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A bilateral contact problem with adhesion and damage between two viscoelastic bodies

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Abstract

This paper deals with the study of a mathematical model which describes the bilateral, frictionless adhesive contact between two viscoelastic bodies with damage. The adhesion of the contact surfaces is considered and is modeled with a surface variable, the bonding field, whose evolution is described by a first order differential equation. We establish a variational formulation for the problem and prove the existence and uniqueness result of the solution. The proofs are based on time-dependent variational equalities, a classical existence and uniqueness result on parabolic equations, differential equations, and fixed-point arguments.

Keywords: Bilateral frictionless contact, adhesion, viscoelastic materials, fixed point, damage, weak solution.
2010 MSC: 74M15, 74F99, 74G25, 74R99.

1. Introduction

In this paper we study a mathematical model which describes the adhesive contact between two viscoelastic bodies, when the frictional tangential traction is negligible in comparison with the traction due to adhesion. As in [1, 3, 5, 6], we use the bonding field as an additional variable, defined on the common

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part of the boundary. We derive a variational formulation of the model then we prove its unique solvability, which provides the existence of a unique weak solution to the adhesive contact problem.

The subject of damage is extremely important in design engineering since. It affects directly the useful life of the designed structure or component. There exists a very large engineering literature on it. Models taking into account the influence of the internal damage of the material on the contact process have been investigated mathematically. General novel models for damage were derived in [8, 9, 11, 12] from the virtual power principle. Mathematical analysis of one-dimensional problems can be found in [7]. In all these papers, the damage of the material is described by a damage function α^ℓ restricted to have values between zero and one, when $\alpha^\ell = 1$, there is no damage in the material, when $\alpha^\ell = 0$ the material is completely damaged, when $0 < \alpha^\ell < 1$ there is a partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [7, 10].

The adhesive contact between bodies, when a glue is added to keep the surfaces from relative motion, has also recently received increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [4, 5, 6, 14, 16] and recently in the monographs [16, 17]. The novelty in all the above papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β, it describes the pointwise fractional density of active bonds on the contact surface, and is sometimes referred to as the intensity of adhesion. Following [5, 6], the bonding field satisfies the restrictions $0 \leq \beta \leq 1$, when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [14, 16].

The paper is organized as follows. In section 2 we present the notation and some preliminaries. In section 3 we present the mechanical problem, we list the assumptions on the data, and give the variational formulation of the problem. In section 4 we state and prove our main existence and uniqueness result, Theorem 4.1. The proof is based on arguments of time-dependent nonlinear equations with monotone operators, a fixed-point argument, and a classical existence and uniqueness result on parabolic equations.

2. Notations and preliminaries

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$; " $\cdot$ " and $|\cdot|$ represent the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ respectively. Thus, for every $\mathbf{u}^\ell, \mathbf{v}^\ell \in \mathbb{R}^d$ and $\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell \in \mathbb{S}^d$ we have:

$$\mathbf{u}^\ell \cdot \mathbf{v}^\ell = u_i^\ell \cdot v_i^\ell, \quad |\mathbf{v}^\ell| = (\mathbf{v}^\ell, \mathbf{v}^\ell)^{1/2},$$
$$\boldsymbol{\sigma}^\ell \cdot \boldsymbol{\tau}^\ell = \sigma_{ij}^\ell \cdot \tau_{ij}^\ell, \quad |\boldsymbol{\tau}^\ell| = (\boldsymbol{\tau}^\ell, \boldsymbol{\tau}^\ell)^{1/2}.$$

Here and below, the indices i and j run between 1 and d the summation convention over repeated indices is adopted.

Let Ω^1 and Ω^2 be two bounded domains in $\mathbb{R}^d$. Everywhere in this paper, we use a superscript ℓ to indicate that a quantity is related to the domain Ω^ℓ, $\ell = 1, 2$. For each domain Ω^ℓ, we assume that its boundary Γ^ℓ is Lipschitz continuous and is partitioned into three disjoint measurable parts Γ_1^ℓ, Γ_2^ℓ and Γ_3^ℓ, with $meas\Gamma_1^\ell > 0$. The unit outward normal to Γ^ℓ is denoted by $\nu^\ell = (\nu_i^\ell)$. We also use the notation

$$H^\ell = \{\mathbf{v}^\ell = (v_i^\ell) \mid v_i^\ell \in L^2(\Omega^\ell)\}, \quad H_1^\ell = \{\mathbf{v}^\ell = (v_i^\ell) \mid v_i^\ell \in H^1(\Omega^\ell)\},$$
$$\mathcal{H}^\ell = \{\boldsymbol{\tau}^\ell = (\tau_{ij}^\ell) \mid \tau_{ij}^\ell = \tau_{ji}^\ell \in L^2(\Omega^\ell)\}, \quad \mathcal{H}_1^\ell = \{\boldsymbol{\tau}^\ell \in \mathcal{H}^\ell \mid \operatorname{Div} \tau_{ij}^\ell \in H^\ell\}.$$

The spaces H^ℓ, H_1^ℓ, $\mathcal{H}^\ell$, and $\mathcal{H}_1^\ell$ are real Hilbert spaces with the canonical inner products given by

$$(\mathbf{u}^\ell, \mathbf{v}^\ell)_{H^\ell} = \int_{\Omega^\ell} \mathbf{u}^\ell \cdot \mathbf{v}^\ell dx, \qquad (\mathbf{u}^\ell, \mathbf{v}^\ell)_{H_1^\ell} = (\mathbf{u}^\ell, \mathbf{v}^\ell)_{H^\ell} + (\nabla \mathbf{u}^\ell, \nabla \mathbf{v}^\ell)_{H^\ell},$$

$$(\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} = \int_{\Omega^\ell} \boldsymbol{\sigma}^\ell \cdot \boldsymbol{\tau}^\ell dx, \qquad (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}_1^\ell} = (\boldsymbol{\sigma}^\ell, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell} + (\mathrm{Div}\, \boldsymbol{\sigma}^\ell, \mathrm{Div}\, \boldsymbol{\tau}^\ell)_{\mathcal{H}^\ell},$$

and the associated norms $\|\cdot\|_{H^\ell}$, $\|\cdot\|_{H_1^\ell}$, $\|\cdot\|_{\mathcal{H}^\ell}$, and $\|\cdot\|_{\mathcal{H}_1^\ell}$ respectively.

Here and below we use the notation

$$\nabla \mathbf{u}^\ell = (u^\ell_{i,j}), \quad \varepsilon(\mathbf{u}^\ell) = (\varepsilon_{ij}(u^\ell)), \quad \varepsilon_{ij}(u^\ell) = \frac{1}{2}(u^\ell_{i,j} + u^\ell_{j,i}), \quad \forall \mathbf{u}^\ell \in H_1^\ell,$$
$$\mathrm{Div}\, \boldsymbol{\sigma}^\ell = (\sigma^\ell_{ij,j}), \quad \forall \boldsymbol{\sigma}^\ell \in \mathcal{H}_1^\ell.$$

Now, we define the space V^ℓ by

$$V^\ell = \{\mathbf{v}^\ell \in H_1^\ell \mid \mathbf{v}^\ell = 0 \text{ on } \Gamma_1^\ell\}.$$

