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On fixed soft element theorems in se-uniform spaces

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Abstract

First we introduce a new structure of uniform spaces, called se-uniform spaces, and provide some of their basic properties. Next, we present the notion of a soft E-distance in se-uniform spaces, which is a soft version of E-distance of Aamri and El Moutawakil [M. Aamri, D. El Moutawakil, Acta Math. Acad. Peadegog. Nyhazi., **20** (2004), 83–91]. Then, by using the soft E-distance, we establish some fixed soft element theorems for various mappings on se-uniform spaces, which are the main results of the paper. This is the first kind of such results in this direction.

Keywords: Soft order, se-uniform space, soft E-distance, fixed soft element.
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1. Introduction

In 1999, Molodtsov [27] introduced the concept of soft set theory as a new approach for coping with uncertainties and also presented the basic results of the new theory. In [27], he successfully applied soft set theory in several directions, such as smoothness of functions, game theory, operations research, Riemann integration, Perron integration and theory of measurement. After presentation of the operations of soft sets [24], the properties and applications of this theory have been studied increasingly [6, 28, 32].

Aktaş and Çağman [5] introduced soft groups and also compared soft sets to fuzzy sets and rough sets. Shabir and Naz [36] initiated the study of soft topological spaces. Das and Samanta [15] presented the notion of a soft metric space by employing soft elements introduced in [13] and investigated some of its fundamental properties. Recently, many papers concerning soft set theory have been published [4, 10, 11, 14, 18, 23, 25, 29].

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The first systematic exposition of uniform spaces was given by Bourbaki in [12]. This structure is a suitable tool for investigations in topology. Also, there exist its remarkable analogies with metrics, but the scope of its applicability is much wider. Therefore, it can be considered as a bridge between metric and topology. Hence, there exists a considerable literature on fixed point theory dealing with results on fixed or common fixed points in uniform spaces.

Fixed point theory plays a fundamental role in mathematics and applied sciences, such as optimization, mathematical models and economic theories. Also, this theory have been applied to show the existence and uniqueness of the solutions of differential equations, integral equations and many other branches of mathematics [16, 30, 31]. A basic result in fixed point theory is the Banach contraction principle. Since the appearance of this principle, there has been a lot of activity in this area. Some authors such as Jachymski [19], Rhoades [34], Suzuki [37], and Wang [41] studied the fixed point or common fixed point theory for contractive selfmappings in complete metric spaces or Banach spaces. Kada et al. [21] introduced the concept of a W-distance on metric spaces and generalized some important results in nonconvex minimizations and in fixed point theory for both W-contractive and W-expansive mappings.

An exact analogue of that principle in uniform spaces was obtained independently by Acharya [3] and Tarafdar [38]. Using the ideas of Kang [22], Montes and Charris [35] established some results on fixed points of mappings by means of appropriate W-contractive or W-expansive assumptions in uniform spaces. Later, Aamri and El Moutawakil [1] proved some common fixed point theorems for some new contractive or expansive mappings in uniform spaces by introducing the notion of E-distance. Altun and Imdad [8] introduced an order relation on uniform spaces by utilizing E-distance mappings and presented some fixed point theorems. Existence and uniqueness of fixed points for various contractive mappings in the setting of uniform spaces, have been investigated by several authors [2, 7, 9, 17, 26, 33, 39, 40].

In this work, with the help of soft elements, we define the concept of se-uniform spaces and investigate some of its properties. We consider the concept of soft E-distance in se-uniform spaces, which is a soft version of the E-distance introduced by Aamri and El Moutawakil [1]. By using the soft E-distance, we prove some fixed soft element theorems for various mappings in se-uniform spaces. Also, we give a soft order relation on se-uniform spaces and utilize this relation to prove some fixed soft element theorems in se-uniform spaces. Moreover, we furnish some examples to demonstrate the validity of the obtained results.

2. Preliminaries

In this section, we recollect some basic notions regarding soft sets.

Throughout this work, let X be an initial universe set and E the set of all parameters for X. Let $P(X)$ denote the power set of X and $A, B \subseteq E$.

Definition 2.1 ([27]). A soft set (F, A) over the universe X with the set A of parameters is defined by the set of ordered pairs

$$(F, A) = \{(a, F(a)) : a \in A, F(a) \in P(X)\},$$

where F is a mapping $F : A \to P(X)$.

Definition 2.2 ([32]). Let (F, A) and (G, B) be two soft sets over a common universe X. Then,

(i) (F, A) is called a soft subset of (G, B) if $A \subseteq B$ and $F(a) \subseteq G(a)$ for all $a \in A$. It is denoted by $(F, A) \tilde{\subseteq} (G, B)$;

(ii) (F, A) and (G, B) are called soft equal if $(F, A) \tilde{\subseteq} (G, B)$ and $(G, B) \tilde{\subseteq} (F, A)$. It is denoted by $(F, A) = (G, B)$.

Definition 2.3 ([24]). Let $E = \{e_1, e_2, ..., e_n\}$ be a set of parameters. The *NOT* set of E, denoted by $\rceil E$, is defined by $\rceil E = \{\rceil e_1, \rceil e_2, ..., \rceil e_n\}$ where $\rceil e_i =$ *not* e_i, for all $i \in \{1, 2, ..., n\}$.

Definition 2.4 ([24]). Let E be a set of parameters and $A, B \subseteq E$. Then,

(i) $\rceil(\rceil A) = A$.
(ii) $\rceil(A \cup B) = \rceil A \cup \rceil B$.
(iii) $\rceil(A \cap B) = \rceil A \cap \rceil B$.

Definition 2.5 ([24])**.** The complement of a soft set (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \rceil A)$, where $F^c : \rceil A \to P(X)$ is a mapping given by $F^c(a) = X - F(\rceil a)$ for all $a \in \rceil A$.

Definition 2.6 ([6])**.** The relative complement of a soft set (F, A) is denoted by $(F, A)^r$ and is defined by $(F, A)^r = (F^r, A)$, where $F^r : A \to P(X)$ is a mapping given by $F^r(a) = X - F(a)$ for all $a \in A$.

Definition 2.7 ([24])**.** Let $(F, A), (G, B)$ be two soft sets over a common universe X. Then,

(i) The soft set (F, A) is called a null soft set, denoted by Φ, if $F(e) = \emptyset$ for all $e \in A$.
(ii) The soft set (F, A) is called an absolute soft set, denoted by $\widetilde{A}$, if $F(e) = X$ for all $e \in A$.

Note that we use the notation $\widetilde{X}$ instead of $\widetilde{A}$ as in [15] throughout this paper.

Definition 2.8 ([24])**.** Let $(F, A), (G, B)$ be two soft sets over a common universe X. Then,

(i) The union of (F, A) and (G, B) is the soft set (H, C) where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e), & \text{if } e \in A - B, \\ G(e), & \text{if } e \in B - A, \\ F(e) \cup G(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(F, A)\widetilde{\cup}(G, B) = (H, C)$.

(ii) The intersection of (F, A) and (G, B) is the soft set (H, C) where $C = A \cap B$ and for all $e \in C$, $H(e) = F(e)$ or $G(e)$ (as both are the same set). We write $(F, A)\widetilde{\cap}(G, B) = (H, C)$.

Pei and Miao [32] gave an alternative definition for intersection of soft sets as follows.

Definition 2.9 ([32])**.** Let (F, A) and (G, B) be two soft sets over a common universe X. The intersection of (F, A) and (G, B) is denoted by $(F, A) \cap (G, B)$, and is defined by $(F, A) \cap (G, B) = (H, C)$, where $C = A \cap B$ and for all $c \in C$, $H(c) = F(c) \cap G(c)$.

Remark 2.10. In the above definitions, the set of parameters may vary from soft set to soft set. However, in order to efficiently discuss, we consider only soft sets (F, A) over a universe X in which all the parameter sets A are the same. The above definitions are also valid for these type of soft sets as a particular case of those definitions. We denote the family of these soft sets by $SS(X, A)$

Definition 2.11 ([13])**.** Let X be a nonempty set and A be a nonempty parameter set. Then a function $\varepsilon : A \to X$ is said to be a soft element of X. A soft element ε of X is said to belong to a soft set (F, A) of X, denoted by $\varepsilon \widetilde{\in} (F, A)$, if $\varepsilon(a) \in F(a)$ for each $a \in A$. Thus a soft set (F, A) can be expressed as $F(a) = \{\varepsilon(a) : \varepsilon \widetilde{\in} F\}$ for each $a \in A$.

