongly} \quad \text{and} \quad \text{a.e.} \quad \text{on} \quad (0,T)\times(L_1,L_2).$$

The proof now can be completed arguing as in Theorem 3.1 of [13]. □

4. General decay of the solution

In this section, we consider the asymptotic behavior of problem (1.1)-(1.3). For the proof of Theorem 2.2, we use the following lemmas.

Lemma 4.1. *Let (u,v,z) be the solution of problem (2.3)-(2.4). Assume that $\mu_2 < \mu_1$. Then we have the inequality*

$$\frac{d}{dt}E(t) \le -c_4\left[\int_\Omega u_t^2(x,t)\mathrm{d}x + \int_\Omega z^2(x,1,t)\mathrm{d}x\right] + \frac{1}{2}\int_\Omega (g'\Box u_x)(t)\mathrm{d}x. \tag{4.1}$$

Proof. Multiplying the first equation of (2.3) by u_t, the second equation of (2.3) by v_t, integrating by parts and (2.4), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega [u_t^2(x,t) + au_x^2(x,t)]\mathrm{d}x\right\} + \frac{1}{2}\frac{d}{dt}\left\{\int_{L_1}^{L_2} [v_t^2(x,t) + bv_x^2(x,t)]\mathrm{d}x\right\}\\ =&-\mu_1\int_\Omega u_t^2(x,t)\mathrm{d}x - \mu_2\int_\Omega u_t(x,t)z(x,1,t)\mathrm{d}x + \int_0^t g(t-s)\int_\Omega u_x(s)u_{xt}(t)\mathrm{d}s\mathrm{d}x \end{aligned} \tag{4.2}$$

for any regular solution. By using standard arguments of density, we can extend the result to weak solutions. From Lemma 2.1, the last term in the right-hand side of (4.2) can be rewritten as

$$\begin{aligned} &\int_0^t g(t-s)\int_\Omega u_x(s)u_{xt}(t)\mathrm{d}s\mathrm{d}x + \frac{1}{2}g(t)\int_\Omega u_x^2\mathrm{d}x\\ =&\frac{1}{2}\frac{d}{dt}\left\{\int_0^t g(s)\int_\Omega u_x^2\mathrm{d}x\mathrm{d}s - \int_\Omega (g\Box u_x)(t)\mathrm{d}x\right\} + \frac{1}{2}\int_\Omega (g'\Box u_x)(t)\mathrm{d}x. \end{aligned}$$

So (4.2) becomes

$$\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega \left[u_t^2(x,t)+\beta(t)u_x^2(x,t)\right]\mathrm{d}x\right\}+\frac{1}{2}\frac{d}{dt}\left\{\int_{L_1}^{L_2}[v_t^2(x,t)+bv_x^2(x,t)]\mathrm{d}x\right\}+\frac{1}{2}\frac{d}{dt}\int_\Omega (g\Box u_x)(t)\mathrm{d}x$$
$$=-\mu_1\int_\Omega u_t^2(x,t)\mathrm{d}x-\mu_2\int_\Omega u_t(x,t)z(x,1,t)\mathrm{d}x-\frac{1}{2}g(t)\int_\Omega u_x^2\mathrm{d}x+\frac{1}{2}\int_\Omega (g'\Box u_x)(t)\mathrm{d}x. \tag{4.3}$$

Now, multiplying the third equation of (2.3) by $\frac{\zeta}{\tau}z$ and integrating the result over $\Omega\times(0,1)$ with respect to x and ρ respectively, we have

$$\frac{\zeta}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x=-\frac{\zeta}{2\tau}\int_\Omega (z^2(x,1)-z^2(x,0))\mathrm{d}x. \tag{4.4}$$

Using (2.5), (4.3) and (4.4), we obtain

$$\frac{d}{dt}E(t)=-\left(\mu_1-\frac{\zeta}{2\tau}\right)\int_\Omega u_t^2(x,t)\mathrm{d}x-\frac{\zeta}{2\tau}\int_\Omega z^2(x,1,t)\mathrm{d}x-\mu_2\int_\Omega u_t(x,t)z(x,1,t)\mathrm{d}x$$
$$-\frac{1}{2}g(t)\int_\Omega u_x^2\mathrm{d}x+\frac{1}{2}\int_\Omega (g'\Box u_x)(t)\mathrm{d}x. \tag{4.5}$$