Since $\mathrm{meas}\, \Gamma_1^\ell > 0$, the following Korn's inequality holds

$$\|\varepsilon(\mathbf{v}^\ell)\|_{\mathcal{H}^\ell} \geq c_K \|\mathbf{v}^\ell\|_{H_1^\ell}, \quad \forall \mathbf{v}^\ell \in V^\ell, \tag{2.1}$$

where the constant c_K denotes a positive constant which may depends only on Ω^ℓ, Γ_1^ℓ (see [13]). Over the space V^ℓ, we consider the inner product given by

$$(\mathbf{u}^\ell, \mathbf{v}^\ell)_{V^\ell} = (\varepsilon(\mathbf{u}^\ell), \varepsilon(\mathbf{v}^\ell))_{\mathcal{H}^\ell}, \quad \forall \mathbf{u}^\ell \mathbf{v}^\ell \in V^\ell, \tag{2.2}$$

and let $\|\cdot\|_{V^\ell}$ be the associated norm. It follows from Korn's inequality (2.1) that the norms $\|\cdot\|_{H_1^\ell}$ and $\|\cdot\|_{V^\ell}$ are equivalent on V^ℓ. Then $(V^\ell, \|\cdot\|_{V^\ell})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (2.2), there exists a constant $c_0^\ell > 0$, depending only the ℓ such that

$$\|\mathbf{v}^\ell\|_{L^2(\Gamma_3)^3} \leq c_0^\ell \|\mathbf{v}^\ell\|_{V^\ell}, \quad \forall \mathbf{v}^\ell \in V^\ell, \tag{2.3}$$

and we denote by c_0 a constant given by

$$c_0 = \max\{c_0^1, c_0^2\}. \tag{2.4}$$

We define the set $\mathbb{V}$ of admissible displacement fields by

$$\mathbb{V} = \{\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in V^1 \times V^2 \mid v^1_\nu + v^2_\nu = 0 \text{ on } \Gamma_3\}.$$

The space V is a real Hilbert space endowed with the canonical inner products $(\cdot,\cdot)_V$ and the associated norm $\|\cdot\|_V$.

Since the boundary Γ^ℓ is Lipschitz continuous, the unit outward normal vector ν^ℓ on the boundary Γ^ℓ is defined a.e. For every vector field, $\mathbf{v}^\ell \in H_1^\ell$ we use the notation $\mathbf{v}^\ell|_{\Gamma^\ell}$ for the trace of $\mathbf{v}^\ell$ on Γ^ℓ and we denote by v^ℓ_ν and $\mathbf{v}^\ell_\tau$ the normal and the tangential components of $\mathbf{v}^\ell$ on the boundary, given by

$$v^\ell_\nu = \mathbf{v}^\ell \cdot \nu^\ell, \quad \mathbf{v}^k_\tau = \mathbf{v}^\ell - v^\ell_\nu \nu^\ell.$$

For a regular (say C^1) stress field $\boldsymbol{\sigma}^\ell$, the application of its trace on the boundary to ν^ℓ is the Cauchy stress vector $\boldsymbol{\sigma}^\ell \nu^\ell$. We define, similarly, the normal and tangential components of the stress on the boundary by the formulas

$$\sigma^\ell_\nu = (\boldsymbol{\sigma}^\ell \nu^\ell) \cdot \nu^\ell, \quad \boldsymbol{\sigma}^\ell_\tau = \boldsymbol{\sigma}^\ell \nu^\ell - \sigma^\ell_\nu \nu^\ell,$$

When $\boldsymbol{\sigma}^\ell$ is a regular function, the following Green's type formula holds,

$$(\boldsymbol{\sigma}^\ell, \varepsilon(\mathbf{v}^\ell))_{\mathcal{H}^\ell} + (\mathrm{Div}\, \boldsymbol{\sigma}^\ell, \mathbf{v}^\ell)_{H^\ell} = \int_{\Gamma^\ell} \boldsymbol{\sigma}^\ell \nu^\ell . \mathbf{v}^\ell da, \quad \forall \mathbf{v}^\ell \in H_1^\ell. \tag{2.5}$$

Here and below we denote by Div the divergence operator for tensor valued functions defined on Ω^1 or Ω^2.

In order to simplify the notations, we define product spaces

$$V = V^1 \times V^2, \qquad H = H^1 \times H^2, \qquad H_1 = H_1^1 \times H_1^2,$$
$$\mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2 \quad \text{and} \quad \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2.$$

They are all Hilbert spaces endowed with the canonical inner products denoted by $(\cdot,\cdot)_V$, $(\cdot,\cdot)_H$, $(\cdot,\cdot)_{H_1}$, $(\cdot,\cdot)_{\mathcal{H}}$, and $(\cdot,\cdot)_{\mathcal{H}_1}$ respectively. Moreover, we denote by $(V', \|\cdot\|_{V'})$ the strong dual of V and $(\cdot,\cdot)_{V'\times V}$ will represent the duality between V' and V.

Finally, for every real Banach space X and $T > 0$, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \leq p \leq +\infty$, $\ell = 1,2$, and we use the dot above to indicate the derivative with respect to the time variable.

3. The model and its variational formulation

We describe the model for the process , we present its variational formulation. The physical setting is the following. We consider two elastic bodies that occupy a bounded domains Ω^1 and Ω^2. The two bodies are in bilateral, frictionless, adhesive contact along the common part $\Gamma_3^1 = \Gamma_3^2$, which will be denoted in what follows. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on $\Gamma_1^\ell \times (0,T)$, so the displacement field vanishes there. A surface tractions of density f_2^ℓ act on $\Gamma_2^\ell \times (0,T)$ and a body force of density f_0^ℓ acts in $\Omega^\ell \times (0,T)$.

We denote by u^ℓ the displacement vectors, by σ^ℓ the stress tensors, by α^ℓ a damage field, and by $\varepsilon^\ell = \varepsilon(u^\ell)$ the linearized strain tensors. We model the materials with nonlinear viscoelastic constitutive law with damage:

$$\boldsymbol{\sigma}^\ell = \mathcal{A}^\ell(\varepsilon(\dot{\mathbf{u}}^\ell)) + \mathcal{G}^\ell(\varepsilon(\mathbf{u}^\ell), \alpha^\ell),$$

where $\mathcal{A}^\ell$ is a given nonlinear viscosity function and $\mathcal{G}^\ell$ is a given nonlinear elasticity function which depends on the internal state variable describing the damage of the material caused by elastic deformation and the dot above represents the time derivative.

The differential inclusion used for the evolution of the damage field is

$$\dot{\alpha}^\ell - k^\ell \Delta \alpha^\ell + \partial\varphi_{K^\ell}\ (\alpha^\ell) \ni \mathcal{S}^\ell(\varepsilon(\mathbf{u}^\ell), \alpha^\ell),$$

where K^ℓ is the set of admissible damage test functions, $\mathcal{S}$ is the source function of the damage

$$K^\ell = \{\zeta \in H^1(\Omega^\ell) \mid 0 \leq \zeta \leq 1 \text{ a.e. } \in \Omega^\ell\},$$

where k^ℓ is a positive coefficient, $\partial_{\varphi_{K^\ell}}$ denotes the subdifferential of the indicator function φ_{K^ℓ}, and $\mathcal{S}^\ell$ is a given constitutive function which describes the sources of the damage in the system.