Let X^A be the family of all soft elements in X with the set A of parameters.

Remark 2.12 ([13]). It is to be noted that every singleton soft set (F, A) (that is, for each $a \in A$, $F(a)$ is a singleton set) can be identified with a soft element by simply identifying the singleton set with the element that it contains for each $a \in A$.

Definition 2.13 ([13])**.** Let $\mathbb{R}$ be the set of real numbers, $\mathfrak{B}(\mathbb{R})$ the collection of all nonempty bounded subsets of $\mathbb{R}$, and A a set of parameters. Then,

$$(F, A) = \{(a, F(a)) : a \in A, F(a) \in \mathfrak{B}(\mathbb{R})\}$$

is called a soft real set. If specifically (F, A) is a singleton soft set, then after identifying (F, A) with the corresponding soft element, it will be called a soft real number.

Definition 2.14 ([13])**.** Let $(F, A), (G, A)$ be soft real numbers. Then,

(i) The sum is defined by $(F + G)(a) = F(a)G(a)$, for each $a \in A$.
(ii) The difference is defined by $(F - G)(a) = F(a) - G(a)$, for each $a \in A$.
(iii) The product is defined by $(F.G)(a) = F(a).G(a)$, for each $a \in A$.
(iv) The modulus of (F, A) is denoted by $(|F|, A)$ and is defined by $|F|(a) = |F(a)|$, for each $a \in A$.

From the above definition of soft real numbers it follows that $(F + G, A), (F - G, A), (F.G, A)$, and $(|F|, A)$ are soft real numbers.

Definition 2.15 ([13])**.** Let (F, A) be a soft real number. Then, (F, A) is said to be a nonnegative soft real number if $F(a)$ is a nonnegative real number for every $a \in A$.

Let $\mathbb{R}(A)^*$ denote the set of all nonnegative soft real numbers.

We use the notations $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ to denote soft elements of a soft set and $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ to denote soft real numbers whereas $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ will denote a particular type of soft real numbers such that $\bar{\alpha}(a) = \alpha$, for all $a \in A$ etc. For example $\overline{0}$ is the soft real number where $\overline{0}(a) = 0$, for all $a \in A$ [15].

Definition 2.16 ([15])**.** Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two soft real numbers. Then,

(i) $\tilde{\alpha} \mathrel{\widetilde{\leq}} \tilde{\beta}$ if $\tilde{\alpha}(a) \leq \tilde{\beta}(a)$, for every $a \in A$.
(ii) $\tilde{\alpha} \mathrel{\widetilde{\geq}} \tilde{\beta}$ if $\tilde{\alpha}(a) \geq \tilde{\beta}(a)$, for every $a \in A$.
(iii) $\tilde{\alpha} \mathrel{\widetilde{<}} \tilde{\beta}$ if $\tilde{\alpha}(a) < \tilde{\beta}(a)$, for every $a \in A$.
(iv) $\tilde{\alpha} \mathrel{\widetilde{>}} \tilde{\beta}$ if $\tilde{\alpha}(a) > \tilde{\beta}(a)$, for every $a \in A$.

Definition 2.17 ([15])**.** Let X be a set and A a set of parameters. A mapping $d : X^A \times X^A \to \mathbb{R}(A)^*$ is called a soft metric on X if it satisfies the following axioms:

(sm_1) $d(\tilde{x}, \tilde{y}) \mathrel{\widetilde{\geq}} \overline{0}$, for all $\tilde{x}, \tilde{y} \mathrel{\widetilde{\in}} \widetilde{X}$.
(sm_2) $d(\tilde{x}, \tilde{y}) = \overline{0}$ if and only if $\tilde{x} = \tilde{y}$.
(sm_3) $d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x})$, for all $\tilde{x}, \tilde{y} \mathrel{\widetilde{\in}} \widetilde{X}$.
(sm_4) $d(\tilde{x}, \tilde{y}) \mathrel{\widetilde{\leq}} d(\tilde{x}, \tilde{z})d(\tilde{z}, \tilde{y})$, for all $\tilde{x}, \tilde{y}, \tilde{z} \mathrel{\widetilde{\in}} \widetilde{X}$.

The triplet (X, d, A) is called a soft metric space on X.

Example 2.18 ([15])**.** Let $X = A = \mathbb{R}$. Then, $d : X^A \times X^A \to \mathbb{R}(A)^*$, where

$$d(\tilde{x}, \tilde{y}) = |\tilde{x} - \tilde{y}| \text{ for every } \tilde{x}, \tilde{y} \mathrel{\widetilde{\in}} \widetilde{X}$$

is a soft metric on X.

Definition 2.19 ([29])**.** Let X and Y be two nonempty sets and $f : X \to Y$ a mapping. Then,

(i) the image of a soft set $(F, A) \in SS(X, A)$ under the mapping f is the soft set $f(F, A) = (f(F), A)$ defined by
$$f(F)(a) = f(F(a)) \text{ for each } a \in A.$$
(ii) the inverse image of a soft set $(G, A) \in SS(Y, A)$ under the mapping f is the soft set $f^{-1}(G, A) = (f^{-1}(G), A)$ defined by
$$f^{-1}(G)(a) = f^{-1}(G(a)) \text{ for each } a \in A.$$

Definition 2.20 ([14])**.** A sequence $\{\tilde{\alpha}_n\}$ of soft real numbers is said to soft converge to $\tilde{\alpha}$, and we write $\lim_{n\to\infty} \tilde{\alpha}_n = \tilde{\alpha}$, if for every $\tilde{\epsilon} \mathrel{\widetilde{>}} \overline{0}$, there exists an $n_0 \in \mathbb{N}$ such that $|\tilde{\alpha}_n - \tilde{\alpha}| \mathrel{\widetilde{<}} \tilde{\epsilon}$ for all $n \geq n_0$.

Definition 2.21. A sequence $\{\tilde{\alpha}_n\}$ of soft real numbers is said to be a soft Cauchy sequence if for every $\tilde{\epsilon} \mathrel{\widetilde{>}} \overline{0}$, there exists an $n_0 \in \mathbb{N}$ such that $|\tilde{\alpha}_n - \tilde{\alpha}_m| \mathrel{\widetilde{<}} \tilde{\epsilon}$ for all $m, n \geq n_0$.

3. *se*-uniform spaces and soft *E*-distance

Now, on the basis of the notion of soft elements, we introduce *se*-uniform spaces and give the concept of a soft E-distance in *se*-uniform spaces, which is a soft version of the E-distance which is proposed by Aamri and El Moutawakil [1].

Definition 3.1. (i) The set $\Delta \subset X^A \times X^A$ is said to be an *se*-diagonal, which is defined by

$$\Delta = \{(\tilde{x}, \tilde{x}) : \tilde{x} \,\widetilde{\in}\, \widetilde{X}\}.$$

(ii) Let $U \subset X^A \times X^A$. Then,

$$U^{-1} = \{(\tilde{x}, \tilde{y}) : (\tilde{y}, \tilde{x}) \in U\}.$$

If $U = U^{-1}$, then U is said to be soft symmetric.

(iii) Let $U, V \subset X^A \times X^A$. Then,

$$U \circ V = \{(\tilde{x}, \tilde{y}) : \text{ for some } \tilde{z} \,\widetilde{\in}\, \widetilde{X}, (\tilde{x}, \tilde{z}) \in V \text{ and } (\tilde{z}, \tilde{y}) \in U\}.$$

Remark 3.2. Let $U, V, W \subset X^A \times X^A$. Then, the following statements are satisfied:

(i) If $U \subseteq V$, then $U^{-1} \subseteq V^{-1}$ and $U \circ W \subseteq V \circ W$.
(ii) $(U \circ V)^{-1} = V^{-1} \circ U^{-1}$.
(iii) $(U \circ V) \circ W = U \circ (V \circ W)$.

Definition 3.3. Let X be a set and A a set of parameters. An *se*-uniformity on X is a structure given by a set $\mathcal{U}$ of subsets of $X^A \times X^A$ which satisfies the following axioms:

(u_1) If $U \in \mathcal{U}$, then $\Delta \subseteq U$.
(u_2) If $U \in \mathcal{U}$, then there is a $V \in \mathcal{U}$ such that $V \circ V \subseteq U$
(u_3) If $U \in \mathcal{U}$, then there is a $V \in \mathcal{U}$ such that $V^{-1} \subseteq U$.
(u_4) If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.
(u_5) If $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$.

