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Common fixed point theorems for six self-maps in b-metric spaces with nonlinear contractive conditions

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Abstract

In the framework of a b-metric space, by using the compatible and weak compatible conditions of self-mapping pair, we discussed the existence and uniqueness of the common fixed point for a class of ϕ-type contraction mapping, some new common fixed point theorems are obtained. In the end of the paper, we give some illustrative examples in support of our new results. The results presented in this paper extend and improve some well-known comparable results in the existing literature.

Keywords: b-metric space, common fixed point, compatible maps, weak compatible maps.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

In 1990, Kang et al. [12] applied the compatibility of mappings to prove common fixed point theorem of φ-contractive mappings. The same year, Liu [15] introduced the notion of weak compatibility of mappings and proved some common fixed point theorems. In 2011, Yu and Gu [20] studied a class of common fixed point problem of φ-contractive mappings and obtained a new common fixed point theorem.

Motivated and inspired by the above results, the aim of the paper is focus on the study of b-metric space proposed by Czerwik[6]. By using the compatible and weak compatible conditions, we prove some new common fixed point theorems for six self-maps satisfying a class of ϕ-type contraction condition. Because of the metric space is a special case of the b-metric space, our results presented in this paper extend and improve some well-known corresponding results in the literature due to Kang et al. [12], Roshan et al. [18], Jungck [10], Diviccaro and Sessa [8], and Ding [7].

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Definition 1.1 ([6])**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric if the following conditions are satisfied:

(b1) $d(x, y) = 0 \Leftrightarrow x = y$;
(b2) $d(x, y) = d(y, x)$;
(b3) $d(x, z) \leq s[d(x, y) + d(y, z)]$

for all $x, y, z \in X$. In this case, the pair (X, d) is called a b-metric space and the number s is called the coefficient of (X, d).

Remark 1.2. The class of b-metric spaces is effectively larger than that of metric spaces. Indeed, b-metric is a metric if and only if $s = 1$. For the counter-example see [2].

In [6], Czerwik extended the Banach contraction principle from metric spaces to b-metric spaces. Since then, a number of authors have investigated fixed point problems in b-metric spaces (see [1–4, 9, 13, 14, 16, 17, 19], and the references therein).

Definition 1.3 ([3])**.** Let (X, d) be a b-metric space, and let $\{x_n\}$ the sequence of points in X.

(a) A sequence $\{x_n\}$ in X is called b-convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.
(b) $\{x_n\}$ in X is said to be b-Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
(c) The b-metric space (X, d) is called b-complete if every b-Cauchy sequence in X is b-convergent.

Proposition 1.4 ([4])**.** *In a b-metric space (X, d) the following assertions hold:*

(i) *a b-convergent sequence has a unique limit.*
(ii) *each b-convergent sequence is a b-Cauchy sequence.*
(iii) *in general, b-metric is not continuous.*

Definition 1.5 ([18])**.** Let (X, d) be a b-metric space. A pair $\{f, g\}$ is said to be compatible if

$$\lim_{n\to\infty} d\left(fgx_n, gfx_n\right) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 1.6 ([11])**.** Let (X, d) be a b-metric space. A pair $\{f, g\}$ is said to be weak compatible if

$$\{t \in X : f(t) = g(t)\} \subset \{t \in X : fg(t) = gf(t)\}.$$

In [18] the authors proved the following result.

Theorem 1.7 ([18, Theorem 2.1])**.** *Suppose that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{T}$ are self-mappings on a b-complete b-metric space (X, d) such that $\mathcal{A}(X) \subset \mathcal{T}(X)$, $\mathcal{B}(X) \subset \mathcal{S}(X)$. Suppose that the condition*

$$d(\mathcal{A}x, \mathcal{B}y) \leq \frac{k}{s^4} \max\left\{d(\mathcal{A}x, \mathcal{S}x), d(\mathcal{B}y, \mathcal{T}y), d(\mathcal{S}x, \mathcal{T}y), \frac{1}{2}\left[d(\mathcal{A}x, \mathcal{T}y) + d(\mathcal{B}y, \mathcal{S}x)\right]\right\} \tag{1.1}$$

holds for all $x, y \in X$ with $0 < k < 1$ and $s \geq 1$ is the coefficient of (X, d). If S and T are continuous and pairs $\{\mathcal{A}, \mathcal{S}\}$ and $\{\mathcal{B}, \mathcal{T}\}$ are compatible, then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{T}$ have a unique common fixed point in X.

The purpose of this article is to further improve and extend Theorem 1.7 to the more general nonlinear contractive type mapping.

To prove our result, we shall use the following lemma.

Lemma 1.8 ([1])**.** *Let (X,d) be a b-metric space with the parameter $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-converge to x and y in X, respectively. Then we have*

$$\frac{1}{s^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, we have $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq sd(x,z).$$

Lemma 1.9 ([18])**.** *Let (X,d) be a b-metric space. If there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim\limits_{n\to\infty} x_n = t$ for some $t \in X$ and $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$, then $\lim\limits_{n\to\infty} y_n = t$.*

Lemma 1.10. *Let (X,d) be a b-metric space. Suppose that the sequence $\{y_n\}$ in X satisfies $\lim\limits_{n\to\infty} d(y_n,y_{n+1}) = 0$. If $\{y_n\}$ is not b-Cauchy in X, then there exists an $\varepsilon_0 > 0$ and positive integer sequences $\{m_i\}$ and $\{n_i\}$ such that*

(i) $m_i > n_i + 1, n_i \to \infty$ $(i \to \infty)$;

(ii) $d(y_{m_i}, y_{n_i}) > \varepsilon_0$; $d(y_{m_i-1}, y_{n_i}) \leq \varepsilon_0$, $i = 1,2,3,\cdots$.