We assume that the normal derivative of α^ℓ represents a homogeneous Newmann boundary condition where

$$\frac{\partial \alpha^\ell}{\partial \nu^\ell} = 0$$

with

$$\alpha = (\alpha^1, \alpha^2).$$

Now we describe the conditions on the contact surface Γ_3. We assume that the contact is bilateral, i.e., there is no separation between the bodies during the process. Therefore

$$u_\nu^1 + u_\nu^2 = 0 \quad \text{on } \Gamma_3 \times [0,T].$$

Moreover,

$$\nu^1 = -\nu^2 \quad \text{on } \Gamma_3, \quad \text{and } \boldsymbol{\sigma}^1\nu^1 = -\boldsymbol{\sigma}^2\nu^2, \quad \text{on } \Gamma_3 \times [0,T].$$

Consequently,

$$\sigma_\nu^1 = \sigma_\nu^2 \quad \text{and} \quad \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \text{ on } \Gamma_3 \times [0,T].$$

Following [4, 5], we introduce a surface state variable β, the bonding field, which is a measure of the fractional intensity of adhesion between the surface and the foundation. This variable is restricted to values $0 \le \beta \le 1$; when $\beta = 0$ all the bonds are severed and there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ it measures the fraction of active bonds, and partial adhesion takes place.

We assume that the resistance to tangential motion is generated by the glue, in comparison to which the frictional traction can be neglected. Moreover, the tangential traction depends only on the bonding field and on the relative tangential displacement, that is

$$-\boldsymbol{\sigma}_\tau^1 = \boldsymbol{\sigma}_\tau^2 = p_\tau(\beta, \mathbf{u}_\tau^1 - \mathbf{u}_\tau^2) \quad \text{on } \Gamma_3 \times [0;T].$$

We assume that the evolution of the bonding field is governed by the differential equation

$$\dot{\beta} = H_{ad}(\beta, \mathbf{R}(|\mathbf{u}_\tau^1 - \mathbf{u}_\tau^2|)).$$

Here, H_{ad} is a general function discussed below, which vanishes when its first argument vanishes. The function $R : \mathbb{R}_+ \to \mathbb{R}_+$ is a truncation and is defined as

$$R(s) = \begin{cases} s & \text{if} \quad 0 \le s \le L \\ L & \text{if} \quad s > L, \end{cases} \tag{3.1}$$

where $L > 0$ is a characteristic length of the bonds (see, e.g., [14]). We use it in H_{ad} since usually, when the glue is stretched beyond the limit L, it does not contribute more to the bond strength.

Let β_0, the initial bonding field. We assume that the process is quasistatic and therefore we neglect the inertial term in the equation of motion. Then, the classical formulation of the mechanical problem may be stated as follows.

Problem P.

Find the displacement field $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$ such that $\mathbf{u}^\ell : \Omega^\ell \times [0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2)$ such that $\boldsymbol{\sigma}^{\boldsymbol{\ell}} : \Omega^\ell \times [0,T] \to \mathbb{S}^d$, a damage field $\alpha = (\alpha^1, \alpha^2)$ such that $\alpha^\ell : \Omega^\ell \times [0,T] \to \mathbb{R}$, and a bonding field $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ such that

$$\boldsymbol{\sigma}^{\boldsymbol{\ell}} = \mathcal{A}^\ell \varepsilon(\dot{\mathbf{u}}^\ell) + \mathcal{G}^\ell(\varepsilon(\mathbf{u}^\ell), \alpha^\ell), \qquad \text{in } \Omega^\ell \times (0,T), \tag{3.2}$$

$$\dot{\alpha}^\ell - k^\ell \Delta \alpha^\ell + \partial \varphi_{K^\ell}(\alpha^\ell) \ni S^\ell(\varepsilon(\mathbf{u}^\ell), \alpha^\ell), \qquad \text{in } \Omega^\ell \times (0,T), \tag{3.3}$$

$$\operatorname{Div} \boldsymbol{\sigma}^\ell + f_0^\ell = \mathbf{0}, \qquad \text{in } \Omega^\ell \times (0,T), \tag{3.4}$$

$$\mathbf{u}^\ell = \mathbf{0}, \qquad \text{on } \Gamma_1^\ell \times (0,T), \tag{3.5}$$

$$\boldsymbol{\sigma}^\ell \boldsymbol{\nu}^\ell = f_2^\ell, \qquad \text{on } \Gamma_2^\ell \times (0,T), \tag{3.6}$$

$$\boldsymbol{\sigma}_\nu^1 = \boldsymbol{\sigma}_\nu^2, \quad \mathbf{u}_\nu^1 + \mathbf{u}_\nu^2 = 0, \qquad \text{on } \Gamma_3 \times (0,T), \tag{3.7}$$

$$-\sigma_\tau^1 = \sigma_\tau^2 = p_\tau(\beta, u_\tau^1 - u_\tau^2), \qquad \text{on } \Gamma_3 \times (0,T), \tag{3.8}$$

$$\dot{\beta} = H_{ad}(\beta, R(|u_\tau^1 - u_\tau^2|)), \qquad \text{on } \Gamma_3 \times (0,T), \tag{3.9}$$

$$\frac{\partial \alpha^\ell}{\partial \nu^\ell} = 0, \qquad \text{on } \Gamma^\ell \times (0,T), \tag{3.10}$$

$$\mathbf{u}^\ell(0) = \mathbf{u}_0^\ell \qquad \alpha^\ell(0) = \alpha_0^\ell, \qquad \text{in } \Omega^\ell, \tag{3.11}$$

$$\beta(0) = \beta_0, \qquad \text{on } \Gamma_3. \tag{3.12}$$

In the study of the Problem **P**, we consider the following assumptions.

Assumptions.

The viscosity function $\mathcal{A}^\ell : \Omega^\ell \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) There exists } L_{\mathcal{A}^\ell} > 0 \text{ such that} \\ \quad |\mathcal{A}^\ell(x,\xi_1) - \mathcal{A}^\ell(x,\xi_2)| \le L_{\mathcal{A}^\ell}|\xi_1 - \xi_2| \\ \quad \forall \xi_1, \xi_2 \in \mathbb{S}^d, \text{a.e.} x \in \Omega^\ell. \\ \text{(b) There exists } m_{\mathcal{A}^\ell} > 0 \text{ such that} \\ \quad (\mathcal{A}^\ell(x,\xi_1) - \mathcal{A}^\ell(x,\xi_2)) \cdot (\xi_1 - \xi_2) \ge m_{\mathcal{A}^\ell}|\xi_1 - \xi_2|^2 \\ \quad \forall \xi_1, \xi_2 \in \mathbb{S}^d, \text{a.e. } x \in \Omega^\ell. \\ \text{(c) The map } x \mapsto \mathcal{A}^\ell(x,\xi) \text{ is Lebesgue measurable on } \Omega^\ell \\ \quad \text{for any } \xi \in \mathbb{S}^d, \\ \text{(d) The map } x \mapsto \mathcal{A}^\ell(x,0) \in \mathcal{H}^\ell. \end{cases} \tag{3.13}$$

The elasticity operator $\mathcal{G}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) There exists } L_{\mathcal{G}^\ell} > 0 \text{ such that} \\ \quad |\mathcal{G}^\ell(x,\xi_1,\alpha_1) - \mathcal{G}^\ell(x,\xi_2,\alpha_2)| \le L_{\mathcal{G}^\ell}(|\xi_1 - \xi_2| + |\alpha_1 - \alpha_2|) \\ \quad \forall \xi_1, \xi_2 \in \mathbb{S}^d,\ \forall \alpha_1, \alpha^2 \in \mathbb{R} \ \text{ a.e. } x \in \Omega^\ell. \\ \text{(b) For any } \xi \in \mathbb{S}^d \text{ and } \alpha \in \mathbb{R} \\ \quad x \mapsto \mathcal{G}^\ell(x,\xi,\alpha) \text{ is Lebesgue measurable on } \Omega^\ell. \\ \text{(c) The map } x \mapsto \mathcal{G}^\ell(x,0,0) \in \mathcal{H}^\ell. \end{cases} \tag{3.14}$$