The triplet $(X, \mathcal{U}, A)$ is called an *se*-uniform space on X.

Definition 3.4. An *se*-uniform space $(X, \mathcal{U}, A)$ is called a Hausdorff space if the intersection of all the $U \in \mathcal{U}$ is equal to Δ, that is, if $(\tilde{x}, \tilde{y}) \in U$ for all $U \in \mathcal{U}$ implies $\tilde{x} = \tilde{y}$.

Example 3.5. Let $X = \{x, y\}$, $A = \{a_1, a_2\}$ and let $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4 \,\widetilde{\in}\, \widetilde{X}$ where

$$\tilde{x}_1(a_1) = x,\ \tilde{x}_1(a_2) = x, \qquad \tilde{x}_2(a_1) = y,\ \tilde{x}_2(a_2) = y,$$
$$\tilde{x}_3(a_1) = x,\ \tilde{x}_3(a_2) = y, \qquad \tilde{x}_4(a_1) = y,\ \tilde{x}_4(a_2) = x.$$

Then, $\mathcal{U} = \{U \subseteq X^A \times X^A : \Delta \subseteq U\}$ is an *se*-uniformity on X and therefore $(X, \mathcal{U}, A)$ is an *se*-uniform space. Moreover, it is a Hausdorff space.

Definition 3.6. A nonvoid subfamily $\mathcal{B}$ of an se-uniformity $\mathcal{U}$ is called an *se*-base of $\mathcal{U}$ if for any $U \in \mathcal{U}$ there is a $V \in \mathcal{B}$ such that $V \subseteq U$.

Theorem 3.7. *A nonvoid family $\mathcal{B} \subseteq \mathcal{P}(X^A \times X^A)$ is an se-base of some se-uniformity on X if it satisfies the following axioms:*

(b_1) If $B \in \mathcal{B}$, then $\Delta \subseteq B$.
(b_2) If $B \in \mathcal{B}$, then there is a $C \in \mathcal{B}$ such that $C \circ C \subseteq B$.
(b_3) If $B \in \mathcal{B}$, then there is a $C \in \mathcal{B}$ such that $C^{-1} \subseteq B$.
(b_4) If $B_1, B_2 \in \mathcal{B}$, then there is a $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Proof. It is easy to see that a nonvoid family $\mathcal{B}$ satisfying $(b_1)-(b_4)$ generates an *se*-uniformity $\mathcal{U} = \{U \subseteq X^A \times X^A :$ for some $B \in \mathcal{B}$, $B \subseteq U\}$. □

Example 3.8. Let (X, d, A) be a soft metric space. Then, $\mathcal{B}_d = \{D_{\tilde{\epsilon}} : \tilde{\epsilon} \tilde{>} \overline{0}\}$ is a *se*-base for some *se*-uniformity $\mathcal{U}_d$ on X such that

$$D_{\tilde{\epsilon}} = \{(\tilde{x}, \tilde{y}) : d(\tilde{x}, \tilde{y}) \tilde{<} \tilde{\epsilon}\} \subset X^A \times X^A.$$

Let us prove that $\mathcal{B}_d$ satisfies axioms $(b_1)-(b_4)$.

(b_1) Let $\tilde{\epsilon} \tilde{>} \overline{0}$. For all $\tilde{x} \tilde{\in} \widetilde{X}$, since $d(\tilde{x}, \tilde{x}) = \overline{0}$, we have $(\tilde{x}, \tilde{x}) \in D_{\tilde{\epsilon}}$.

(b_2) For every $\tilde{\epsilon} \tilde{>} \overline{0}$, we have $D_{\tilde{\epsilon}/2} \circ D_{\tilde{\epsilon}/2} \subseteq D_{\tilde{\epsilon}}$. Indeed, let $(\tilde{x}, \tilde{y}) \in D_{\tilde{\epsilon}/2} \circ D_{\tilde{\epsilon}/2}$. Then, there exists a $\tilde{z} \tilde{\in} \widetilde{X}$ such that $(\tilde{x}, \tilde{z}) \in D_{\tilde{\epsilon}/2}$ and $(\tilde{z}, \tilde{y}) \in D_{\tilde{\epsilon}/2}$. Hence, $d(\tilde{x}, \tilde{y}) \tilde{\leq} d(\tilde{x}, \tilde{z}) + d(\tilde{z}, \tilde{y}) \tilde{<} \tilde{\epsilon}/2 + \tilde{\epsilon}/2 = \tilde{\epsilon}$. So, $(\tilde{x}, \tilde{y}) \in D_{\tilde{\epsilon}}$.

(b_3) It is clear because $D_{\tilde{\epsilon}} = D_{\tilde{\epsilon}}^{-1}$ for any $\tilde{\epsilon} \tilde{>} \overline{0}$.

(b_4) Let $\tilde{\epsilon_1}, \tilde{\epsilon_2} \tilde{>} \overline{0}$. Now, let us take $\tilde{\epsilon} \tilde{>} \overline{0}$ such that $\tilde{\epsilon}(a) = \min\{\tilde{\epsilon_1}(a), \tilde{\epsilon_2}(a)\}$ for each $a \in A$. Therefore, it is easy to see that $D_{\tilde{\epsilon}} \subseteq D_{\tilde{\epsilon_1}} \cap D_{\tilde{\epsilon_2}}$.

This *se*-uniformity $\mathcal{U}_d$ is called the *se*-uniformity generated by the soft metric d. Also, one can readily check that $(X, \mathcal{U}_d, A)$ is a Hausdorff space.

Definition 3.9. Let $(X, \mathcal{U}, A)$ be an *se*-uniform space. If $U \in \mathcal{U}$ and $(\tilde{x}, \tilde{y}) \in U, (\tilde{y}, \tilde{x}) \in U$, then $\tilde{x}$ and $\tilde{y}$ are said to be soft U-close.

Remark 3.10. Let $(X, \mathcal{U}, A)$ be an *se*-uniform space. Clearly, if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$. Hence, each $U \in \mathcal{U}$ contains a soft symmetric element $V \in \mathcal{U}$. In fact, let $U \in \mathcal{U}$. From (u_4) it follows that $V = U \cap U^{-1}$ is a soft symmetric element of $\mathcal{U}$ contained in U. Therefore, if $(\tilde{x}, \tilde{y}) \in V$, then $\tilde{x}$ and $\tilde{y}$ are both soft V- and soft U-close and so for our purpose one may assume that each $U \in \mathcal{U}$ is soft symmetric.

Definition 3.11. Let $(X, \mathcal{U}, A)$ be an *se*-uniform space. A mapping $p : X^A \times X^A \to \mathbb{R}(A)^+$, where $\mathbb{R}(A)^+ = \{\tilde{\alpha} \tilde{\in} \widetilde{\mathbb{R}} : \tilde{\alpha} \tilde{>} \overline{0}$ or $\tilde{\alpha} = \overline{0}\}$, is called a soft E-distance on X if

(p_1) for any $U \in \mathcal{U}$, there exists a $\tilde{\delta} \tilde{>} \overline{0}$ such that $p(\tilde{z}, \tilde{x}) \tilde{<} \tilde{\delta}$ and $p(\tilde{z}, \tilde{y}) \tilde{<} \tilde{\delta}$ for some $\tilde{z} \tilde{\in} \widetilde{X}$, imply $(\tilde{x}, \tilde{y}) \in U$.

(p_2) $p(\tilde{x}, \tilde{y}) \tilde{\leq} p(\tilde{x}, \tilde{z}) + p(\tilde{z}, \tilde{y})$ for all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \widetilde{X}$.