Proof. The proof is similar to the proof of Lemma 2.1 in [5], hence it is deleted. □

2. Main results

In this section, suppose that Φ_1 be the set of functions $\phi : [0,\infty)^5 \to [0,\infty)$ satisfying the conditions:

(ϕ_1) ϕ is non-decrease and upper semicontinuous about each variable.

(ϕ_2) For all $t > 0$,
$$\psi(t) = \max\{\phi(0,0,t,t,t), \phi(t,t,t,0,2t), \phi(t,t,t,2t,0)\} < t. \tag{2.1}$$

Let Φ_2 be the set of functions $\phi : [0,\infty)^5 \to [0,\infty)$ satisfying the condition (ϕ_1) and

(ϕ_3) for all $t > 0$,

$$\psi(t) = \max\{\phi(t,t,t,t,t), \phi(t,t,t,0,2t), \phi(t,t,t,2t,0)\} < t. \tag{2.2}$$

Clearly we can get: If $t \leq \psi(t)$, then $t = 0$.

Theorem 2.1. *Let A, B, S, T, F, and G be six self-mappings on a b-complete b-metric space (X,d), and the following conditions hold:*

(i) *$A(X) \subset TG(X)$, $B(X) \subset SF(X)$;*

(ii) *$AF = FA$, $SF = FS$, $BG = GB$, $TG = GT$;*

(iii) *For all $x, y \in X$,*

$$d(Ax,By) \leq \frac{1}{s^4}\phi\begin{pmatrix} d(Ax,SFx), d(By,TGy), d(SFx,TGy), \\ d(Ax,TGy), d(By,SFx) \end{pmatrix}, \tag{2.3}$$

where $\phi \in \Phi_1$ and $s \geq 1$ is the coefficient of (X,d).

If it satisfies one of the following conditions, then A, B, S, T, F, and G have a unique common fixed point z in X. Moreover, z is also a unique common fixed point of the pairs $\{A, SF\}$ and $\{B, TG\}$, respectively.

1) *Either A or SF is continuous, $\{S, SF\}$ is compatible and $\{B, TG\}$ is weak compatible;*
2) *Either B or TG is continuous, $\{B, TG\}$ is compatible and $\{A, SF\}$ is weak compatible;*
3) *Either SF or TG is surjection, and $\{A, SF\}$ and $\{B, TG\}$ are weak compatible.*

Proof. Let $x_0 \in X$, as $A(X) \subset TG(X), B(X) \subset SF(X)$, there exist $\{x_n\}, \{y_n\} \subset X$ such that

$$y_{2n} = Ax_{2n} = TGx_{2n+1}, y_{2n+1} = Bx_{2n+1} = SFx_{2n+2}, \quad n = 0, 1, 2, 3, \cdots .$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $y_{2n_0} = y_{2n_0+1}$, then from (2.3) we have

$$\begin{aligned} d(y_{2n_0+1}, y_{2n_0+2}) &= d(Ax_{2n_0+2}, Bx_{2n_0+1}) \\ &\leq \frac{1}{s^4}\phi\begin{pmatrix} d(Ax_{2n_0+2}, SFx_{2n_0+2}), d(Bx_{2n_0+1}, TGx_{2n_0+1}), \\ d(SFx_{2n_0+2}, TGx_{2n_0+1}), d(Ax_{2n_0+2}, TGx_{2n_0+1}), \\ d(Bx_{2n_0+1}, SFx_{2n_0+2}) \end{pmatrix} \\ &= \frac{1}{s^4}\phi\begin{pmatrix} d(y_{2n_0+2}, y_{2n_0+1}), d(y_{2n_0+1}, y_{2n_0}), d(y_{2n_0+1}, y_{2n_0}), \\ d(y_{2n_0+2}, y_{2n_0}), d(y_{2n_0+1}, y_{2n_0+1}) \end{pmatrix} \\ &= \frac{1}{s^4}\phi\left(d(y_{2n_0+1}, y_{2n_0+2}), 0, 0, d(y_{2n_0+1}, y_{2n_0+2}), 0\right) \\ &\leq \frac{1}{s^4}\phi\begin{pmatrix} d(d(y_{2n_0+1}, y_{2n_0+2}), d(y_{2n_0+1}, y_{2n_0+2}), \\ d(y_{2n_0+1}, y_{2n_0+2}), 2d(y_{2n_0+1}, y_{2n_0+2}), 0 \end{pmatrix} \\ &\leq \frac{1}{s^4}\psi\left(d(y_{2n_0+1}, y_{2n_0+2})\right) \\ &\leq \psi\left(d(y_{2n_0+1}, y_{2n_0+2})\right). \end{aligned}$$

By property of ψ, we obtain $d(y_{2n_0+1}, y_{2n_0+2}) = 0$. Consequently, $y_{2n_0+1} = y_{2n_0+2}$.

Similarly, we can get $y_{2n_0+2} = y_{2n_0+3}$. Hence, by the mathematical induction, we obtain $y_{2n_0} = y_{2n_0+1} = y_{2n_0+2} = \cdots$. This implies that $\{y_n\}_{n\geq n_0}$ is a constant sequence. Therefore, the sequence $\{y_n\}$ is a b-Cauchy sequence in (X, d). The same conclusion holds if we suppose that there exists $n_0 \in \mathbb{N}$ such that $y_{2n_0+1} = y_{2n_0+2}$. Without loss of generality, we can suppose that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Then $d(y_n, y_{n+1}) > 0$ for all $n \in \mathbb{N}$. Hence, from (2.3) we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Ax_{2n}, Bx_{2n+1}) \\ &\leq \frac{1}{s^4}\phi\begin{pmatrix} d(Ax_{2n}, SFx_{2n}), d(Bx_{2n+1}, TGx_{2n+1}), d(SFx_{2n}, TGx_{2n+1}), \\ d(Ax_{2n}, TGx_{2n+1}), d(Bx_{2n+1}, SFx_{2n}) \end{pmatrix} \\ &= \frac{1}{s^4}\phi\begin{pmatrix} d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n-1}, y_{2n}), \\ d(y_{2n}, y_{2n}), d(y_{2n+1}, y_{2n-1}) \end{pmatrix} \\ &\leq \frac{1}{s^4}\phi\begin{pmatrix} d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), \\ 0, sd(y_{2n-1}, y_{2n}) + sd(y_{2n}, y_{2n+1}) \end{pmatrix}. \end{aligned} \tag{2.4}$$