The damage source function $\mathcal{S}^\ell : \Omega^\ell \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists } M_{\mathcal{S}^\ell} > 0 \text{ such that} \\ \quad |\mathcal{S}^\ell(\mathbf{x},\boldsymbol{\varepsilon}_1,\alpha_1) - \mathcal{S}^\ell(\mathbf{x},\boldsymbol{\varepsilon}_2,\alpha_2)| \le M_{\mathcal{S}^\ell}(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\alpha_1 - \alpha_2|). \\ \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}, \ \text{a.e. } \mathbf{x} \in \Omega^\ell. \\ \text{(b) For any } \boldsymbol{\varepsilon} \in \mathbb{S}^d,\ \alpha \in \mathbb{R}, \\ \quad \mathbf{x} \mapsto \mathcal{S}^\ell(\mathbf{x},\boldsymbol{\varepsilon},\alpha) \text{ is Lebesgue measurable on } \Omega^\ell. \\ \text{(c) The mapping } \mathbf{x} \mapsto \mathcal{S}^\ell(\mathbf{x},\mathbf{0},\mathbf{0}) \in \mathbf{L^2}(\boldsymbol{\Omega}^\ell). \end{cases} \tag{3.15}$$

The tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(a) There exists } L_\tau > 0 \ \text{ such that} \\ \quad |p_\tau(x,\beta_1,r_1) - \ p_\tau(x,\beta_2,r_2)| \le L_\tau(|\beta_1 - \beta_2| + |r_1 - r_2|) \\ \quad \forall \beta_1, \beta_2 \in \mathbb{R}, r_1, r_2 \in \mathbb{R}^d, \text{a.e.} x \in \Gamma_3. \\ \text{(b) The map } x \mapsto p_\tau(x,\beta,r) \text{ is Lebesgue measurable on } \Gamma_3 \\ \quad \forall \beta \in \mathbb{R}, r \in \mathbb{R}^d. \\ \text{(c) The map } x \mapsto p_\tau(x,0,0) \in L^\infty(\Gamma_3)^d. \\ \text{(d) } p_\tau(x,\beta,r).\nu(x) = 0 \quad \forall r \in \mathbb{R}^d \text{ such that } r.\nu(x) = 0, \text{a.e.} x \in \Gamma_3. \end{cases} \tag{3.16}$$

The adhesion function $H_{ad} : \Gamma_3 \times \mathbb{R} \times [0,L] \to \mathbb{R}$ satisfait

$$\begin{cases} \text{(a) There exists } L_{Had} > 0 \text{ such that} \\ \quad |H_{ad}(x,b_1,r_1) - H_{ad}(x,b_2,r_2)| \le L_{Had}(|b_1 - b_2| + |r_1 - r_2|) \\ \quad \forall b_1, b_2 \in \mathbb{R}\ \forall r_1, r_2 \in [0,L] \text{ a.e. } x \in \Gamma_3. \\ \text{(b) The map } x \to H_{ad}(x,b,r) \text{ is Lebesgue measurable on } \Gamma_3 \\ \quad \forall\, b \in \mathbb{R}\, r \in [0,L]. \\ \text{(c) The map } (b,r) \mapsto H_{ad}(x,b,r) \ \text{ is continuous on } \mathbb{R} \times [0,L] \\ \quad \text{a.e.} x \in \Gamma_3. \\ \text{(d) } H_{ad}(x,0,r) = 0 \quad \forall\, r \in [0,L], \ \text{a.e. } x \in \Gamma_3. \\ \text{(e) } H_{ad}(x,b,r) \ge 0 \quad \forall\, b \le 0, r \in [0,L], \text{a.e.} x \in \Gamma_3 \text{ and} \\ \quad H_{ad}(x,b,r) \le 0 \quad \forall\, b \ge 1,\ r \in [0,L], \ \text{a.e. } x \in \Gamma_3. \end{cases} \tag{3.17}$$

We also suppose that the body forces and surface tractions satisfy

$$\mathbf{f}_0^\ell \in C(0,T;H^\ell), \quad \mathbf{f}_2^\ell \in C(0,T;L^2(\Gamma_2^\ell)^d), \tag{3.18}$$

and, finally, the initial data satisfies

$$\beta_0 \in L^\infty(\Gamma_3), \quad 0 \le \beta_0 \le 1 \text{ a.e. } x \in \Gamma_3. \tag{3.19}$$

Finally we assume that the initial data satisfy the following conditions

$$\mathbf{u}_0^\ell \in V^\ell, \tag{3.20}$$

$$\alpha_0^\ell \in K^\ell. \tag{3.21}$$

We define the bilinear form $a : H^1(\Omega^\ell) \times H^1(\Omega^\ell) \to \mathbb{R}$ by

$$a(\zeta,\varphi) = \sum_{\ell=1}^{2} k^\ell \int_{\Omega^\ell} \nabla\zeta^\ell \cdot \nabla\varphi^\ell dx. \tag{3.22}$$

The microcrack diffusion coefficient verifies

$$k^\ell > 0. \tag{3.23}$$

Using (2.5) and (3.4), we deduce that for $\ell = 1,2$ we have

$$\begin{aligned}(\boldsymbol{\sigma}^\ell(t), \boldsymbol{\varepsilon}(\mathbf{v}^\ell))_{\mathcal{H}^\ell} =& (\mathbf{f}_0^\ell(t), \mathbf{v}^\ell)_{H^\ell} + \int_{\Gamma_2^\ell} \mathbf{f}_2^\ell(t).\mathbf{v}^\ell da + \\ & \int_{\Gamma_3} (\sigma_\nu^\ell(t) v_\nu^\ell + \boldsymbol{\sigma}_\tau^\ell(t).\mathbf{v}_\tau^\ell) da \quad \forall \mathbf{v}^\ell \in V^\ell \text{ a.e. } t \in (0;T).\end{aligned} \tag{3.24}$$

We define the map $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2) : [0,T] \to V$ by the equality

$$(\mathbf{f}(t), \mathbf{v})_V = \sum_{\ell=1}^{2} \Big((\mathbf{f}_0^\ell(t), \mathbf{v}^\ell)_{H^\ell} + \int_{\Gamma_2^\ell} \mathbf{f}_2^\ell(t).\mathbf{v}^\ell da \Big) \tag{3.25}$$

for all $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in V$, a.e.$t \in (0;T)$. We note that, using (3.18) we obtain the following regularity

$$\mathbf{f} \in C(0,T;V). \tag{3.26}$$

From (3.24) and (3.25), we deduce

$$\begin{aligned}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} =& \sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell(t), \boldsymbol{\varepsilon}(\mathbf{v}^\ell))_{\mathcal{H}^\ell} \\ =& (\mathbf{f}(t), \mathbf{v})_V + \sum_{\ell=1}^{2} \int_{\Gamma_3} \sigma_\nu^\ell(t).v_\nu^\ell da + \sum_{\ell=1}^{2} \int_{\Gamma_3} \boldsymbol{\sigma}_\tau^\ell(t).\mathbf{v}_\tau^\ell da \quad \forall \mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in V, \text{ a.e. } t \in (0,T).\end{aligned} \tag{3.27}$$

Keeping in mind (3.7) and (3.8), we deduce

$$\sum_{\ell=1}^{2} \int_{\Gamma_3} (\sigma_\nu^\ell v_\nu^\ell + \boldsymbol{\sigma}_\tau^\ell.\mathbf{v}_\tau^\ell) da = -\int_{\Gamma_3} p_\tau(\beta, \mathbf{u}_\tau^1 - \mathbf{u}_\tau^2).(\mathbf{v}_\tau^1 - \mathbf{v}_\tau^2) da. \tag{3.28}$$

Let us define the functional $j : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j(\beta, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \mathbf{p}_\tau(\beta, \mathbf{u}_\tau^1 - \mathbf{u}_\tau^2).(\mathbf{v}_\tau^1 - \mathbf{v}_\tau^2) da. \tag{3.29}$$

for all $\beta \in L^\infty(\Gamma_3)$ and $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2) \in V$ and $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2) \in V$. Taking into account (3.27)–(3.29), we can write

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j(\beta(t), \mathbf{u}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v}). \quad \forall \mathbf{v} \in V, \text{ a.e. } t \in [0,T].$$

Then, the variational formulation of the Problem $\mathbf{P}$ may be stated as follows.