Example 3.12. Let (X, d, A) be a soft metric space, where $X = A = [0, 1]$ and $d(\tilde{x}, \tilde{y}) = |\tilde{x} - \tilde{y}|$ for every $\tilde{x}, \tilde{y} \tilde{\in} \widetilde{X}$. It is easy to see that $(X, \mathcal{U}_d, A)$ is an *se*-uniform space, where $\mathcal{U}_d$ is the *se*-uniformity generated by the soft metric d. Define a mapping $p : X^A \times X^A \to \mathbb{R}(A)^+$ by

$$p(\tilde{x}, \tilde{y}) = \begin{cases} \overline{0}, & \text{if } \tilde{y} = \overline{0}; \\ \tilde{y}, & \text{if } \overline{0} \tilde{<} \tilde{y} \tilde{<} \overline{1} \\ \overline{1}, & \text{otherwise} \end{cases}$$

for all $\tilde{x}, \tilde{y} \tilde{\in} \widetilde{X}$. Then, p is a soft E-distance on X. In fact, since for all $\tilde{x}, \tilde{y}, \tilde{z} \tilde{\in} \widetilde{X}$, $p(\tilde{x}, \tilde{y}) = p(\tilde{z}, \tilde{y})$, (p_2) holds. Let $U \in \mathcal{U}_d$. Then, there exists an $\tilde{\epsilon} \tilde{>} \overline{0}$ such that $D_{\tilde{\epsilon}} \subseteq U$. If $p(\tilde{z}, \tilde{x}) \tilde{<} \tilde{\epsilon}$ and $p(\tilde{z}, \tilde{y}) \tilde{<} \tilde{\epsilon}$, then we have

$$d(\tilde{x}, \tilde{y}) = |\tilde{x} - \tilde{y}| \tilde{<} \tilde{\epsilon}.$$

Therefore, we get $(\tilde{x}, \tilde{y}) \in D_{\tilde{\epsilon}}$ and so $(\tilde{x}, \tilde{y}) \in U$. This shows that p satisfies (p_1).

Example 3.13. Let $(X, \mathcal{U}, A)$ be an *se*-uniform space which is defined in Example 3.5. Let a mapping $p : X^A \times X^A \to \mathbb{R}(A)^+$ be defined by $p(\tilde{x}_i, \tilde{x}_i) = \overline{0}$, for $i = 1, 2, 3, 4$,

$$p(\tilde{x}_i, \tilde{x}_{i+1}) = p(\tilde{x}_{i+1}, \tilde{x}_i) = \overline{\tfrac{1}{4}}, \text{ for } i = 1, 2, 3,$$
$$p(\tilde{x}_i, \tilde{x}_{i+2}) = p(\tilde{x}_{i+2}, \tilde{x}_i) = \overline{\tfrac{1}{3}}, \text{ for } i = 1, 2 \text{ and}$$
$$p(\tilde{x}_i, \tilde{x}_{i+3}) = p(\tilde{x}_{i+3}, \tilde{x}_i) = \overline{\tfrac{1}{2}}, \text{ for } i = 1.$$

Then, one can easily verify that p is a soft E-distance on X.

Definition 3.14. Let $(X,\mathcal{U},A)$ be an se-uniform space and p a soft E-distance on X.

(i) A sequence $\{\tilde{x}_n\}$ of soft elements in X is said to soft p-converge to $\tilde{x} \tilde{\in} \widetilde{X}$ if for every $\tilde{\epsilon} \tilde{>} \overline{0}$, there exists an $n_0 \in \mathbb{N}$ such that $p(\tilde{x}_n,\tilde{x}) \tilde{<} \tilde{\epsilon}$ for all $n \geq n_0$. We denote this by $\lim_{n\to\infty} p(\tilde{x}_n,\tilde{x}) = \overline{0}$.

(ii) A sequence $\{\tilde{x}_n\}$ of soft elements in X is said to be a soft p-Cauchy sequence if for every $\tilde{\epsilon} \tilde{>} \overline{0}$, there exists an $n_0 \in \mathbb{N}$ such that $p(\tilde{x}_n,\tilde{x}_m) \tilde{<} \tilde{\epsilon}$ for all $m,n \geq n_0$.

(iii) $(X,\mathcal{U},A)$ is said to be a soft S-complete space if for every soft p-Cauchy sequence $\{\tilde{x}_n\}$ in X there exists an $\tilde{x} \tilde{\in} \widetilde{X}$ such that $\lim_{n\to\infty} p(\tilde{x}_n,\tilde{x}) = \overline{0}$.

(iv) $(X,\mathcal{U},A)$ is said to be a soft p-bounded space if

$$\delta_p(\widetilde{X})(a) = \sup\{p(\tilde{x},\tilde{y})(a) \mid \tilde{x},\, \tilde{y} \tilde{\in} \widetilde{X}\} < \infty \text{ for every } a \in A.$$

Definition 3.15. Let $(X,\mathcal{U},A)$ be an se-uniform space, p a soft E-distance on X, and Z a nonempty subset of X. Z is said to be a soft S-complete subspace of X if for every soft p-Cauchy sequence $\{\tilde{z}_n\}$ in Z there exists a $\tilde{z} \tilde{\in} \widetilde{Z}$ such that $\lim_{n\to\infty} p(\tilde{z}_n,\tilde{z}) = \overline{0}$.

Definition 3.16. Let $(X,\mathcal{U},A)$ be an se-uniform space. A sequence $\{\tilde{x}_n\}$ of soft elements in X is said to be a soft Cauchy sequence for $\mathcal{U}$ if for any $U \in \mathcal{U}$, there exists an $n_0 \geq 1$ such that $\tilde{x}_n$ and $\tilde{x}_m$ are soft U-close for all $m,n \geq n_0$.

Jungck and Rhoades [20] defined the concept of weak compatibility. We introduce this concept in soft setting as follows:

Definition 3.17. Let $(X,\mathcal{U},A)$ be a Hausdorff se-uniform space and p a soft E-distance on X. Two mappings $f,g : (X,\mathcal{U},A) \to (X,\mathcal{U},A)$ are said to be soft weakly compatible if $f(g(\tilde{x})) = g(f(\tilde{x}))$ for all $\tilde{x} \tilde{\in} \widetilde{X}$ which satisfy $f(\tilde{x}) = g(\tilde{x})$.

Example 3.18. Consider Example 3.12. Let us define mappings $f,g : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} x, & \text{if } x \in [0,\frac{1}{3}); \\ 1, & \text{if } x \in [\frac{1}{3},1]. \end{cases} \qquad g(x) = \begin{cases} 1-x, & \text{if } x \in [0,\frac{1}{3}); \\ 1, & \text{if } x \in [\frac{1}{3},1]. \end{cases}$$

Then for every $\tilde{x}$ satisfying $\overline{\frac{1}{3}} \tilde{\leq} \tilde{x} \tilde{\leq} \overline{1}$, we have $f(g(\tilde{x})) = g(f(\tilde{x}))$, showing that f,g are soft weakly compatible mappings.

Definition 3.19. Let $(X,\mathcal{U},A)$ be a Hausdorff se-uniform space and let $f : (X,\mathcal{U},A) \to (X,\mathcal{U},A)$ be a mapping. A soft element $\tilde{x} \tilde{\in} \widetilde{X}$ is called a fixed soft element of f if $f(\tilde{x}) = \tilde{x}$.

Example 3.20. Let f be the mapping defined in Example 3.18. Then every $\tilde{x}$ satisfying $\overline{0} \tilde{\leq} \tilde{x} \tilde{<} \overline{\frac{1}{3}}$ and $\tilde{x} = \overline{1}$ is a fixed soft element of f.

Definition 3.21. Let X be a set, A a set of parameters, and $\mathfrak{R}$ a soft relation on X, i.e., a subset of $X^A \times X^A$. We say that $\mathfrak{R}$ is a soft order in X if it has the following properties (we write $\tilde{x}\,\mathfrak{R}\,\tilde{y}$ instead of $(\tilde{x},\tilde{y}) \in \mathfrak{R}$):

(so_1) For every $\tilde{x} \tilde{\in} \widetilde{X}$, $\tilde{x}\,\mathfrak{R}\,\tilde{x}$.
(so_2) If $\tilde{x}\,\mathfrak{R}\,\tilde{y}$ and $\tilde{y}\,\mathfrak{R}\,\tilde{x}$, then $\tilde{x} = \tilde{y}$.
(so_3) If $\tilde{x}\,\mathfrak{R}\,\tilde{y}$ and $\tilde{y}\,\mathfrak{R}\,\tilde{z}$, then $\tilde{x}\,\mathfrak{R}\,\tilde{z}$.

A set X together with a soft order $\mathfrak{R}$ in X is called a soft ordered set, denoted by $(X,\mathfrak{R},A)$. For example, $(\mathbb{R},\tilde{\leq},A)$ is a soft ordered set.

Definition 3.22. Let $(X,\tilde{\preceq},A)$ and $(Y,\tilde{\preceq}',A)$ be two soft ordered sets. A mapping $f : (X,\tilde{\preceq},A) \to (Y,\tilde{\preceq}',A)$ is said to be a soft nondecreasing if $f(\tilde{x}) \tilde{\preceq}' f(\tilde{y})$ for every pair $\tilde{x},\tilde{y}$ of soft elements in X satisfying $\tilde{x} \tilde{\preceq} \tilde{y}$.