If $d(y_{2n-1}, y_{2n}) < d(y_{2n}, y_{2n+1})$, then $d(y_{2n}, y_{2n+1}) > 0$ (otherwise, we have $d(y_{2n-1}, y_{2n}) < 0$, which is a contradiction). In this case, from (2.3) and the property of ϕ and ψ, we deduce that

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &\leq \frac{1}{s^4}\phi\left(sd(y_{2n}, y_{2n+1}), sd(y_{2n}, y_{2n+1}), sd(y_{2n}, y_{2n+1}), 0, 2sd(y_{2n}, y_{2n+1})\right) \\ &\leq \frac{1}{s^4}\psi\left(sd(y_{2n}, y_{2n+1})\right) < \frac{1}{s^4}\left(sd(y_{2n}, y_{2n+1})\right) = \frac{1}{s^3}d(y_{2n}, y_{2n+1}), \end{aligned}$$

which is a contradiction, hence $d(y_{2n-1}, y_{2n}) \geq d(y_{2n}, y_{2n+1})$. Again, by (2.1), (2.4), and the property of ϕ and ψ, we get

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &\leq \frac{1}{s^4}\phi\left(sd(y_{2n-1}, y_{2n}), sd(y_{2n-1}, y_{2n}), sd(y_{2n-1}, y_{2n}), 0, 2sd(y_{2n-1}, y_{2n})\right) \\ &\leq \frac{1}{s^4}\psi\left(sd(y_{2n-1}, y_{2n})\right) < \frac{1}{s^4}(sd(y_{2n-1}, y_{2n})) = \frac{1}{s^3}d(y_{2n-1}, y_{2n}). \end{aligned} \tag{2.5}$$

Also, applying (2.3) and the property of ϕ and ψ, we proceed similarly as above and obtain

$$d(y_{2n+1}, y_{2n+2}) < \frac{1}{s^3} d(y_{2n}, y_{2n+1}). \tag{2.6}$$

Combining (2.5) and (2.6), we get

$$d(y_n, y_{n+1}) < \frac{1}{s^3} d(y_{n-1}, y_n). \tag{2.7}$$

Applying the above inequality (2.7) n times, we obtain

$$d(y_n, y_{n+1}) < \frac{1}{s^3} d(y_{n-1}, y_n) < \cdots < \left(\frac{1}{s^3}\right)^n d(y_0, y_1). \tag{2.8}$$

Taking limit as $n \to \infty$ in (2.8), we have

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.9}$$

Next, we shall show that $\{y_n\}$ is a b-Cauchy sequence in X. Otherwise, from Lemma 1.10, there exists $\varepsilon_0 > 0$ and two positive integer sequences $\{m_i\}$ and $\{n_i\}$ such that

(a) $m_i > n_i + 1$, $n_i \to \infty$ $(i \to \infty)$;
(b) $d(y_{m_i}, y_{n_i}) > \varepsilon_0$, $d(y_{m_i-1}, y_{n_i}) \le \varepsilon_0$, $i = 1, 2, 3, \cdots$.

From the condition (b) and using the triangular inequality, we have

$$d(y_{m_i}, y_{n_i}) \le sd(y_{m_i}, y_{m_i-1}) + sd(y_{m_i-1}, y_{n_i}) \le sd(y_{m_i}, y_{m_i-1}) + s\varepsilon_0, \tag{2.10}$$

$$d(y_{m_{i+1}}, y_{n_i}) \le sd(y_{m_{i+1}}, y_{m_i-1}) + sd(y_{m_i-1}, y_{n_i}) \le s^2 d(y_{m_{i+1}}, y_{m_i}) + s^2 d(y_{m_i}, y_{m_i-1}) + s\varepsilon_0, \tag{2.11}$$

$$d(y_{m_i-1}, y_{n_i+1}) \le sd(y_{m_{i-1}}, y_{n_i}) + sd(y_{n_i}, y_{n_i+1}) \le s\varepsilon_0 + sd(y_{n_i}, y_{n_i+1}), \tag{2.12}$$

$$\begin{aligned} d(y_{m_i}, y_{n_i+1}) &\le sd(y_{m_i}, y_{m_i-1}) + sd(y_{m_i-1}, y_{n_i+1}) \\ &\le sd(y_{m_i}, y_{m_i-1}) + s^2 d(y_{m_i-1}, y_{n_i}) + s^2 d(y_{n_i}, y_{n_i+1}) \\ &\le sd(y_{m_i}, y_{m_i-1}) + s^2 \varepsilon_0 + s^2 d(y_{n_i}, y_{n_i+1}). \end{aligned} \tag{2.13}$$