Problem PV.

Find a displacement field $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2) : [0,T] \to \mathbb{V}$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) : [0,T] \to \mathcal{H}$, a damage field $\alpha = (\alpha^1, \alpha^2) : [0,T] \to H^1(\Omega)$, and a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\boldsymbol{\sigma}^\ell(t) = \mathcal{A}^\ell \varepsilon(\dot{\mathbf{u}}^\ell(t)) + \mathcal{G}^\ell \varepsilon(\mathbf{u}^\ell(t), \alpha^\ell) \tag{3.30}$$

$$\dot{\beta}(t) = H_{ad}(\beta(t), R(|\mathbf{u}^1_\tau(t) - \mathbf{u}^2_\tau(t)|)), \quad 0 \le \beta(t) \le 1, \tag{3.31}$$

$$\sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell(t), \varepsilon(v^\ell))_{\mathcal{H}^\ell} + j(\beta(t), \mathbf{u}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V, \quad \forall \mathbf{v} \in V, \tag{3.32}$$

$$\left\{ \begin{aligned} &\alpha(t) \in K, \ \sum_{\ell=1}^{2} (\dot{\alpha}^\ell(t), \xi^\ell - \alpha^\ell(t))_{L^2(\Omega^\ell)} + a(\alpha(t), \xi - \alpha(t)) \\ &\qquad \ge \sum_{\ell=1}^{2} (S^\ell(\varepsilon(\mathbf{u}^\ell(t)), \alpha^\ell(t)), \xi^\ell - \alpha^\ell(t))_{L^2(\Omega^\ell)}, \ \xi \in K, \end{aligned} \right. \tag{3.33}$$

a.e $t \in [0,T]$,

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \beta(0) = \beta_0, \quad \alpha(0) = \alpha_0. \tag{3.34}$$

We notice that the variational Problem **PV** is formulated in terms of displacement, stress field, damage field, and bonding field. The existence of the unique solution Problem **PV** is stated and proved in the next following section.

4. Well posedness of the problem

Our main existence and uniqueness result is the following.

Theorem 4.1. *Assume that* (3.13)–(3.21) *hold. Then there exists a unique solution to Problem PV. Moreover, the solution satisfies*

$$\mathbf{u} \in C^1(0,T;V), \tag{4.1}$$

$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1), \tag{4.2}$$

$$\alpha \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{4.3}$$

$$\beta \in W^{1,\infty}(0,T;L^\infty(\Gamma_3)). \tag{4.4}$$

A quadruplet $(\mathbf{u}, \boldsymbol{\sigma}, \beta, \alpha)$ which satisfies (3.30)–(3.34) is called a weak solution to the compliance contact problem **P**. We conclude that under the stated assumptions, problem (3.2)–(3.12) has a unique weak solution satisfying (4.1)–(4.4).

We turn now to the proof of Theorem 4.1 which carried out in several steps. To this end, we assume in the following that (3.13)–(3.21) hold. Below, C denotes a generic positive constant which may depend on Ω^ℓ, Γ_1^ℓ, Γ_3, $\mathcal{A}^\ell$, and T, but does not depend on t nor of the rest of input data, and whose value may change from place to place. Moreover, for the sake of simplicity, we suppress, in what follows, the explicit dependence of various functions on $x^\ell \in \Omega^\ell \cup \Gamma^\ell$. The proof of Theorem 4.1 will be carried out in several steps. In the first step we solve the differential equation in (3.33) for the adhesion field, where $\mathbf{u}^\ell$ is given, and study the continuous dependence of the adhesion solution with respect to $\mathbf{u}^\ell$.

Lemma 4.2. *For every* $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2) \in C(0,T;V)$*, there exists a unique solution*

$$\beta_{\mathbf{u}} \in W^{1,\infty}(0,T;L^\infty(\Gamma_3)),$$

satisfying

$$\dot{\beta}_u(t) = H_{ad}(\beta_{\mathbf{u}}(t), R(|\mathbf{u}^1_\tau(t) - \mathbf{u}^2_\tau(t)|)), \qquad a.e.\ t \in (0,T), \tag{4.5}$$

$$\beta_{\mathbf{u}}(0) = \beta_0. \tag{4.6}$$

Moreover,

$$0 \le \beta_u(t) \le 1, \quad \forall t \in [0,T], \ \text{a.e. on } \Gamma_3, \tag{4.7}$$

and there exists a constant $C > 0$, such that, for all $\mathbf{u}_i = (\mathbf{u}^{1,i}, \mathbf{u}^{2,i}) \in C(0,T;V)$,

$$\|\beta_{\mathbf{u}_1}(t) - \beta_{\mathbf{u}_2}(t)\|^2_{L^2(\Gamma_3)} \le C \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|^2_V ds.$$

Proof. We consider the map $H : [0,T] \times L^\infty(\Gamma_3) \to L^\infty(\Gamma_3)$ defined by

$$H(t,\beta) = H_{ad}(\beta, \mathbf{R}(|\mathbf{u}^1_\tau(t) - \mathbf{u}^2_\tau(t)|), \quad \text{a.e. } t \in (0,T) \ \forall \beta \in L^\infty(\Gamma_3).$$

It is easy to check that H is Lipschitz continuous with respect to the second variable, uniformly in time; moreover, for all $t \in [0,T]$ and $\beta \in L^\infty(\Gamma_3)$, $t \to H(t,\beta)$ belongs to $L^\infty(0;T;L^\infty(\Gamma_3))$. Thus, the existence of a unique function β_u which satisfies (4.5)–(4.6) follows from a version of the Cauchy–Lipschitz theorem.