4. Main Results

In this section, inspired by the works [2, 7, 8], we prove some fixed soft element theorems for various mappings with the help of soft E-distance. We begin with the following lemma that will play a crucial role in the proofs of the main theorems.

Lemma 4.1. *Let $(X,\mathcal{U},A)$ be a Hausdorff se-uniform space and p a soft E-distance on X. Let $\{\tilde{x}_n\}$ and $\{\tilde{y}_n\}$ be arbitrary sequences of soft elements in X and $\{\tilde{\alpha}_n\}$, $\{\tilde{\beta}_n\}$ sequences of nonnegative soft real numbers soft converging to $\overline{0}$. Then, for $\tilde{x}$, $\tilde{y}$, $\tilde{z}\,\widetilde{\in}\,\widetilde{X}$, the following results hold:*

(a) If $p(\tilde{x}_n,\tilde{y})\ \widetilde{\leq}\ \tilde{\alpha}_n$ and $p(\tilde{x}_n,\tilde{z})\ \widetilde{\leq}\ \tilde{\beta}_n$ for all $n\in\mathbb{N}$, then $\tilde{y}=\tilde{z}$. In particular, if $p(\tilde{x},\tilde{y})=\overline{0}$ and $p(\tilde{x},\tilde{z})=\overline{0}$, then $\tilde{y}=\tilde{z}$.

(b) If $p(\tilde{x}_n,\tilde{x}_m)\,\widetilde{\leq}\,\tilde{\alpha}_n$ for all $m>n$, then $\{\tilde{x}_n\}$ is a soft Cauchy sequence for $\mathcal{U}$.

Proof. (a) Let $p(\tilde{x}_n,\tilde{y})\ \widetilde{\leq}\ \tilde{\alpha}_n$ and $p(\tilde{x}_n,\tilde{z})\ \widetilde{\leq}\ \tilde{\beta}_n$ for all $n\in\mathbb{N}$. Then, for every $\tilde{\epsilon}\,\widetilde{>}\,\overline{0}$, there exists an $n_0\in\mathbb{N}$ such that $p(\tilde{x}_n,\tilde{y})\,\widetilde{<}\,\tilde{\epsilon}$ and $p(\tilde{x}_n,\tilde{z})\,\widetilde{<}\,\tilde{\epsilon}$ for all $n\geq n_0$ since $\{\tilde{\alpha}_n\}$ and $\{\tilde{\beta}_n\}$ soft converge to $\overline{0}$. From the condition (p_1) of Definition 3.11 it follows that $(\tilde{y},\tilde{z})\in U$ for every $U\in\mathcal{U}$. Thus, since $(X,\mathcal{U},A)$ is a Hausdorff space, we have $\tilde{y}=\tilde{z}$. (b) Let $U\in\mathcal{U}$. Then, there exists a $\tilde{\delta}\,\widetilde{>}\,\overline{0}$ such that $p(\tilde{z},\tilde{x})\,\widetilde{<}\,\tilde{\delta}$ and $p(\tilde{z},\tilde{y})\,\widetilde{<}\,\tilde{\delta}$ for some $\tilde{z}\,\widetilde{\in}\,\widetilde{X}$ imply $(\tilde{x},\tilde{y})\in U$. Since $\{\tilde{\alpha}_n\}$ soft converge to $\overline{0}$, there exists an $n_0\in\mathbb{N}$ such that $\tilde{\alpha}_n\,\widetilde{<}\,\tilde{\delta}$ for all $n\geq n_0$. By hypothesis, we have $p(\tilde{x}_n,\tilde{x}_m)\,\widetilde{\leq}\,\tilde{\alpha}_n\,\widetilde{<}\,\tilde{\delta}$ for all $m>n\geq n_0$. Now, choose an $m\geq 1$ with $m>n\geq n_0$. From the fact that $p(\tilde{x}_n,\tilde{x}_u)\,\widetilde{<}\,\tilde{\delta}$ and $p(\tilde{x}_n,\tilde{x}_v)\,\widetilde{<}\,\tilde{\delta}$ for every $u,v\geq m$ it follows that $\tilde{x}_u$ and $\tilde{x}_v$ are soft U-close. □

Let $\psi:(\mathbb{R}^+,\widetilde{\leq},A)\to(\mathbb{R}^+,\widetilde{\leq},A)$ be a soft nondecreasing mapping satisfying the following conditions:

(ψ_1) $\overline{0}\,\widetilde{<}\,\psi(\tilde{\alpha})\,\widetilde{<}\,\tilde{\alpha}$, for each $\tilde{\alpha}\,\widetilde{>}\,\overline{0}$.

(ψ_2) $\lim_{n\to\infty}\psi^n(\tilde{\alpha})=\overline{0}$, for each $\tilde{\alpha}\,\widetilde{>}\,\overline{0}$, where ψ^n is the nth iterate of ψ.

Theorem 4.2. *Let $(X,\mathcal{U},A)$ be a Hausdorff se-uniform space and p be a soft E-distance on X such that $(X,\mathcal{U},A)$ is a soft p-bounded space. Let $f,g:(X,\mathcal{U},A)\to(X,\mathcal{U},A)$ be two soft weakly compatible mappings satisfying the following conditions:*

(i) $f(X)\subseteq g(X)$,

(ii) $p(f(\tilde{x}),f(\tilde{y}))\,\widetilde{\leq}\,\psi\big(p(g(\tilde{x}),g(\tilde{y}))\big)$, for all $\tilde{x},\tilde{y}\,\widetilde{\in}\,\widetilde{X}$,

(iii) Either $f(X)$ or $g(X)$ is a soft S-complete subspace of X. Then f and g have a unique common fixed soft element.

Proof. Let $\tilde{x}_0\,\widetilde{\in}\,\widetilde{X}$. Since $f(X)\subseteq g(X)$, we can choose a $\tilde{x}_1\,\widetilde{\in}\,\widetilde{X}$ such that $f(\tilde{x}_0)=g(\tilde{x}_1)$. Again, we can choose a $\tilde{x}_2\,\widetilde{\in}\,\widetilde{X}$ such that $f(\tilde{x}_1)=g(\tilde{x}_2)$. Continuing this process, we can choose a $\tilde{x}_n\,\widetilde{\in}\,\widetilde{X}$ such that $f(\tilde{x}_{n-1})=g(\tilde{x}_n)$. Now, we shall show that the sequence $\{f(\tilde{x}_n)\}$ of soft elements in $f(X)$ is a soft p-Cauchy sequence. Let $\tilde{\epsilon}\,\widetilde{>}\,\overline{0}$. Since $\delta_p(\widetilde{X})\,\widetilde{>}\,\overline{0}$, by (ψ_2), there exists an $n_0\in\mathbb{N}$ such that $\psi^n(\delta_p(\widetilde{X}))\,\widetilde{<}\,\tilde{\epsilon}$ for all $n\geq n_0$. Then, by using the condition (ii), we get

$$\begin{aligned}p(f(\tilde{x}_n),f(\tilde{x}_m))&\,\widetilde{\leq}\,\psi\big(p(g(\tilde{x}_n),g(\tilde{x}_m))\big)=\psi\big(p(f(\tilde{x}_{n-1}),f(\tilde{x}_{m-1}))\big)\\&\,\widetilde{\leq}\,\psi^2\big(p(g(\tilde{x}_{n-1}),g(\tilde{x}_{m-1}))\big)=\psi^2\big(p(f(\tilde{x}_{n-2}),f(\tilde{x}_{m-2}))\big)\\&\ \vdots\\&\,\widetilde{\leq}\,\psi^n\big(p(f(\tilde{x}_0),f(\tilde{x}_{m-n}))\big)\end{aligned}$$

for $m\geq n\geq n_0$. From the fact that the mapping ψ is soft nondecreasing, it follows that

$$p(f(\tilde{x}_n),f(\tilde{x}_m))\,\widetilde{\leq}\,\psi^n\big(p(f(\tilde{x}_0),f(\tilde{x}_{m-n}))\big)\,\widetilde{\leq}\,\psi^n(\delta_p(\widetilde{X}))\,\widetilde{<}\,\tilde{\epsilon}.$$

Since p is not symmetrical, by repeating the same argument we get

$$p(f(\tilde{x}_m),f(\tilde{x}_n))\,\widetilde{\leq}\,\psi^n(\delta_p(\widetilde{X}))\,\widetilde{<}\,\tilde{\epsilon}.$$

Hence the sequence $\{f(\tilde{x}_n)\}$ is a soft p-Cauchy sequence.