By Young's inequality in (4.5), we get

$$\frac{d}{dt}E(t)\le-\left(\mu_1-\frac{\zeta}{2\tau}-\frac{\mu_2}{2}\right)\int_\Omega u_t^2(x,t)\mathrm{d}x-\left(\frac{\zeta}{2\tau}-\frac{\mu_2}{2}\right)\int_\Omega z^2(x,1,t)\mathrm{d}x+\frac{1}{2}\int_\Omega (g'\Box u_x)(t)\mathrm{d}x.$$

Then exploiting (2.6) our conclusion holds. The proof is complete. □

Now, we define the functional $\mathscr{D}(t)$ as follows

$$\mathscr{D}(t)=\int_\Omega uu_t\mathrm{d}x+\frac{\mu_1}{2}\int_\Omega u^2\mathrm{d}x+\int_{L_1}^{L_2} vv_t\mathrm{d}x.$$

Then we have the following estimate.

Lemma 4.2. *The functional $\mathscr{D}(t)$ satisfies*

$$\frac{d}{dt}\mathscr{D}(t)\le\int_\Omega u_t^2\mathrm{d}x+\int_{L_1}^{L_2} v_t^2\mathrm{d}x+(L^2\varepsilon+\varepsilon-\beta(t))\int_\Omega u_x^2\mathrm{d}x+\frac{1}{4\varepsilon}(a-\beta(t))\int_\Omega (g\Box u_x)\mathrm{d}x$$
$$+\frac{\mu_2^2}{4\varepsilon}\int_\Omega z^2(x,1,t)\mathrm{d}x-\int_{L_1}^{L_2} bv_x^2\mathrm{d}x. \tag{4.6}$$

Proof. Taking the derivative of $\mathscr{D}(t)$ with respect to t and using (2.3), we have

$$\frac{d}{dt}\mathscr{D}(t)=\int_\Omega u_t^2\mathrm{d}x-\int_\Omega (au_x-g*u_x)u_x\mathrm{d}x-\mu_2\int_\Omega z(x,1,t)u\mathrm{d}x+\int_{L_1}^{L_2} v_t^2\mathrm{d}x-\int_{L_1}^{L_2} bv_x^2\mathrm{d}x$$
$$=\int_\Omega u_t^2\mathrm{d}x-\beta(t)\int_\Omega u_x^2\mathrm{d}x-\int_\Omega (g\diamond u_x)u_x\mathrm{d}x-\mu_2\int_\Omega z(x,1,t)u\mathrm{d}x+\int_{L_1}^{L_2} v_t^2\mathrm{d}x-\int_{L_1}^{L_2} bv_x^2\mathrm{d}x. \tag{4.7}$$

By the boundary condition (1.2), we have

$$u^2(x,t)=\left(\int_0^x u_x(x,t)\mathrm{d}x\right)^2\le L_1\int_0^{L_1} u_x^2(x,t)\mathrm{d}x,\quad x\in[0,L_1],$$

$$u^2(x,t)\le(L_3-L_2)\int_{L_2}^{L_3} u_x^2(x,t)\mathrm{d}x,\quad x\in[L_2,L_3],$$

which implies

$$\int_\Omega u^2(x,t)\mathrm{d}x \le L^2 \int_\Omega u_x^2 \mathrm{d}x, \quad x \in \Omega, \tag{4.8}$$

where $L = \max\{L_1, L_3 - L_2\}$. By exploiting Young's inequality and (4.8), we get for any $\varepsilon > 0$

$$\mu_2 \int_\Omega z(x,1,t)u\mathrm{d}x \le \frac{\mu_2^2}{4\varepsilon}\int_\Omega z^2(x,1,t)\mathrm{d}x + L^2\varepsilon \int_\Omega u_x^2\mathrm{d}x. \tag{4.9}$$