Finally, the proof of (4.7) is a consequence of the assumptions (3.17) and (3.19), see [15] for detail. Now let $\mathbf{u}_1$, $\mathbf{u}_2 \in C(0,T;V)$ and let $t \in [0,T]$. We have, for $i = 1,2$,

$$\beta_{u_i}(t) = \beta_0 + \int_0^t H_{ad}\Big(\beta_{u_i}(s), R(|u^{1i}_\tau(s) - u^{2i}_\tau(s)|)\Big) ds, \quad i = 1,\, 2,$$

where $\mathbf{u}_i = (\mathbf{u}^{1i}, \mathbf{u}^{2i})$ and $\beta_{\mathbf{u}_i} = \beta_i$. Using now (3.17) and (3.1), we obtain

$$|\beta_1(t) - \beta_2(t)| \le C\Big(\int_0^t |\beta_1(s) - \beta_2(s)| ds + \int_0^t |u^{11}_\tau(s) - u^{21}_\tau(s) - (u^{12}_\tau(s) - u^{22}_\tau(s))| ds\Big).$$

Next, we apply Gronwall's inequality to deduce

$$|\beta_1(t) - \beta_2(t)| \le C \int_0^t |u^{11}_\tau(s) - u^{21}_\tau(s) - (u^{12}_\tau(s) - u^{22}_\tau(s))|\, ds,$$

which implies

$$|\beta_1(t) - \beta_2(t)|^2 \le C \int_0^t (|u^{11}(s) - u^{12}(s)|^2 + |u^{21}(s) - u^{22}(s)|^2) ds.$$

Integrating the last inequality over Γ_3 and keeping in mind (2.3), we find

$$\|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)} \le C \int_0^t \Big(\|u^{11}(s) - u^{12}(s)\|^2_{V^1} + \|u^{21}(s) - u^{22}(s)\|^2_{V^2}\Big) ds.$$

Taking into account (2.4), we deduce

$$\|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)} \le C \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|^2_V ds,$$

the second part of lemma 4.2. □

Now we consider the following viscoelastic problem and we prove the existence and uniqueness result for (3.30), (3.31), and (3.32) with the corresponding initial condition.

Problem QV.

Find a displacement field $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2) : [0,T] \to \mathbb{V}$, a damage field $\alpha = (\alpha^1, \alpha^2) : [0,T] \to H^1(\Omega)$, and a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) : [0,T] \to \mathcal{H}$, satisfying (3.30), (3.33), and

$$\sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell(t), \varepsilon(v^\ell))_{\mathcal{H}^\ell} + j(\beta(t), \mathbf{u}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V, \quad \forall \mathbf{v} \in V, \quad t \in [0,T], \tag{4.8}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \qquad \alpha(0) = \alpha_0. \tag{4.9}$$

Let $\eta \in C(0,T;H)$, and consider the following variational problem.

Problem $\mathbf{QV}_\eta$.

Find a displacement field $\mathbf{u}_\eta = (\mathbf{u}^1_\eta, \mathbf{u}^2_\eta) : [0, T] \to \mathbb{V}$ and a stress field $\boldsymbol{\sigma}_\eta = (\boldsymbol{\sigma}^1_\eta, \boldsymbol{\sigma}^2_\eta) : [0, T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}^\ell_\eta(t) = \mathcal{A}^\ell \varepsilon(\dot{\mathbf{u}}^\ell_\eta(t)) + \eta(t)^\ell, \tag{4.10}$$

$$\sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell_\eta(t), \varepsilon(v^\ell))_{\mathcal{H}^\ell} j(\beta_{u_\eta}(t), \mathbf{u}_\eta(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_V, \quad \forall \mathbf{v} \in V, \quad t \in [0, T], \tag{4.11}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{4.12}$$

To solve problem $\mathbf{QV}_\eta$, we consider $\theta \in C(0, T; V)$ and we construct the following intermediate problem.

Problem $\mathbf{QV}_{\eta\theta}$.

Find a displacement field $\mathbf{u}_{\eta\theta} = (\mathbf{u}^1_{\eta\theta}, \mathbf{u}^2_{\eta\theta}) : [0, T] \to \mathbb{V}$ and a stress field $\boldsymbol{\sigma}_{\eta\theta} = (\boldsymbol{\sigma}^1_{\eta\theta}, \boldsymbol{\sigma}^2_{\eta\theta}) : [0, T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}^\ell_{\eta\theta}(t) = \mathcal{A}^\ell \varepsilon(\dot{\mathbf{u}}^\ell_{\eta\theta}(t)) + \eta^\ell(t), \tag{4.13}$$

$$\sum_{\ell=1}^{2} (\boldsymbol{\sigma}^\ell_{\eta\theta}(t), \varepsilon(v^\ell))_{\mathcal{H}^\ell} + (\theta(t), v)_V = (\mathbf{f}(t), \mathbf{v})_V, \quad \forall \mathbf{v} \in V, \quad t \in [0, T], \tag{4.14}$$

$$\mathbf{u}_{\eta\theta}(0) = \mathbf{u}_0 \tag{4.15}$$

Lemma 4.3. *There exists a unique solution $\mathbf{u}_{\eta\theta}, \boldsymbol{\sigma}_{\eta\theta}$ of problem $\mathbf{PQ}_{\eta\theta}$ which satisfies* (4.1)–(4.2).

Proof. We use Riesz's representation theorem to define the operator $A : V \to V$ by

$$(Au, v)_V = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{4.16}$$

$$(\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}} = \sum_{\ell=1}^{2} (\mathcal{A}^\ell(\varepsilon(\mathbf{u}^\ell)), \varepsilon(\mathbf{v}^\ell))_{\mathcal{H}^\ell} \quad \forall\, \mathbf{u}^\ell, \mathbf{v}^\ell \in V^\ell. \tag{4.17}$$

Using (3.13), it follows that A is a strongly monotone Lipschitz operator, thus A is invertible and $A^{-1} : V \to V$ is also a strongly monotone Lipschitz operator. It follows that there exists a unique function $v_{\eta\theta}$ which satisfies

$$\mathbf{v}_{\eta\theta} \in C(0, T; V), \tag{4.18}$$

$$A\mathbf{v}_{\eta\theta}(t) = h_{\eta\theta}(t), \tag{4.19}$$

where $h_{\eta\theta} \in C(0, T; V)$ is such that

$$(h_{\eta\theta}(t), v)_V = (\mathbf{f}(t), \mathbf{v})_V - (\eta(t), \varepsilon(\mathbf{v}))_{\mathcal{H}} - (\theta(t), \mathbf{v})_V, \quad \forall \mathbf{v} \in V, \tag{4.20}$$

Let $\mathbf{u}_{\eta\theta} : [0, T] \to V$ be a function defined by

$$\mathbf{u}_{\eta\theta} = \int_0^t \mathbf{v}_{\eta\theta} ds + \mathbf{u}_0 \quad \forall t \in [0, T]. \tag{4.21}$$

It follows from (4.21), (4.18), and (4.19) that $\mathbf{u}_{\eta\theta} \in C(0, T; V)$. Consider $\boldsymbol{\sigma}_{\eta\theta}$ defined in (4.13). Since $\eta \in C(0, T; \mathcal{H})$, $\mathbf{u}_{\eta\theta} \in C^1(0, T; V)$ and from the relations (3.13) we deduce that $\boldsymbol{\sigma}_{\eta\theta} \in C(0, T; \mathcal{H})$. Since $\operatorname{Div} \boldsymbol{\sigma}_{\eta\theta} = -\mathbf{f}_0 \in C(0, T; H)$, we further have $\boldsymbol{\sigma}_{\eta\theta} \in C(0, T; \mathcal{H}_1)$. This concludes the existence part of lemma 4.4.

The uniqueness of the solution follows from the unique solvability of time-dependent equation (4.19). Finally $(\mathbf{u}_{\eta\theta}, \boldsymbol{\sigma}_{\eta\theta})$ is the unique solution to the problem $\mathbf{QV}_{\eta\theta}$ obtained in Lemma 4.4, which concludes the proof. □

Now we consider the operator $\Lambda : C(0,T;V) \to C(0,T;V)$, given by

$$(\Lambda\boldsymbol{\theta}(t),\mathbf{v})_V = j(\beta_{u_{\eta\theta}}(t),\mathbf{u}_{\eta\theta}(t),\mathbf{v}), \quad \forall\mathbf{v}\in V, \quad t\in[0,T], \tag{4.22}$$

We have the following result.