Taking the upper limit as $i \to \infty$ in (2.10), (2.11), (2.12), and (2.13), we obtain

$$\limsup_{i\to\infty} d(y_{m_i}, y_{n_i}) \le s\varepsilon_0, \tag{2.14}$$

$$\limsup_{i\to\infty} d(y_{m_{i+1}}, y_{n_i}) \le s\varepsilon_0, \tag{2.15}$$

$$\limsup_{i\to\infty} d(y_{m_i-1}, y_{n_i+1}) \le s\varepsilon_0, \tag{2.16}$$

$$\limsup_{i\to\infty} d(y_{m_i}, y_{n_i+1}) \le s^2\varepsilon_0. \tag{2.17}$$

Again, from the condition (b) and using the triangular inequality, we have

$$\begin{aligned} \varepsilon_0 < d(y_{m_i}, y_{n_i}) &\le sd(y_{m_i}, y_{m_i+1}) + sd(y_{m_i+1}, y_{n_i}) \\ &\le sd(y_{m_i}, y_{m_i+1}) + s^2 d(y_{m_i+1}, y_{n_i+1}) + s^2 d(y_{n_i+1}, y_{n_i}), \end{aligned} \tag{2.18}$$

$$\varepsilon_0 < d(y_{m_i}, y_{n_i}) \le sd(y_{m_i}, y_{n_i+1}) + sd(y_{n_i+1}, y_{n_i}). \tag{2.19}$$

Taking the upper limit as $i \to \infty$ in (2.18) and (2.19), we obtain

$$\begin{aligned} \limsup_{i\to\infty} d(y_{m_i-1}, y_{n_i+1}) &\ge \frac{\varepsilon_0}{s^2}, \\ \limsup_{i\to\infty} d(y_{m_i}, y_{n_i+1}) &\ge \frac{\varepsilon_0}{s}. \end{aligned} \tag{2.20}$$

Next, we discuss in following cases.

(I) Suppose that m_i is even number and n_i is odd number. It follows from (2.3) that,

$$\begin{aligned} d(y_{n_i+1}, y_{m_i+1}) &= d(Ax_{n_i+1}, Bx_{m_i+1}) \\ &\leq \frac{1}{s^4}\phi\left(\begin{array}{c} d(Ax_{n_i+1}, SFx_{n_i+1}), d(Bx_{m_i+1}, TGx_{m_i+1}), \\ d(SFx_{n_i+1}, TGx_{m_i+1}), d(Ax_{n_i+1}, TGx_{m_i+1}), \\ d(Bx_{m_i+1}, SFx_{n_i+1}) \end{array}\right) \\ &= \frac{1}{s^4}\phi\left(\begin{array}{c} d(y_{n_i+1}, y_{n_i}), d(y_{m_i+1}, y_{m_i}), d(y_{n_i}, y_{m_i}), \\ d(y_{n_i+1}, y_{m_i}), d(y_{m_i+1}, y_{n_i}) \end{array}\right). \end{aligned}$$

Taking the upper limit as $i \to \infty$ in the above inequality, and using (2.9), (2.14), (2.15), (2.17), the condition (b), and the property of ϕ and ψ, we get

$$\begin{aligned} \frac{\varepsilon_0}{s^2} &\leq \limsup_{i\to\infty} d(y_{n_i+1}, y_{m_i+1}) \\ &\leq \frac{1}{s^4}\limsup_{i\to\infty}\phi\left(\begin{array}{c} d(y_{n_i+1}, y_{n_i}), d(y_{m_i+1}, y_{m_i}), d(y_{n_i}, y_{m_i}), \\ d(y_{n_i+1}, y_{m_i}), d(y_{m_i+1}, y_{n_i}) \end{array}\right) \\ &\leq \frac{1}{s^4}\phi\left(0, 0, s\varepsilon_0, s^2\varepsilon_0, s\varepsilon_0\right) \leq \frac{1}{s^4}\phi\left(0, 0, s^2\varepsilon_0, s^2\varepsilon_0, s^2\varepsilon_0\right) \\ &\leq \frac{1}{s^4}\psi\left(s^2\varepsilon_0\right) < \frac{1}{s^4}(s^2\varepsilon_0) = \frac{\varepsilon_0}{s^2}, \end{aligned}$$

which is a contradiction.

(II) Suppose that m_i and n_i are both even numbers. It follows from (2.3) that

$$\begin{aligned} d(y_{m_i}, y_{n_i+1}) &= d(Ax_{m_i}, Bx_{n_i+1}) \\ &\leq \frac{1}{s^4}\phi\left(\begin{array}{c} d(Ax_{m_i}, SFx_{m_i}), d(Bx_{n_i+1}, TGx_{n_i+1}), \\ d(SFx_{m_i}, TGx_{n_i+1}), d(Ax_{m_i}, TGx_{n_i+1}), \\ d(Bx_{n_i+1}, SFx_{m_i}) \end{array}\right) \\ &= \frac{1}{s^4}\phi\left(\begin{array}{c} d(y_{m_i}, y_{m_i-1}), d(y_{n_i+1}, y_{n_i}), d(y_{m_i-1}, y_{n_i}), \\ d(y_{m_i}, y_{n_i}), d(y_{n_i+1}, y_{m_i-1}) \end{array}\right). \end{aligned}$$

Taking the upper limit as $i \to \infty$ in the above inequality, and using (2.9), (2.14), (2.16), (2.20), the condition (b), and the property of ϕ and ψ, we get

$$\begin{aligned} \varepsilon_0 s &\leq \limsup_{i\to\infty} d(y_{m_i}, y_{n_i+1}) \\ &\leq \frac{1}{s^4}\limsup_{i\to\infty}\phi\left(\begin{array}{c} d(y_{m_i}, y_{m_i-1}), d(y_{n_i+1}, y_{n_i}), d(y_{m_i-1}, y_{n_i}), \\ d(y_{m_i}, y_{n_i}), d(y_{n_i+1}, y_{m_i-1}) \end{array}\right) \\ &\leq \frac{1}{s^4}\phi\left(0, 0, \varepsilon_0, s\varepsilon_0, s\varepsilon_0\right) \\ &\leq \frac{1}{s^4}\phi\left(0, 0, s\varepsilon_0, s\varepsilon_0, s\varepsilon_0\right) \\ &\leq \frac{1}{s^4}\psi\left(s\varepsilon_0\right) < \frac{1}{s^4}(s\varepsilon_0) = \frac{1}{s^3}\varepsilon_0, \end{aligned}$$

which is a contradiction.