Young's inequality and (G1) imply that

$$\begin{aligned}\int_\Omega (g\diamond u_x)u_x\mathrm{d}x &\le \varepsilon\int_\Omega u_x^2\mathrm{d}x + \frac{1}{4\varepsilon}\int_\Omega (g\diamond u_x)^2\mathrm{d}x \\ &\le \varepsilon\int_\Omega u_x^2\mathrm{d}x + \frac{1}{4\varepsilon}(a-\beta(t))\int_\Omega (g\Box u_x)\mathrm{d}x. \end{aligned}\tag{4.10}$$

Inserting the estimates (4.9) and (4.10) into (4.7), then (4.6) is fulfilled. The proof is complete. □

Now, as in Lemma 4.5 of [21], we introduce the function

$$q(x) = \begin{cases} x - \dfrac{L_1}{2}, & x \in [0, L_1], \\ \dfrac{L_1}{2} - \dfrac{L_1 + L_3 - L_2}{2(L_2 - L_1)}(x - L_1), & x \in (L_1, L_2), \\ x - \dfrac{L_2 + L_3}{2}, & x \in [L_2, L_3]. \end{cases}\tag{4.11}$$

It is easy to see that $q(x)$ is bounded, that is $|q(x)| \le M$, where $M = \max\left\{\dfrac{L_1}{2}, \dfrac{L_3 - L_2}{2}\right\}$ is a positive constant. And we define the functionals

$$\mathscr{F}_1(t) = -\int_\Omega q(x)u_t(au_x - g*u_x)\mathrm{d}x, \quad \mathscr{F}_2(t) = -\int_{L_1}^{L_2} q(x)v_xv_t\mathrm{d}x. \tag{4.12}$$

Then we have the following estimates.

Lemma 4.3. *The functionals $\mathscr{F}_1(t)$ and $\mathscr{F}_2(t)$ satisfy*

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_1(t) \le &\left[-\frac{q(x)}{2}(au_x - g*u_x)^2\right]_{\partial\Omega} - \left[\frac{a}{2}q(x)u_t^2\right]_{\partial\Omega} + \left[\frac{a}{2} + \frac{\mu_1^2}{2\varepsilon_1} + \frac{M^2}{4\varepsilon_1}\right]\int_\Omega u_t^2\mathrm{d}x \\ &+ \left[\varepsilon_1M^2a^2 + \beta^2(t) + 2M^2\varepsilon_1(a-\beta(t))^2 + c_5^2\varepsilon_1\right]\int_\Omega u_x^2\mathrm{d}x + \frac{\mu_2^2}{2\varepsilon_1}\int_\Omega z^2(x,1,t)\mathrm{d}x \\ &+ (1 + 2M^2\varepsilon_1)(a-\beta(t))\int_\Omega (g\Box u_x)\mathrm{d}x + (a-\beta(t))\varepsilon_1\int_\Omega (g'\Box u_x)\mathrm{d}x \end{aligned}\tag{4.13}$$

and

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_2(t) \le &-\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)}\left(\int_{L_1}^{L_2} v_t^2\mathrm{d}x + \int_{L_1}^{L_2} bv_x^2\mathrm{d}x\right) + \frac{L_1}{4}v_t^2(L_1) + \frac{L_3 - L_2}{4}v_t^2(L_2) \\ &+ \frac{b}{4}\left((L_3 - L_2)v_x^2(L_2,t) + L_1v_x^2(L_1,t)\right). \end{aligned}\tag{4.14}$$

Proof. Taking the derivative of $\mathscr{F}_1(t)$ with respect to t and using (2.3), we get

$$\frac{d}{dt}\mathscr{F}_1(t) = -\int_\Omega q(x)u_{tt}(au_x - g*u_x)\mathrm{d}x - \int_\Omega q(x)u_t\left(au_{xt} - g(t)u_x(t) + (g'\diamond u_x)(t)\right)\mathrm{d}x$$

$$
\begin{aligned}
&= \left[-\frac{q(x)}{2}(au_x - g * u_x)^2\right]_{\partial\Omega} + \frac{1}{2}\int_\Omega q'(x)(au_x - g * u_x)^2 \mathrm{d}x - \left[\frac{a}{2}q(x)u_t^2\right]_{\partial\Omega} \\
&\quad + \frac{a}{2}\int_\Omega q'(x)u_t^2 \mathrm{d}x - \int_\Omega q(x)(\mu_1 u_t(x,t) + \mu_2 z(x,1,t))(g * u_x)\mathrm{d}x \\
&\quad + \int_\Omega q(x) a u_x(\mu_1 u_t(x,t) + \mu_2 z(x,1,t))\mathrm{d}x - \int_\Omega q(x) u_t[(g' \diamond u_x)(t) - g(t)u_x]\mathrm{d}x. \qquad (4.15)
\end{aligned}
$$