Lemma 4.4. *For each $\theta \in C(0,T;V)$ the function $\Lambda\theta : [0,T] \to V$ belongs to $C(0,T;V)$. Moreover, there exists a unique element $\theta^* \in C(0,T;V)$ such that $\Lambda\theta^* = \theta^*$.*

Proof. Let $\theta_i \in C(0,T;V)$. We use the notation $\mathbf{u}_i = (u^{1i}, u^{2i})$, and $\beta_{u_i} = \beta_i$ for $i = 1, 2$. The equalities and inequalities below are valid for all $v \in V$ a.e. $t \in (0,T)$. Using (4.22), (3.29), and the properties of the function p_τ, after some computation, we obtain

$$\begin{aligned}|(\Lambda\boldsymbol{\theta}_1(t)-\Lambda\boldsymbol{\theta}_2(t),v)_V| \le & C\Big(\|u^{11}(t)-u^{12}(t)\|_{L^2(\Gamma_3)^d}+\|u^{21}(t)-u^{22}(t))\|_{L^2(\Gamma_3)^d}\\ &+\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\Big)(\|v^1\|_{L^2(\Gamma_3)^d}+\|v^2\|_{L^2(\Gamma_3)^d}).\end{aligned}$$

Moreover, keeping in mind (2.3) and (2.4), we can write

$$\begin{aligned}|(\Lambda\boldsymbol{\theta}_1(t)-\Lambda\boldsymbol{\theta}_2(t),\mathbf{v})_V| \le C(&\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\\ &+(\|u^{11}(t)-u^{12}(t)\|_{V^1}+\|u^{21}(t)-u^{22}(t)\|_{V^2})(\|v^1\|_{V^1}+\|v^2\|_{V^2})),\end{aligned}$$

and form this inequality we find

$$\|\Lambda\boldsymbol{\theta}_1(t)-\Lambda\boldsymbol{\theta}_2(t)\|_V \le C\Big(\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}+\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V\Big). \tag{4.23}$$

Then by Lemma 4.3, we have

$$\|\Lambda\boldsymbol{\theta}_1(t)-\Lambda\boldsymbol{\theta}_2(t)\|_V^2 \le C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V^2 \; + \int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2 ds\Big),$$

$$\|\Lambda\boldsymbol{\theta}_1(t)-\Lambda\boldsymbol{\theta}_2(t)\|_V^2 \le C\int_0^t \|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V^2 ds. \tag{4.24}$$

Moreover, from (4.14) it follows that

$$(\mathcal{A}\varepsilon(\mathbf{v}_1)-\mathcal{A}\varepsilon(\mathbf{v}_2),\varepsilon(\mathbf{v}_1-\mathbf{v}_2))_{\mathcal{H}}+(\boldsymbol{\theta}_1-\boldsymbol{\theta}_2,\mathbf{v}_1-\mathbf{v}_2)_V = 0 \text{ on } (0,T). \tag{4.25}$$

Hence

$$\|\mathbf{v}_1(s)-\mathbf{v}_2(s)\|_V \le C\|\boldsymbol{\theta}_1(s)-\boldsymbol{\theta}_2(s)\|_V, \quad \forall s\in[0,T]. \tag{4.26}$$

Now from the inequalities (4.24) and (4.26) we have

$$\|\Lambda\boldsymbol{\theta}_1(t)-\Lambda\boldsymbol{\theta}_2(t)\|_V^2 \le C\int_0^t \|\boldsymbol{\theta}_1(s)-\boldsymbol{\theta}_2(s)\|_V^2, \quad \forall s\in[0,T].$$

Reiterating this inequality n times yields

$$\|\Lambda^n\boldsymbol{\theta}_1-\Lambda^n\boldsymbol{\theta}_2\|_{C(0,T;V)}^2 \le \frac{(CT)^n}{n!}\|\boldsymbol{\theta}_1-\boldsymbol{\theta}_2\|_{C(0,T;V)}^2.$$

We conclude that for a sufficiently large n, the mapping Λ^n of Λ is a contraction in the Banach space $C(0,T;V)$. Therefore, there exists a unique $\boldsymbol{\theta}^* \in C(0,T;V)$ such that $\Lambda^n\boldsymbol{\theta}^* = \theta^*$ and, moreover, $\boldsymbol{\theta}^*$ is the unique fixed point of the mapping Λ. □

Lemma 4.5. *There exists a unique solution of problem $\mathbf{QV}_\eta$ satisfying* (4.1)–(4.2)*.*

Proof. Let $\theta^* \in C(0,T;V)$ be the fixed point of Λ. Lemma 4.4 implies that $(\mathbf{u}_{\eta\theta^*},\boldsymbol{\sigma}_{\eta\theta^*}) \in C(0,T;V) \times C(0,T;\mathcal{H}_1)$ is the unique solution of $\mathbf{QV}_{\eta\theta}$ for $\theta = \theta^*$. Since $\Lambda\theta^* = \theta^*$ and from the relations (4.22), (4.10), (4.11), and (4.12), we obtain that $(\mathbf{u}_\eta,\boldsymbol{\sigma}_\eta) = (\mathbf{u}_{\eta\theta^*},\boldsymbol{\sigma}_{\eta\theta^*})$ is the unique solution of $\mathbf{QV}_\eta$. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ given in (4.22). □

Let $\omega \in C(0,T;L^2(\Omega))$. We suppose that the assumptions of Theorem 4.1 hold and we consider the following intermediate problem for the damage field.

Problem $\mathbf{PV}_\omega$.

Find the damage field $\alpha_\omega = (\alpha^1_\omega, \alpha^2_\omega) : [0,T] \to H^1(\Omega)$ such that $\alpha_\omega(t) \in K$ for all $t \in [0,T]$ and

$$(\dot{\alpha}_\omega(t), \xi - \alpha_\omega)_{L^2(\Omega)} + a(\alpha_\omega(t), \xi - \alpha_\omega(t)) \geq (\omega(t), \xi - \alpha_\omega(t))_{L^2(\Omega)} \quad \forall \xi \in K, \text{ a.e. } t \in (0,T), \tag{4.27}$$

$$\alpha_\omega(0) = \alpha_0, \tag{4.28}$$

where $K = K^1 \times K^2$, $(\dot{\alpha}_\omega(t), \xi - \alpha_\omega)_{L^2(\Omega)} = \sum_{\ell=1}^{2} (\dot{\alpha}^\ell_\omega(t), \xi^\ell - \alpha^\ell_\omega(t))_{L^2(\Omega^\ell)}$ and $(\omega(t), \xi - \alpha_\omega(t))_{L^2(\Omega)} = \sum_{\ell=1}^{2} (\omega^\ell(t), \xi^\ell - \alpha^\ell_\omega(t))_{L^2(\Omega^\ell)}$. To solve problem $\mathbf{PV}_\omega$, we recall the following standard result for parabolic variational inequalities (see, e.g.,[17], page 47).

Lemma 4.6. *Problem $\mathbf{PV}_\omega$ has a unique solution $\alpha_\omega(t)$ such that*

$$\alpha_\omega \in W^{1,2}(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)). \tag{4.29}$$

Proof. We use (3.21), (3.22), and a classical existence and uniqueness result on parabolic equations (see for instance [2, p. 124]. □

As a consequence of solving the problems $\mathbf{QV}_\eta$ and $\mathbf{PV}_\omega$, we may define the operator $\mathcal{L} : C(0,T; V \times L^2(\Omega)) \to C(0,T; V \times L^2(\Omega))$ by

$$\mathcal{L}(\boldsymbol{\eta}, \omega) = (\mathcal{G}(\varepsilon(\mathbf{u}_\eta), \alpha_\omega), \mathcal{S}(\varepsilon(\mathbf{u}_\eta), \alpha_\omega)) \tag{4.30}$$

for all $(\boldsymbol{\eta}, \omega) \in C(0,T; V \times L^2(\Omega))$, then we have the following lemma.