(III) Suppose that m_i and n_i are both odd numbers. (IV) Suppose that m_i is odd number and n_i is even number. Similarly, such two cases can deduce a contradiction. This implies $\{y_n\}$ is a b-Cauchy sequence in X.

As X is b-complete, there exists $z \in X$ such that $y_n \to z$ $(n \to \infty)$, then $\{y_{2n-1}\}$ and $\{y_{2n}\}$ b-convergent to z, that is,

$$Ax_{2n} = y_{2n} \to z, \quad SFx_{2n} = y_{2n-1} \to z \ (n \to \infty).$$

1) Let either A or SF is continuous, $\{A, SF\}$ is compatible and $\{B, TG\}$ is weak compatible.

First, suppose that SF is continuous, then $\{(SF)SFx_{2n}\}$ and $\{(SF)Ax_{2n}\}$ b-converge to SFz, since $\{A, SF\}$ is compatible, then we have

$$\lim_{n\to\infty} d((SF)Ax_{2n}, A(SF)x_{2n}) = 0.$$

Using Lemma 1.9, we obtain $\lim\limits_{n\to\infty} A(SF)x_{2n} = SFz$.

By (2.3), we have

$$d(A(SF)x_{2n}, Bx_{2n-1}) \leq \frac{1}{s^4}\phi\begin{pmatrix} d(A(SF)x_{2n}, (SF)(SF)x_{2n}), d(Bx_{2n-1}, TGx_{2n-1}), \\ d((SF)(SF)x_{2n}, TGx_{2n-1}), d(A(SF)x_{2n}, TGx_{2n-1}), \\ d(Bx_{2n-1}, (SF)(SF)x_{2n}) \end{pmatrix}. \tag{2.21}$$

Taking the upper limit as $i \to \infty$ in (2.21), using Lemma 1.8 and the property of ϕ and ψ, we obtain

$$\begin{aligned}
\frac{1}{s^2}d(SFz, z) &\leq \limsup_{n\to\infty} d(A(SF)x_{2n}, Bx_{2n-1}) \\
&\leq \frac{1}{s^4}\limsup_{n\to\infty}\phi\begin{pmatrix} d(A(SF)x_{2n}, (SF)(SF)x_{2n}), d(Bx_{2n-1}, TGx_{2n-1}), \\ d((SF)(SF)x_{2n}, TGx_{2n-1}), d(A(SF)x_{2n}, TGx_{2n-1}), \\ d(Bx_{2n-1}, (SF)(SF)x_{2n}) \end{pmatrix} \\
&\leq \frac{1}{s^4}\phi\left(s^2d(SFz, SFz), s^2d(z, z), s^2d(SFz, z), s^2d(SFz, z), s^2d(SFz, z)\right) \\
&= \frac{1}{s^4}\phi\left(0, 0, s^2d(SFz, z), s^2d(SFz, z), s^2d(SFz, z)\right) \\
&\leq \frac{1}{s^4}\psi\left(s^2d(SFz, z)\right).
\end{aligned}$$

The above inequality becomes

$$s^2d(SFz, z) \leq \psi\left(s^2d(SFz, z)\right).$$

By the property of ψ, we get $s^2d(SFz, z) = 0$, hence $SFz = z$. Again from (2.3), we get

$$d(Az, Bx_{2n-1}) \leq \frac{1}{s^4}\phi\begin{pmatrix} d(Az, SFz), d(Bx_{2n-1}, TGx_{2n-1}), d(SFz, TGx_{2n-1}), \\ d(Az, TGx_{2n-1}), d(Bx_{2n-1}, SFz) \end{pmatrix}. \tag{2.22}$$

Taking the upper limit as $i \to \infty$ in (2.22), using Lemma 1.8, $SFz = z$, and the property of ϕ and ψ, we obtain

$$\begin{aligned}
\frac{1}{s}d(Az, z) &\leq \limsup_{n\to\infty} d(Az, Bx_{2n-1}) \\
&\leq \frac{1}{s^4}\limsup_{n\to\infty}\phi\begin{pmatrix} d(Az, SFz), d(Bx_{2n-1}, TGx_{2n-1}), d(SFz, TGx_{2n-1}), \\ d(Az, TGx_{2n-1}), d(Bx_{2n-1}, SFz) \end{pmatrix} \\
&\leq \frac{1}{s^4}\phi\left(d(Az, SFz), s^2d(z, z), sd(SFz, z), sd(Az, z), sd(z, SFz)\right) \\
&= \frac{1}{s^4}\phi\left(d(Az, z), s^2d(z, z), sd(z, z), sd(Az, z), sd(z, z)\right) \\
&= \frac{1}{s^4}\phi\left(d(Az, z), 0, 0, sd(Az, z), 0\right) \\
&\leq \frac{1}{s^4}\phi\left(sd(Az, z), sd(Az, z), sd(Az, z), 2sd(Az, z), 0\right) \\
&\leq \frac{1}{s^4}\psi\left(sd(Az, z)\right) \leq \frac{1}{s^2}\psi\left(sd(Az, z)\right).
\end{aligned}$$

The above inequality becomes

$$sd(Az,z) \leq \psi\left(sd(Az,z)\right).$$

By the property (ϕ_3), we get $sd(Az,z)=0$, which means $Az=z$.