Suppose that $g(X)$ is a soft S-complete space. Then, there exists an $\tilde{r}\,\tilde{\in}\,\widetilde{X}$ such that $\lim_{n\to\infty} p(f(\tilde{x}_n), g(\tilde{r})) = \overline{0}$. Therefore, we have $\lim_{n\to\infty} p(g(\tilde{x}_n), g(\tilde{r})) = \overline{0}$. Since

$$p(f(\tilde{x}_n), f(\tilde{r}))\,\widetilde{\leq}\,\psi\big(p(g(\tilde{x}_n), g(\tilde{r}))\big),$$

we obtain

$$\lim_{n\to\infty} p(f(\tilde{x}_n), f(\tilde{r})) = \lim_{n\to\infty} p(f(\tilde{x}_n), g(\tilde{r})) = \overline{0}.$$

By applying Lemma 4.1(a), we get $f(\tilde{r}) = g(\tilde{r})$. Since f and g are soft weakly compatible mappings, we have $f(g(\tilde{r})) = g(f(\tilde{r}))$. Also, $f(f(\tilde{r})) = f(g(\tilde{r})) = g(f(\tilde{r})) = g(g(\tilde{r}))$. Suppose that $p\big(f(\tilde{r}), f(f(\tilde{r}))\big)\,\widetilde{>}\,\overline{0}$. From ($ii$) and ($\psi_1$), it follows that

$$p\big(f(\tilde{r}), f(f(\tilde{r}))\big)\,\widetilde{\leq}\,\psi\big(p\big(g(\tilde{r}), g(f(\tilde{r}))\big)\big) = \psi\big(p\big(f(\tilde{r}), f(f(\tilde{r}))\big)\big)\,\widetilde{<}\,p\big(f(\tilde{r}), f(f(\tilde{r}))\big),$$

which yields a contradiction. Thus, $p\big(f(\tilde{r}), f(f(\tilde{r}))\big) = \overline{0}$. Similarly, it can be shown that $p\big(f(\tilde{r}), f(\tilde{r})\big) = \overline{0}$. Because $p\big(f(\tilde{r}), f(f(\tilde{r}))\big) = p\big(f(\tilde{r}), f(\tilde{r})\big) = \overline{0}$, by Lemma 4.1($a$), we get $f(\tilde{r}) = f(f(\tilde{r}))$. Therefore, $g(f(\tilde{r})) = f(f(\tilde{r})) = f(\tilde{r})$, which implies that $f(\tilde{r})$ is a common fixed soft element of f and g.

If $f(X)$ is a soft S-complete space, then there exists a $\tilde{u}\,\tilde{\in}\,\widetilde{X}$ such that $\lim_{n\to\infty} p(f(\tilde{x}_n), f(\tilde{u})) = \overline{0}$. Since $f(X) \subseteq g(X)$, there exists an $\tilde{r}\,\tilde{\in}\,\widetilde{X}$ such that $f(\tilde{u}) = g(\tilde{r})$ and the proof that $g(\tilde{r})$ is a common fixed soft element of f and g is the same as that given when $g(X)$ is soft S-complete space.

To investigate the uniqueness of the common fixed soft element of f and g, suppose that there exist $\tilde{u}, \tilde{v}\,\tilde{\in}\,\widetilde{X}$ such that $f(\tilde{u}) = g(\tilde{u}) = \tilde{u}$ and $f(\tilde{v}) = g(\tilde{v}) = \tilde{v}$. If $p(\tilde{u}, \tilde{v})\,\widetilde{>}\,\overline{0}$, then from ($ii$) and ($\psi_1$), we have

$$p(\tilde{u}, \tilde{v}) = p(f(\tilde{u}), f(\tilde{v}))\,\widetilde{\leq}\,\psi\big(p(g(\tilde{u}), g(\tilde{v}))\big) = \psi(p(\tilde{u}, \tilde{v}))\,\widetilde{<}\,p(\tilde{u}, \tilde{v})$$

which is a contradiction. Consequently, we have $p(\tilde{u}, \tilde{v}) = \overline{0}$. Analogously, one can show that $p(\tilde{u}, \tilde{u}) = \overline{0}$. Thus, by Lemma 4.1($a$), we obtain $\tilde{u} = \tilde{v}$. Hence, f and g have a unique common fixed soft element. □

Condition (ii) and the assumption concerning soft weak compatibility of f and g in Theorem 4.2 is not superfluous. To see this, consider the following example.

Example 4.3. Consider Example 3.12. Clearly, $(X, \mathcal{U}_d, A)$ is a soft p-bounded space. Next, let mappings f and g be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{3}) \text{ and } x = 1; \\ \frac{1}{5}, & \text{if } x \in [\frac{1}{3}, 1). \end{cases} \qquad g(x) = \begin{cases} \frac{1}{3} - x, & \text{if } x \in [0, \frac{1}{3}); \\ \frac{1}{2}, & \text{if } x \in [\frac{1}{3}, 1); \\ 0, & \text{if } x = 1. \end{cases}$$

It is easy to see that $f(X) \subseteq g(X)$ and $f(X)$ is a soft S-complete space. But

$$f(g(\overline{1})) = f(\overline{0}) = \overline{0} \text{ and } g(f(\overline{1})) = g(\overline{0}) = \overline{\frac{1}{3}}$$

and so f and g are not soft weakly compatible mappings. Consider the soft nondecreasing mapping ψ defined by

$$\psi(x) = \begin{cases} x^2, & \text{if } x \in [0, \frac{1}{3}); \\ \frac{x}{3}, & \text{if } x \in [\frac{1}{3}, 1]. \end{cases}$$

Then, it satisfies (ψ_1) and (ψ_2). On the other hand, for $\tilde{x}\,\tilde{\in}\,\widetilde{X}$, we have

$$p(f(\tilde{x}), f(\overline{\frac{1}{3}})) = \overline{\frac{1}{5}}\,\widetilde{>}\,\psi\big(p(g(\tilde{x}), g(\overline{\frac{1}{3}}))\big) = \overline{\frac{1}{6}},$$

which implies that condition (ii) in Theorem 4.2 does not hold for all $\tilde{x}, \tilde{y}\,\tilde{\in}\,\widetilde{X}$. Then, f and g do not have a common fixed soft element.

Next, we recall the following example to show the validity of Theorem 4.2.

Example 4.4. Consider Example 3.13. It is obvious that $(X,\mathcal{U},A)$ is a soft p-bounded space. Take mappings $f,g:X\to X$ such that

$$f(x)=f(y)=x \text{ and } g(x)=x,\ g(y)=y.$$

One can readily verify that $f(X)\subseteq g(X)$ and $f(X)$ is a soft S-complete space. Also, f and g are soft weakly compatible mappings, because

$$f(g(\tilde{x}_1))=g(f(\tilde{x}_1)) \text{ whenever } f(\tilde{x}_1)=g(\tilde{x}_1).$$

Now, let ψ be the mapping defined in Example 4.3. Then, the condition (ii) in Theorem 4.2 holds for all $\tilde{x},\tilde{y}\,\tilde{\in}\,\widetilde{X}$. Hence, all the conditions of Theorem 4.2 are satisfied and $\tilde{x}_1$ is the unique common fixed soft element of f and g.

In Theorem 4.2, letting $g=Id_X$ (resp. $f=Id_X$), where Id_X is the identity mapping on X, we obtain the following results:

Corollary 4.5. *Let $(X,\mathcal{U},A)$ be a Hausdorff se-uniform space and p a soft E-distance on X such that $(X,\mathcal{U},A)$ is a soft p-bounded space. Let $f:(X,\mathcal{U},A)\to(X,\mathcal{U},A)$ be a mapping satisfying the following conditions:*

(i) $p(f(\tilde{x}),f(\tilde{y}))\,\tilde{\leq}\,\psi(p(\tilde{x},\tilde{y}))$, *for all* $\tilde{x},\tilde{y}\,\tilde{\in}\,\widetilde{X}$,
(ii) $f(X)$ *is a soft S-complete space.*

Then f has a unique fixed soft element.