We note that

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega q'(x)(au_x - g * u_x)^2 \mathrm{d}x = \frac{1}{2}\int_\Omega \left[\left(a - \int_0^t g(s)\mathrm{d}s\right) u_x + g \diamond u_x\right]^2 \mathrm{d}x \\
&\le \int_\Omega |\beta(t)|^2 u_x^2 \mathrm{d}x + \int_\Omega |g \diamond u_x|^2 \mathrm{d}x \le \int_\Omega |\beta(t)|^2 u_x^2 \mathrm{d}x + (a - \beta(t)) \int_\Omega (g \Box u_x)\mathrm{d}x. \qquad (4.16)
\end{aligned}
$$

Young's inequality gives us for any $\varepsilon_1 > 0$,

$$
\int_\Omega q(x) a u_x(\mu_1 u_t(x,t) + \mu_2 z(x,1,t))\mathrm{d}x \le \varepsilon_1 M^2 a^2 \int_\Omega u_x^2 \mathrm{d}x + \frac{\mu_1^2}{4\varepsilon_1}\int_\Omega u_t^2 \mathrm{d}x + \frac{\mu_2^2}{4\varepsilon_1}\int_\Omega z^2(x,1,t)\mathrm{d}x, \qquad (4.17)
$$

$$
\begin{aligned}
&\int_\Omega q(x)(\mu_1 u_t(x,t) + \mu_2 z(x,1,t))(g * u_x)\mathrm{d}x \\
\le& \varepsilon_1 M^2 \int_\Omega (g * u_x)^2 \mathrm{d}x + \frac{\mu_1^2}{4\varepsilon_1}\int_\Omega u_t^2 \mathrm{d}x + \frac{\mu_2^2}{4\varepsilon_1}\int_\Omega z^2(x,1,t)\mathrm{d}x \\
\le& 2\varepsilon_1 M^2 (a - \beta(t))^2 \int_\Omega u_x^2 \mathrm{d}x + 2M^2\varepsilon_1(a - \beta(t))\int_\Omega (g \Box u_x)\mathrm{d}x + \frac{\mu_1^2}{4\varepsilon_1}\int_\Omega u_t^2 \mathrm{d}x + \frac{\mu_2^2}{4\varepsilon_1}\int_\Omega z^2(x,1,t)\mathrm{d}x \qquad (4.18)
\end{aligned}
$$

and

$$
\int_\Omega q(x) u_t[(g' \diamond u_x)(t) - g(t)u_x]\mathrm{d}x \le \frac{M^2}{4\varepsilon_1}\int_\Omega u_t^2 \mathrm{d}x + c_5^2 \varepsilon_1 \int_\Omega u_x^2 \mathrm{d}x + (a - \beta(t))\varepsilon_1 \int_\Omega (g' \Box u_x)\mathrm{d}x. \qquad (4.19)
$$

Inserting (4.16)-(4.19) into (4.15), we get (4.13).

By the same method, taking the derivative of $\mathscr{F}_1(t)$ with respect to t, we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathscr{F}_2(t) &= -\int_{L_1}^{L_2} q(x) v_{xt} v_t \mathrm{d}x - \int_{L_1}^{L_2} q(x) v_x v_{tt} \mathrm{d}x \\
&= \left[-\frac{1}{2}q(x)v_t^2\right]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} q'(x) v_t^2 \mathrm{d}x + \frac{1}{2}\int_{L_1}^{L_2} b q'(x) v_x^2 \mathrm{d}x + \left[-\frac{b}{2}q(x)v_x^2\right]_{L_1}^{L_2} \\
&\le -\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)}\left(\int_{L_1}^{L_2} v_t^2 \mathrm{d}x + \int_{L_1}^{L_2} b v_x^2 \mathrm{d}x\right) + \frac{L_1}{4}v_t^2(L_1) + \frac{L_3 - L_2}{4}v_t^2(L_2) \\
&\quad + \frac{b}{4}\left((L_3 - L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\right).
\end{aligned}
$$

Thus, the proof of Lemma 4.3 is finished. □

As in [2], we define the functional

$$
\mathscr{F}_3(t) = \tau \int_\Omega \int_0^1 e^{-\tau\rho} z^2(x,\rho,t)\mathrm{d}\rho \mathrm{d}x,
$$

then we have the following estimate.