Lemma 4.7. *The operator $\mathcal{L}$ has a unique fixed point*

$$(\boldsymbol{\eta}^*, \omega^*) \in C(0,T; V \times L^2(\Omega)).$$

Proof. Let $(\boldsymbol{\eta}_1, \omega_1)$, $(\boldsymbol{\eta}_2, \omega_2) \in C(0,T; V \times L^2(\Omega))$, $t \in [0,T]$ and use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i} = \mathbf{v}_{\eta_i} = \mathbf{v}_i$, and $\alpha_{\omega_i} = \alpha_i$ for $i = 1,2$. Taking into account the relations (3.14), (3.15), and (4.30), we deduce that

$$\|\mathcal{L}(\boldsymbol{\eta}_1, \omega_1) - \mathcal{L}(\boldsymbol{\eta}_2, \omega_2)\|_{V \times L^2(\Omega)} \leq C\Big(\|\mathbf{u}_1 - \mathbf{u}_2\|_V + \|\alpha_1 - \alpha_2\|_{L^2(\Omega)}\Big). \tag{4.31}$$

Moreover, using (4.11) we obtain

$$\begin{aligned}(\mathcal{A}\varepsilon(\mathbf{v}_1) - \mathcal{A}\varepsilon(\mathbf{v}_2), \varepsilon(\mathbf{v}_1) - \varepsilon(\mathbf{v}_2))_{\mathcal{H}} &= j(\beta_{u^2}, \mathbf{u}_2, \mathbf{v}_1 - \mathbf{v}_2) \\ &- j(\beta_{u_1}, \mathbf{u}_1, \mathbf{v}_1 - \mathbf{v}_2) + (\boldsymbol{\eta}_2 - \boldsymbol{\eta}_1, \varepsilon(\mathbf{v}_1) - \varepsilon(\mathbf{v}_2))_{\mathcal{H}} \text{ a.e. } t \in [0,T].\end{aligned} \tag{4.32}$$

Keeping in mind (3.13) and (3.16), we find

$$\|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|^2_V \leq C\Big(\|\beta_{\mathbf{u}_1}(t) - \beta_{\mathbf{u}_2}(t)\|^2_{L^2(\Gamma_3)} + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_V + \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|^2_V\Big). \tag{4.33}$$

By Lemma 4.3, we obtain

$$\begin{aligned}\|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|^2_V &\leq C\Big(\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|^2_V + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_V \\ &\quad + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|^2_V ds\Big) \\ &\leq C\Big(\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|^2_V + \int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|^2_V ds\Big).\end{aligned} \tag{4.34}$$

Applying Gronwall inequality yields

$$\|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V^2 \le C\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_{\mathcal{H}}^2. \tag{4.35}$$

Since $\mathbf{u}_1(0) = \mathbf{u}_2(0)$, we have

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \le C\int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V ds.$$

From the two previous inequalities, we find

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \le C\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_V ds. \tag{4.36}$$

From (4.27), we deduce that

$$(\dot{\alpha}_1 - \dot{\alpha}_2, \alpha_1 - \alpha_2)_{L^2(\Omega)} + a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2)) \le (\omega_1 - \omega_2, \alpha_1 - \alpha_2)_{L^2(\Omega)}, \quad \text{a.e. } \in (0,T).$$

Integrating the inequality with respect to time, using the initial conditions $\alpha_1(0) = \alpha_2(0) = \alpha_0$ and the inequality $a(\alpha_1 - \alpha_2, \alpha_1 - \alpha_2) \ge 0$, we find

$$\frac{1}{2}|\alpha_1(t) - \alpha_2(t)|_{L^2(\Omega)}^2 \le C\int_0^t (\omega_1(s) - \omega_2(s), \alpha_1(s) - \alpha_2(s))_{L^2(\Omega)} ds,$$

which implies that

$$|\alpha_1(t) - \alpha_2(t)|_{L^2(\Omega)}^2 \le C\int_0^t |\omega_1(s) - \omega_2(s)|_{L^2(\Omega)}^2 ds + \int_0^t |\alpha_1(s) - \alpha_2(s)|_{L^2(\Omega)}^2 ds.$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Omega)} \le C\int_0^t \|\omega_1(s) - \omega_2(s)\|_{L^2(\Omega)} ds, \quad \forall t \in [0,T]. \tag{4.37}$$

Substituting (4.36) and (4.37) in (4.31), we obtain

$$\|\mathcal{L}(\boldsymbol{\eta}_1, \omega_1) - \mathcal{L}(\boldsymbol{\eta}_2, \omega_2)\|_{V\times L^2(\Omega)} \le C\int_0^t \|(\boldsymbol{\eta}_1, \omega_1)(s) - (\boldsymbol{\eta}_2, \omega_2)(s)\|_{V\times L^2(\Omega)} ds. \tag{4.38}$$

Lemma 4.8 is a consequence of the result (4.38) and Banach's fixed-point Theorem. □

Now, we have all the ingredients to solve **QV** .

Lemma 4.8. *There exists a unique solution* $\{\mathbf{u}, \sigma, \alpha\}$ *of problem* **QV** *satisfying* (4.1)–(4.4).

Proof. We start the proof by the existence of the weak solution:

Existence

Let $(\boldsymbol{\eta}^*, \omega^*) \in C(0,T; V \times L^2(\Omega))$ be the fixed point of $\mathcal{L}$ given by (4.30); by lemma 4.6, we conclude that $\{\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta\} = \{\mathbf{u}_{\eta\theta^*}, \boldsymbol{\sigma}_{\eta\theta^*}\} \in C(0,T;V) \times C(0,T;\mathcal{H}_1)$ is the unique solution of $\mathbf{QV}_\eta$. Since $\mathcal{L}(\boldsymbol{\eta}^*, \omega^*) = (\boldsymbol{\eta}^*, \omega^*)$, from the relations (4.10), (4.11), (4.12), and lemma 4.7 we obtain that $\{\mathbf{u}, \boldsymbol{\sigma}, \alpha\} = \{\mathbf{u}_{\eta^*\theta^*}, \boldsymbol{\sigma}_{\eta^*\theta^*}, \alpha_{\eta^*\theta^*}\}$ is the unique solution of **QV**. The regularity of the solution follows from lemmas 4.6 and lemma 4.7. [5mm]

Uniqueness

The uniqueness of the solution results from the uniqueness of the fixed point of the operator $\mathcal{L}$ defined by (4.30). □

Now, we have all the ingredients to prove Theorem 4.1.

Proof of Theorem 4.1. In this proof we give the existence and uniqueness of the weak solution.

Existence

Let $\{\mathbf{u}, \boldsymbol{\sigma}, \alpha\}$ is the solution of the problem QV given by Lemmas 4.5, 4.6, and $\beta = \beta_{\mathbf{u}}$ the solution of (4.5) given by Lemma 4.2. It follows that $\{\mathbf{u}, \boldsymbol{\sigma}, \beta, \alpha\}$ is the solution of problem PV satisfying (3.30)–(3.34). This concludes part existence.

Uniqueness

The uniqueness of the solution follows from the uniqueness of the solution of Cauchy problems (4.8)–(4.9) and (4.11), guaranteed by Lemmas 4.5 and 4.2 respectively. □

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