Corollary 4.6. *Let $(X,\mathcal{U},A)$ be a Hausdorff se-uniform space and p a soft E-distance on X such that $(X,\mathcal{U},A)$ is a soft p-bounded and soft S-complete space. Let $g:(X,\mathcal{U},A)\to(X,\mathcal{U},A)$ be a surjective mapping such that*

$$p(\tilde{x},\tilde{y})\,\tilde{\leq}\,\psi\big(p(g(\tilde{x}),g(\tilde{y}))\big), \text{ for all } \tilde{x},\tilde{y}\,\tilde{\in}\,\widetilde{X}.$$

Then g has a unique fixed soft element.

Definition 4.7. Let X be a set and A a set of parameters. A mapping $f:X\to\mathbb{R}$ is called a soft semi-sequentially continuous if for any sequence $\{\tilde{x}_n\}$ of soft elements in X, the sequence $\{f(\tilde{x}_n)\}$ is a soft convergent sequence in $\mathbb{R}$.

Example 4.8. It is easily seen that every constant mapping $f:X\to\mathbb{R}$ is a soft semi-sequentially continuous.

Dhage [16] defined the notion of a weak increasing. We introduce this notion in soft setting as follows:

Definition 4.9. Let $(X,\tilde{\preceq},A)$ be a soft ordered set. Two mappings $f,g:X\to X$ are said to be soft weakly increasing if $f(\tilde{x})\,\tilde{\preceq}\,g(f(\tilde{x}))$ and $g(\tilde{x})\,\tilde{\preceq}\,f(g(\tilde{x}))$ hold for all $\tilde{x}\,\tilde{\in}\,\widetilde{X}$.

Example 4.10. Take as the soft ordered set $([0,1],\tilde{\leq},A)$ and let $f,g:[0,1]\to[0,1]$ be the mappings defined in Example 3.18. Then,

$$f(\tilde{x})=\tilde{x}\,\tilde{\leq}\,g(f(\tilde{x}))=g(\tilde{x})=\overline{1}-\tilde{x} \quad\text{and}\quad g(\tilde{x})=\overline{1}-\tilde{x}\,\tilde{\leq}\,f(g(\tilde{x}))=f(\overline{1}-\tilde{x})=\overline{1}$$

for every $\tilde{x}$ satisfying $\overline{0}\,\tilde{\leq}\,\tilde{x}\,\tilde{<}\,\overline{\tfrac{1}{3}}$. Similarly,

$$f(\tilde{x})=\overline{1}\,\tilde{\leq}\,g(f(\tilde{x}))=g(\overline{1})=\overline{1} \quad\text{and}\quad g(\tilde{x})=\overline{1}\,\tilde{\leq}\,f(g(\tilde{x}))=f(\overline{1})=\overline{1}$$

for every $\tilde{x}$ satisfying $\overline{\tfrac{1}{3}}\,\tilde{\leq}\,\tilde{x}\,\tilde{\leq}\,\overline{1}$. The same also holds for the other $\tilde{x}\,\tilde{\in}\,\widetilde{X}$, so that f and g are soft weakly increasing mappings.

Definition 4.11. Let $(X,\mathcal{U},A)$ be an se-uniform space and p a soft E-distance on X.

(i) $(X,\mathcal{U},A)$ is said to be a soft p-complete space if for every soft Cauchy sequence $\{\tilde{x}_n\}$ for $\mathcal{U}$, there exists an $\tilde{x}\tilde{\in}\widetilde{X}$ such that $\lim_{n\to\infty} p(\tilde{x}_n,\tilde{x})=\overline{0}$.

(ii) $f:(X,\mathcal{U},A)\to(X,\mathcal{U},A)$ is soft p-continuous if $\lim_{n\to\infty} p(\tilde{x}_n,\tilde{x})=\overline{0}$ implies

$$\lim_{n\to\infty} p(f(\tilde{x}_n),f(\tilde{x}))=\overline{0}.$$

Lemma 4.12. *Let $(X,\mathcal{U},A)$ be a Hausdorff se-uniform space, p a soft E-distance on X, and $\varphi:X\to\mathbb{R}$. By letting for $\tilde{x},\tilde{y}\tilde{\in}\widetilde{X}$*

$$\tilde{x}\,\tilde{\preceq}\,\tilde{y} \text{ iff } \tilde{x}=\tilde{y} \text{ or } p(\tilde{x},\tilde{y})\,\tilde{\leq}\,\varphi(\tilde{x})-\varphi(\tilde{y})$$

we define a soft order $\tilde{\preceq}$ on X induced by φ.

Proof. We shall show that $\tilde{\preceq}$ satisfies

(so$_1$) Since for every $\tilde{x}\tilde{\in}\widetilde{X}$, $\tilde{x}=\tilde{x}$, we have $\tilde{x}\,\tilde{\preceq}\,\tilde{x}$.

(so$_2$) For $\tilde{x},\tilde{y}\tilde{\in}\widetilde{X}$, let $\tilde{x}\,\tilde{\preceq}\,\tilde{y}$ and $\tilde{y}\,\tilde{\preceq}\,\tilde{x}$. Then,

$$\tilde{x}=\tilde{y} \text{ or } p(\tilde{x},\tilde{y})\,\tilde{\leq}\,\varphi(\tilde{x})-\varphi(\tilde{y}) \quad \text{and} \quad \tilde{y}=\tilde{x} \text{ or } p(\tilde{y},\tilde{x})\,\tilde{\leq}\,\varphi(\tilde{y})-\varphi(\tilde{x}).$$

Therefore we either have $\tilde{x}=\tilde{y}$ or $p(\tilde{x},\tilde{y})+p(\tilde{y},\tilde{x})=\overline{0}$. If $p(\tilde{x},\tilde{y})+p(\tilde{y},\tilde{x})=\overline{0}$, then we get $p(\tilde{x},\tilde{y})=\overline{0}$ and $p(\tilde{y},\tilde{x})=\overline{0}$. It follows from the condition (p$_2$) of Definition 3.11 that $p(\tilde{x},\tilde{x})=\overline{0}$. Thus, applying Lemma 4.1(a), we infer that $\tilde{x}=\tilde{y}$.

(so$_3$) For $\tilde{x},\tilde{y},\tilde{z}\tilde{\in}\widetilde{X}$, let $\tilde{x}\,\tilde{\preceq}\,\tilde{y}$ and $\tilde{y}\,\tilde{\preceq}\,\tilde{z}$. Then,

$$\tilde{x}=\tilde{y} \text{ or } p(\tilde{x},\tilde{y})\,\tilde{\leq}\,\varphi(\tilde{x})-\varphi(\tilde{y}) \quad \text{and} \quad \tilde{y}=\tilde{z} \text{ or } p(\tilde{y},\tilde{z})\,\tilde{\leq}\,\varphi(\tilde{y})-\varphi(\tilde{z}).$$

Observe that

(i) If $\tilde{x}=\tilde{y}$ and $\tilde{y}=\tilde{z}$, then $\tilde{x}=\tilde{z}$, i.e., $\tilde{x}\,\tilde{\preceq}\,\tilde{z}$.

(ii) If $\tilde{x}=\tilde{y}$ and $p(\tilde{y},\tilde{z})\,\tilde{\leq}\,\varphi(\tilde{y})-\varphi(\tilde{z})$, then $p(\tilde{x},\tilde{z})\,\tilde{\leq}\,\varphi(\tilde{x})-\varphi(\tilde{z})$, i.e., $\tilde{x}\,\tilde{\preceq}\,\tilde{z}$.

(iii) If $p(\tilde{x},\tilde{y})\,\tilde{\leq}\,\varphi(\tilde{x})-\varphi(\tilde{y})$ and $\tilde{y}=\tilde{z}$, then $p(\tilde{x},\tilde{z})\,\tilde{\leq}\,\varphi(\tilde{x})-\varphi(\tilde{z})$, i.e., $\tilde{x}\,\tilde{\preceq}\,\tilde{z}$.

(iv) If $p(\tilde{x},\tilde{y})\,\tilde{\leq}\,\varphi(\tilde{x})-\varphi(\tilde{y})$ and $p(\tilde{y},\tilde{z})\,\tilde{\leq}\,\varphi(\tilde{y})-\varphi(\tilde{z})$, then

$$p(\tilde{x},\tilde{z})\,\tilde{\leq}\,p(\tilde{x},\tilde{y})+p(\tilde{y},\tilde{z})\tilde{\leq}\,\varphi(\tilde{x})-\varphi(\tilde{y})+\varphi(\tilde{y})-\varphi(\tilde{z})=\varphi(\tilde{x})-\varphi(\tilde{z})$$

i.e., $\tilde{x}\,\tilde{\preceq}\,\tilde{z}$, which completes the proof.