Lemma 4.4 ([2]). *The functionals $\mathscr{F}_3(t)$ satisfies*

$$\frac{d}{dt}\mathscr{F}_3(t) \leq -c_6\left(\int_\Omega z^2(x,1,t)\mathrm{d}x + \tau\int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\right) + \int_\Omega u_t^2(x,t)\mathrm{d}x.$$

Now, we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. We define the Lyapunov functional

$$L(t) = N_1E(t) + N_2\mathscr{D}(t) + N_3\mathscr{F}_1(t) + N_4\mathscr{F}_2(t) + \mathscr{F}_3(t), \tag{4.20}$$

where N_1, N_2, N_3 and N_4 are positive constants that will be fixed later.

Taking the derivative of (4.20) with respect to t and making use of the above lemmas, we have

$$\begin{aligned}
\frac{d}{dt}L(t) \leq & \left\{-N_1c_4 + 1 + N_2 + N_3\left(\frac{a}{2} + \frac{\mu_1^2}{2\varepsilon_1} + \frac{M^2}{4\varepsilon_1}\right)\right\}\int_\Omega u_t^2\mathrm{d}x \\
& + \left\{-N_1c_4 - c_6 + \frac{\mu_2^2N_2}{4\varepsilon} + \frac{\mu_2^2N_3}{2\varepsilon_1}\right\}\int_\Omega z^2(x,1,t)\mathrm{d}x \\
& + \left\{-N_2(\beta(t) - L^2\varepsilon - \varepsilon) + N_3\left(\varepsilon_1M^2a^2 + \beta(t)^2 + 2M^2\varepsilon_1(a-\beta(t))^2 + c_5^2\varepsilon_1\right)\right\}\int_\Omega u_x^2\mathrm{d}x \\
& + \left\{-\frac{b(L_1+L_3-L_2)}{4(L_2-L_1)}N_4 - N_2b\right\}\int_{L_1}^{L_2} v_x^2\mathrm{d}x + \left\{-\frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4 + N_2\right\}\int_{L_1}^{L_2} v_t^2\mathrm{d}x \\
& + (N_4 - bN_3)\frac{b}{4}\left((L_3-L_2)v_x^2(L_2,t) + L_1v_x^2(L_1,t)\right) \\
& + (N_4 - aN_3)\left[\frac{L_1}{4}v_t^2(L_1,t) + \frac{L_3-L_2}{4}v_t^2(L_2,t)\right] \\
& + c(N_2,N_3)\int_\Omega(g\Box u_x)\mathrm{d}x + \left(\frac{N_1}{2} - c(N_3)\right)\int_\Omega(g'\Box u_x)\mathrm{d}x.
\end{aligned} \tag{4.21}$$

At this moment, we wish all coefficients except the last two in (4.21) will be negative. In fact, under assumption (2.7), we can find N_2, N_3 and N_4 such that

$$N_2 < \frac{L_1+L_3-L_2}{4(L_2-L_1)}N_4, \quad N_4 < bN_3, \quad N_4 < aN_3, \quad N_2 > 2N_3\beta_0.$$

Once the above constants are fixed, we may choose ε and ε_1 small enough such that

$$N_2(L^2\varepsilon + \varepsilon) + N_3\left(\varepsilon_1M^2a^2 + 2M^2\varepsilon_1(a-\beta(t))^2 + c_5^2\varepsilon_1\right) < N_2 - N_3\beta(t).$$