As $z \in A(X) \subset TG(X)$, there exists $\mu \in X$ such that $z = Az = TG\mu$. Using (2.3) and the property of ϕ and ψ, we get

$$\begin{aligned} d(z,B\mu) &= d(Az,B\mu) \\ &\leq \frac{1}{s^4}\phi\left(d(Az,SFz),d(B\mu,TG\mu),d(SFz,TG\mu),d(Az,TG\mu),d(B\mu,SFz)\right) \\ &= \frac{1}{s^4}\phi\left(d(z,z),d(B\mu,z),d(z,z),d(z,z),d(B\mu,z)\right) \\ &= \frac{1}{s^4}\phi\left(0,d(B\mu,z),0,0,d(B\mu,z)\right) \\ &\leq \frac{1}{s^4}\phi\left(d(B\mu,z),d(B\mu,z),d(B\mu,z),0,2d(B\mu,z)\right) \\ &\leq \frac{1}{s^4}\psi\left(d(B\mu,z)\right) \leq \psi\left(d(B\mu,z)\right). \end{aligned}$$

By the property of ψ, we get $d(B\mu,z)=0$, this implies that $B\mu = z$, and so $TG\mu = B\mu = z$.

By the weak compatibility of $\{B,TG\}$, we get

$$TGz = (TG)B\mu = B(TG)\mu = Bz.$$

Further, from (2.3) and the property of ϕ and ψ,

$$\begin{aligned} d(z,TGz) &= d(Az,Bz) \\ &\leq \frac{1}{s^4}\phi\left(d(Az,SFz),d(Bz,TGz),d(SFz,TGz),d(Az,TGz),d(Bz,SFz)\right) \\ &= \frac{1}{s^4}\phi\left(d(z,z),d(TGz,TGz),d(z,TGz),d(z,TGz),d(TGz,z)\right) \\ &= \frac{1}{s^4}\phi\left(0,0,d(z,TGz),d(z,TGz),d(z,TGz)\right) \\ &\leq \frac{1}{s^4}\psi\left(d(z,TGz)\right) \leq \psi\left(d(z,TGz)\right). \end{aligned}$$

By the property of ψ, we get $d(z,TGz)=0$, this implies that $z=TGz$, and so $z=TGz=Bz$. Therefore, $z=TGz=Bz=Az=SFz$.

Actually, since $AF=FA, SF=FS$, then

$$AFz = FAz = Fz, \quad (SF)Fz = F(SF)z = Fz.$$

Using (2.2), $z=TGz=Bz$, and the property of ϕ and ψ, we have

$$\begin{aligned} d(Fz,z) &= d(FAz,Bz) = d(AFz,Bz) \\ &\leq \frac{1}{s^4}\phi\begin{pmatrix} d(AFz,(SF)Fz),d(Bz,TGz),d((SF)Fz,TGz), \\ d(AFz,TGz),d(Bz,(SF)Fz) \end{pmatrix} \\ &= \frac{1}{s^4}\phi\left(d(Fz,Fz),d(z,z),d(Fz,z),d(Fz,z),d(z,Fz)\right) \\ &= \frac{1}{s^4}\phi\left(0,0,d(Fz,z),d(Fz,z),d(Fz,z)\right) \\ &\leq \frac{1}{s^4}\psi\left(d(Fz,z)\right) \leq \psi\left(d(Fz,z)\right). \end{aligned}$$

By the property of ψ, we get $d(Fz,z)=0$, this implies that $Fz=z$. As $SFz=z$, we know $Sz=z$. So $Fz=Sz=z$.

Since $BG=GB,\ TG=GT$, then

$$BGz=GBz=Gz,\quad (TG)Gz=G(TG)z=Gz.$$

By (2.3) and the property of ϕ and ψ, we get

$$\begin{aligned} d(z,Gz)&=d(Az,GBz)=d(Az,BGz)\\ &\le \frac{1}{s^4}\phi\begin{pmatrix} d(Az,SFz),d(BGz,(TG)Gz),d(SFz,(TG)Gz),\\ d(Az,(TG)Gz),d(BGz,SFz)\end{pmatrix}\\ &\le \frac{1}{s^4}\phi\left(d(z,z),d(Gz,Gz),d(z,Gz),d(z,Gz),d(Gz,z)\right)\\ &= \frac{1}{s^4}\phi\left(0,0,d(z,Gz),d(z,Gz),d(Gz,z)\right)\\ &\le \frac{1}{s^4}\psi\left(d(z,Gz)\right)\le\psi\left(d(z,Gz)\right).\end{aligned}$$

Using the property of ψ, we have $d(z,Gz)=0$, this is $z=Gz$. Since $z=TGz$, then $z=Tz$, so, $z=Tz=Gz$.

In the above proof, having that

$$z=Tz=Gz=Az=Bz=Sz=Fz,$$

we arrive z is the common fixed point of A,B,S,T,F, and G in X.

Second, suppose that A is continuous, then $\{A^2x_{2n}\}$ and $\{A(SF)x_{2n}\}$ converge to Az, using the compatibility of $\{A,SF\}$, having that

$$\lim_{n\to\infty} d((SF)Ax_{2n},A(SF)x_{2n})=0.$$

Using Lemma 1.9, we obtain $\lim\limits_{n\to\infty}(SF)Ax_{2n}=Az$.