□

Theorem 4.13. *Let $(X,\mathcal{U},A)$ be a Hausdorff se-uniform space, p a soft E-distance on soft p-complete space X such that $p(\tilde{x},\tilde{x})=\overline{0}$ for each $\tilde{x}\tilde{\in}\widetilde{X}$ and let $\varphi:X\to\mathbb{R}$ be a soft semi-sequentially continuous mapping. If $\tilde{\preceq}$ is the soft order on X induced by φ and mappings $f,g:(X,\mathcal{U},A)\to(X,\mathcal{U},A)$ are both soft p-continuous and soft weakly increasing, then f and g have a common fixed soft element.*

Proof. Let $\tilde{x}_0\tilde{\in}\widetilde{X}$ and define a sequence $\{\tilde{x}_n\}$ of soft elements in X by

$$f(\tilde{x}_{2n})=\tilde{x}_{2n+1} \quad and \quad g(\tilde{x}_{2n+1})=\tilde{x}_{2n+2} \;\; for\; n\in\{0,1,...\}.$$

From the fact that f and g are soft weakly increasing, it follows that

$$\tilde{x}_1=f(\tilde{x}_0)\,\tilde{\preceq}\,g(f(\tilde{x}_0))=g(\tilde{x}_1)=\tilde{x}_2$$
$$\tilde{x}_2=g(\tilde{x}_1)\,\tilde{\preceq}\,f(g(\tilde{x}_1))=f(\tilde{x}_2)=\tilde{x}_3$$

and by continuing this process we get

$$\tilde{x}_1\,\tilde{\preceq}\,\tilde{x}_2\,\tilde{\preceq}\,...\,\tilde{x}_n\,\tilde{\preceq}\,\tilde{x}_{n+1}\,\tilde{\preceq}\,....$$

Now, since the mapping φ is soft semi-sequentially continuous, it follows that the sequence $\{\varphi(\tilde{x}_n)\}$ of soft real numbers is a soft convergent sequence. Therefore, one can readily verify that it is a soft Cauchy sequence in $\mathbb{R}$. It turns out that $\{\tilde{x}_n\}$ is a soft Cauchy sequence for $\mathcal{U}$. Indeed, let $U\in\mathcal{U}$. Then, there exists a $\tilde{\delta}\,\tilde{>}\,\overline{0}$

such that $p(\tilde{z},\tilde{x}) \tilde{<} \tilde{\delta}$ and $p(\tilde{z},\tilde{y}) \tilde{<} \tilde{\delta}$ for some $\tilde{z} \tilde{\in} \widetilde{X}$ imply $(\tilde{x},\tilde{y}) \in U$. Because the sequence $\{\varphi(\tilde{x}_n)\}$ is soft Cauchy in $\mathbb{R}$, there is an $n_0 \geq 1$ such that $|\varphi(\tilde{x}_m) - \varphi(\tilde{x}_n)| \tilde{<} \tilde{\delta}$ for all $m,n \geq n_0$. If $m = n$, then it is obvious that $\tilde{x}_n$ and $\tilde{x}_m$ are soft U-close. If $n < m$, then by $\tilde{x}_n \tilde{\preceq} \tilde{x}_m$, we have

$$\tilde{x}_n = \tilde{x}_m \ \text{ or } \ p(\tilde{x}_n,\tilde{x}_m) \tilde{\leq} \varphi(\tilde{x}_n) - \varphi(\tilde{x}_m).$$

This implies that

$$p(\tilde{x}_n,\tilde{x}_m) \tilde{\leq} \varphi(\tilde{x}_n) - \varphi(\tilde{x}_m) = |\varphi(\tilde{x}_m) - \varphi(\tilde{x}_n)| \tilde{<} \tilde{\delta}.$$

Hence, from the fact that $p(\tilde{x},\tilde{x}) = \overline{0}$ for each $\tilde{x} \tilde{\in} \widetilde{X}$, it follows that $\tilde{x}_n$ and $\tilde{x}_m$ are soft U-close. Since $(X,\mathcal{U},A)$ is a soft p-complete space, there exists an $\tilde{x} \tilde{\in} \widetilde{X}$ such that $\lim_{n\to\infty} p(\tilde{x}_n,\tilde{x}) = \overline{0}$. By the soft p-continuity of f, we have $\lim_{n\to\infty} p(\tilde{x}_{2n+1},\tilde{x}) = \lim_{n\to\infty} p(\tilde{x}_{2n+1}, f(\tilde{x})) = \overline{0}$. Thus, by Lemma 4.1($a$), we obtain $f(\tilde{x}) = \tilde{x}$. In the same manner, we can show that $g(\tilde{x}) = \tilde{x}$, because the mapping g is soft p-continuous. It follows that $f(\tilde{x}) = g(\tilde{x}) = \tilde{x}$, hence f and g have a common fixed soft element. □

Theorem 4.14. *Let $(X,\mathcal{U},A)$ be a Hausdorff se-uniform space, p a soft E-distance on soft p-complete space X such that $p(\tilde{x},\tilde{x}) = \overline{0}$ for each $\tilde{x} \tilde{\in} \widetilde{X}$ and let $\varphi : X \to \mathbb{R}$ be a soft semi-sequentially continuous mapping. If $\tilde{\preceq}$ is the soft order on X induced by φ and a mapping $f : (X,\mathcal{U},A) \to (X,\mathcal{U},A)$ is both soft p-continuous and soft nondecreasing with $\tilde{x}_0 \tilde{\preceq} f(\tilde{x}_0)$ for some $\tilde{x}_0 \tilde{\in} \widetilde{X}$, then f has a fixed soft element.*

Proof. Let us take an $\tilde{x}_0 \tilde{\in} \widetilde{X}$ such that $\tilde{x}_0 \tilde{\preceq} f(\tilde{x}_0)$ and consider a sequence $\{\tilde{x}_n\}$ of soft elements in X satisfying $f(\tilde{x}_{n-1}) = \tilde{x}_n$ for $n \in \{1,2,...\}$. Since the mapping f is soft nondecreasing, we have $\tilde{x}_0 \tilde{\preceq} \tilde{x}_1 \tilde{\preceq} \tilde{x}_2 \tilde{\preceq} ....$ Therefore, as is shown in the proof of Theorem 4.13, $\{\tilde{x}_n\}$ is a soft Cauchy sequence for $\mathcal{U}$. From the fact that $(X,\mathcal{U},A)$ is a soft p-complete space, it follows that there exists an $\tilde{x} \tilde{\in} \widetilde{X}$ such that $\lim_{n\to\infty} p(\tilde{x}_n,\tilde{x}) = \overline{0}$. By the soft p-continuity of f, we have $\lim_{n\to\infty} p(\tilde{x}_n,\tilde{x}) = \lim_{n\to\infty} p(\tilde{x}_n, f(\tilde{x})) = \overline{0}$. Thus, by Lemma 4.1($a$), we obtain $f(\tilde{x}) = \tilde{x}$. □

Example 4.15. Consider Example 3.13. Using the fact that $\Delta \in \mathcal{U}$, we get that $(X,\mathcal{U},A)$ is a soft p-complete space. Now, define mappings $f : X \to X$ and $\varphi : X \to \mathbb{R}$ by $f(x) = f(y) = x$ and $\varphi(x) = \varphi(y) = 0$. Since φ is the constant mapping, it is a soft semi-sequentially continuous. Also, the soft order on X induced by φ is $\tilde{\preceq} := \{(\tilde{x}_1,\tilde{x}_1),(\tilde{x}_2,\tilde{x}_2),(\tilde{x}_3,\tilde{x}_3),(\tilde{x}_4,\tilde{x}_4)\}$. As it can be easily seen, the mapping f is soft nondecreasing with respect to $\tilde{\preceq}$ and soft p-continuous. Moreover, we have $\tilde{x}_1 \tilde{\preceq} f(\tilde{x}_1)$ for $\tilde{x}_1 \tilde{\in} \widetilde{X}$. So, all the conditions of Theorem 4.14 are satisfied and $\tilde{x}_1$ is a fixed soft element of f.

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