Finally, choosing N_1 large enough such that the first two coefficients in (4.21) are negative and the last coefficient is positive. From the above, we deduce that, there exists two positive constants α_1 and α_2 such that (4.21) becomes

$$\frac{d}{dt}L(t) \leq -\alpha_1E(t) + \alpha_2\int_\Omega(g\Box u_x)\mathrm{d}x. \tag{4.22}$$

On the other hand, by the definition of the functionals $\mathscr{D}(t)$, $\mathscr{F}_1(t)$, $\mathscr{F}_2(t)$, $\mathscr{F}_3(t)$ and $E(t)$, for N_1 large enough, there exists a positive constant α_3 satisfying

$$|N_2\mathscr{D}(t) + N_3\mathscr{F}_1(t) + N_4\mathscr{F}_2(t) + \mathscr{F}_3(t)| \leq \alpha_3E(t),$$

which implies that

$$(N_1 - \alpha_3)E(t) \leq L(t) \leq (N_1 + \alpha_3)E(t).$$

In order to finish the proof of the stability estimates, we need to estimate the last term in (4.22). Exploiting (G2) and (4.1), we have

$$\xi(t)\int_\Omega (g\Box u_x)\mathrm{d}x \le \int_\Omega [(\xi g)\Box u_x]\mathrm{d}x \le -\int_\Omega (g'\Box u_x)\mathrm{d}x \le -2\frac{d}{dt}E(t). \tag{4.23}$$

Now, we define functionals $\mathscr{L}(t)$ as

$$\mathscr{L}(t) = \xi(t)L(t) + 2\alpha_2 E(t).$$

The fact that $L(t)$ and $E(t)$ are equivalent and (G2) imply that, for some positive constants η_1 and η_2,

$$\eta_1 E(t) \le \mathscr{L}(t) \le \eta_2 E(t), \tag{4.24}$$

Using (4.23), (4.24) and (G2), we obtain

$$\begin{aligned}\frac{d}{dt}\mathscr{L}(t) &= \xi'(t)L(t) + \xi(t)\frac{d}{dt}L(t) + 2\alpha_2\frac{d}{dt}E(t)\\ &\le \xi(t)\left(-\alpha_1 E(t) + \alpha_2\int_\Omega (g\Box u_x)\mathrm{d}x\right) + 2\alpha_2\frac{d}{dt}E(t)\\ &\le -\alpha_1\xi(t)E(t)\\ &\le -\gamma_0\xi(t)\mathscr{L}(t),\end{aligned}$$

where $\gamma_0 = \dfrac{\alpha_1}{\eta_2}$. We conclude that, for any $\gamma_1 \in (0, \gamma_0)$,

$$\frac{d}{dt}\mathscr{L}(t) \le -\gamma_1\xi(t)\mathscr{L}(t). \tag{4.25}$$

A simple integration of (4.25) leads to

$$\mathscr{L}(t) \le \mathscr{L}(0)e^{-\gamma_1\int_0^t \xi(s)\mathrm{d}s}, \quad \forall t \ge 0. \tag{4.26}$$

Again, use of (4.24) and (4.26) yields the desired result (2.8). This completes the proof of Theorem 2.2. □

Remark 4.5. Here we consider some examples to illustrate the energy decay rates obtained by Theorem 2.2.

Example 1. Let

$$g(t) = k_1 e^{-k_2(1+t)^q}, \quad 0 < q < 1, \quad k_1 > 0, \quad k_2 > 0,$$

then it is clear that (G2) holds for $\xi(t) = k_2 q(1+t)^{q-1}$. Consequently, by (2.8), we obtain the decay result

$$E(t) \le \tilde{c_6}e^{-\tilde{c_7}k_2(1+t)^q},$$

where $\tilde{c_6}$ and $\tilde{c_7}$ are positive constants.

Example 2. If

$$g(t) = k_3 e^{-k_4[\ln(1+t)]^p}, \quad k_3 > 0, \quad k_4 > 0,$$

then our assumption (G2) holds with $\xi(t) = \dfrac{k_4 p[\ln(1+t)]^{p-1}}{1+t}$. Eq. (2.8) gives us

$$E(t) \le \tilde{c_8}e^{-\tilde{c_9}k_4[\ln(1+t)]^p},$$

where $\tilde{c_8}$ and $\tilde{c_9}$ are positive constants.

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