By (2.2), we get

$$d(A^2x_{2n},Bx_{2n-1})\le\frac{1}{s^4}\phi\begin{pmatrix} d(A^2x_{2n},(SF)Ax_{2n}),d(Bx_{2n-1},TGx_{2n-1}),\\ d((SF)Ax_{2n},TGx_{2n-1}),d(A^2x_{2n},TGx_{2n-1}),\\ d(Bx_{2n-1},(SF)Ax_{2n})\end{pmatrix}. \tag{2.23}$$

Taking the upper limit as $i\to\infty$ in (2.23), using Lemma 1.8 and the property of ϕ and ψ, we obtain

$$\begin{aligned} \frac{1}{s^2}d(Az,z)&\le\limsup_{n\to\infty} d(A^2x_{2n},Bx_{2n-1})\\ &\le\frac{1}{s^4}\limsup_{n\to\infty}\phi\begin{pmatrix} d(A^2x_{2n},(SF)Ax_{2n}),d(Bx_{2n-1},TGx_{2n-1}),\\ d((SF)Ax_{2n},TGx_{2n-1}),d(A^2x_{2n},TGx_{2n-1}),\\ d(Bx_{2n-1},(SF)Ax_{2n})\end{pmatrix}\\ &\le\frac{1}{s^4}\phi\left(s^2d(Az,Az),s^2d(z,z),s^2d(Az,z),s^2d(Az,z),s^2d(z,Az)\right)\\ &=\frac{1}{s^4}\phi\left(0,0,s^2d(Az,z),s^2d(Az,z),s^2d(Az,z)\right)\\ &\le\frac{1}{s^4}\psi\left(s^2d(Az,z)\right).\end{aligned}$$

The above inequality becomes

$$s^2d(Az,z)\le\psi\left(s^2d(Az,z)\right).$$

By the property of ψ, we get $s^2d(Az,z)=0$, this is $Az=z$.

Since $z \in A(X) \subset TG(X)$, there exists $\mu \in X$ such that $z = Az = TG\mu$. By (2.3), we get

$$d(Ax_{2n}, B\mu) \le \frac{1}{s^4}\phi\begin{pmatrix} d(Ax_{2n}, SFx_{2n}), d(B\mu, TG\mu), d(SFx_{2n}, TG\mu), \\ d(Ax_{2n}, TG\mu), d(B\mu, SFx_{2n}) \end{pmatrix}. \tag{2.24}$$

Taking the upper limit as $i \to \infty$ in (2.24), using Lemma 1.8 and the property of ϕ and ψ, we obtain

$$\begin{aligned}
\frac{1}{s}d(z, B\mu) &\le \limsup_{n\to\infty} d(Ax_{2n}, B\mu) \\
&\le \frac{1}{s^4}\limsup_{n\to\infty}\phi\begin{pmatrix} d(Ax_{2n}, SFx_{2n}), d(B\mu, TG\mu), d(SFx_{2n}, TG\mu), \\ d(Ax_{2n}, TG\mu), d(B\mu, SFx_{2n}) \end{pmatrix} \\
&\le \frac{1}{s^4}\phi\left(s^2 d(z,z), d(B\mu, z), sd(z,z), sd(z,z), sd(B\mu, z)\right) \\
&= \frac{1}{s^4}\phi\left(0, d(B\mu, z), 0, 0, sd(B\mu, z)\right) \\
&\le \frac{1}{s^4}\phi\left(sd(B\mu, z), sd(B\mu, z), sd(B\mu, z), 0, 2sd(B\mu, z)\right) \\
&\le \frac{1}{s^4}\psi\left(sd(B\mu, z)\right) \le \frac{1}{s^2}\psi\left(sd(B\mu, z)\right).
\end{aligned}$$

The above inequality becomes $sd(B\mu, z) \le \psi(sd(B\mu, z))$. By the property of ψ, we get $sd(B\mu, z) = 0$, this is $B\mu = z$.

Thus $TG\mu = B\mu = z$. Using the weak compatibility of $\{B, TG\}$, we obtain

$$TGz = (TG)B\mu = B(TG)\mu = Bz.$$

By (2.3), we have

$$d(Ax_{2n}, Bz) \le \frac{1}{s^4}\phi\begin{pmatrix} d(Ax_{2n}, SFx_{2n}), d(Bz, TGz), d(SFx_{2n}, TGz), \\ d(Ax_{2n}, TGz), d(Bz, SFx_{2n}) \end{pmatrix}. \tag{2.25}$$

Taking the upper limit as $i \to \infty$ in (2.25), using Lemma 1.8, $TGz = Bz$, and the property of ϕ and ψ, we obtain

$$\begin{aligned}
\frac{1}{s}d(z, Bz) &\le \limsup_{n\to\infty} d(Ax_{2n}, Bz) \\
&\le \frac{1}{s^4}\limsup_{n\to\infty}\phi\begin{pmatrix} d(Ax_{2n}, SFx_{2n}), d(Bz, TGz), d(SFx_{2n}, TGz), \\ d(Ax_{2n}, TGz), d(Bz, SFx_{2n}) \end{pmatrix} \\
&\le \frac{1}{s^4}\limsup_{n\to\infty}\phi\begin{pmatrix} s^2 d(z,z), d(Bz, Bz), sd(z, Bz), \\ sd(z, Bz), sd(Bz, z) \end{pmatrix} \\
&= \frac{1}{s^4}\phi\left(0, 0, sd(z, Bz), sd(z, Bz), sd(z, Bz)\right) \\
&\le \frac{1}{s^4}\psi\left(sd(z, Bz)\right) \le \frac{1}{s^2}\psi\left(sd(z, Bz)\right).
\end{aligned}$$

The above inequality becomes $sd(z, Bz) \le \psi(sd(z, Bz))$. By the property of ψ, we obtain $sd(z, Bz) = 0$, this is $z = Bz$, and so $z = TGz = Bz$.

Since $z \in B(X) \subset SF(X)$, so, there exists $\omega \in X$ such that $z = Bz = SF\omega$. By (2.3) and the property of ϕ and ψ, we have

$$\begin{aligned}
d(A\omega, z) &= d(A\omega, Bz) \\
&\le \frac{1}{s^4}\phi\begin{pmatrix} d(A\omega, SF\omega), d(Bz, TGz), d(SP\omega, TGz), \\ d(A\omega, TGz), d(Bz, SF\omega) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{s^4}\phi\left(d(A\omega,z),d(z,z),d(z,z),d(A\omega,z),d(z,z)\right)\\
&= \frac{1}{s^4}\phi\left(d(A\omega,z),0,0,d(A\omega,z),0\right)\\
&\le \frac{1}{s^4}\phi\left(d(A\omega,z),d(A\omega,z),d(A\omega,z),2d(A\omega,z),0\right)\\
&\le \frac{1}{s^4}\psi\left(d(A\omega,z)\right) \le \psi\left(d(A\omega,z)\right).
\end{aligned}
$$

By the property of ψ, we get $d(A\omega,z)=0$, this is $A\omega=z$, and so $A\omega=SP\omega=z$. Using the compatibility of $\{A,SF\}$, we have

$$Az=A(SF)\omega=(SF)A\omega=SFz.$$

In the above proof, we get, $z=TGz=Bz=Az=SFz$, so, z is the common fixed point of A,B,SF, and TG.

Similarly, we can also prove that z is the common fixed point of A,B,S,T,F, and G in X.

Finally, we prove that z is the unique common fixed point of A,B,S,T,F, and G in X, furthermore, z is also the unique common point of the pairs $\{A,SF\}$ and $\{B,TG\}$, respectively.

Suppose on the contrary, that there exists $z^*\in X$ such that $z^*\neq z$ and z^* is also the common fixed point of the pair $\{B,TG\}$ in X. Then, by (2.3) and the property of ϕ and ψ, we obtain

$$
\begin{aligned}
d(z,z^*) &= d(Az,Bz^*)\\
&\le \frac{1}{s^4}\phi\begin{pmatrix} d(Az,SFz),d(Bz^*,TGz^*),d(SFz,TGz^*),\\ d(Az,TGz^*),d(Bz^*,SFz)\end{pmatrix}\\
&= \frac{1}{s^4}\phi\left(d(z,z),d(z^*,z^*),d(z,z^*),d(z,z^*),d(z^*,z)\right)\\
&= \frac{1}{s^4}\phi\left(0,0,d(z,z^*),d(z,z^*),d(z,z^*)\right)\\
&\le \frac{1}{s^4}\psi\left(d(z,z^*)\right) \le \psi\left(d(z,z^*)\right).
\end{aligned}
$$

Then, by the property of ψ, we get $d(z,z^*)=0$, this is $z=z^*$. Thus, z is the unique common fixed point of the pair $\{B,TG\}$ in X. Similarly, z is the unique common fixed point of the pair $\{A,SF\}$ in X. Thus, z is the unique common fixed point of A,B,S,T,F, and G in X.

2) We show that B or TG is continuous, $\{B,TG\}$ is compatible and $\{A,SF\}$ weak compatible, the proof is similar with 1).

3) We show that SF or TG is onto mapping, furthermore, $\{A,SF\}$ and $\{B,TG\}$ are both weak compatible. Suppose that SF is surjection, then there exists $\nu\in X$, such that $SF\nu=z$. Using (2.3), we get

$$d(A\nu,Bx_{2n-1}) \le \frac{1}{s^4}\phi\begin{pmatrix} d(A\nu,SF\nu),d(Bx_{2n-1},TGx_{2n-1}),d(SF\nu,TGx_{2n-1}),\\ d(A\nu,TGx_{2n-1}),d(Bx_{2n-1},SF\nu)\end{pmatrix}. \tag{2.26}$$

Taking the upper limit as $i\to\infty$ in (2.26), using Lemma 1.8, and the property of ϕ and ψ, we get

$$
\begin{aligned}
\frac{1}{s}d(A\nu,z) &\le \limsup_{n\to\infty} d(A\nu,Bx_{2n-1})\\
&\le \frac{1}{s^4}\limsup_{n\to\infty}\phi\begin{pmatrix} d(A\nu,SF\nu),d(Bx_{2n-1},TGx_{2n-1}),d(SF\nu,TGx_{2n-1}),\\ d(A\nu,TGx_{2n-1}),d(Bx_{2n-1},SF\nu)\end{pmatrix}\\
&\le \frac{1}{s^4}\phi\left(d(A\nu,z),s^2d(z,z),sd(z,z),sd(A\nu,z),sd(z,z)\right)\\
&= \frac{1}{s^4}\phi\left(d(A\nu,z),0,0,sd(A\nu,z),0\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{s^4}\phi\left(sd(A\nu, z), sd(A\nu, z), sd(A\nu, z), 2sd(A\nu, z), 0\right)\\
&\leq \frac{1}{s^4}\psi\left(sd(A\nu, z)\right) \leq \frac{1}{s^2}\psi\left(sd(A\nu, z)\right).
\end{aligned}
$$

The above inequality becomes $sd(A\nu, z) \leq \psi\left(sd(A\nu, z)\right)$. Thus, by the property of ψ, we get $sd(A\nu, z) = 0$, that is $A\nu = z$. So $z = A\nu = SF\nu$. Since $\{A, SF\}$ is weak compatible, we have $SFz = (SF)A\nu = A(SF)\nu = Az$.

We replace z with ν in (2.26), then we get

$$
d(Az, Bx_{2n-1}) \leq \frac{1}{s^4}\phi\begin{pmatrix} d(Az, SFz), d(Bx_{2n-1}, TGx_{2n-1}), d(SFz, TGx_{2n-1}), \\ d(Az, TGx_{2n-1}), d(Bx_{2n-1}, SFz) \end{pmatrix}. \tag{2.27}
$$

Taking the upper limit as $i \to \infty$ in (2.27), using Lemma 1.8 and the property of ϕ and ψ, we obtain

$$
\begin{aligned}
\frac{1}{s}d(Az, z) &\leq \limsup_{n\to\infty} d(Az, Bx_{2n-1})\\
&\leq \frac{1}{s^4}\limsup_{n\to\infty}\phi\begin{pmatrix} d(Az, SFz), d(Bx_{2n-1}, TGx_{2n-1}), d(SFz, TGx_{2n-1}), \\ d(Az, TGx_{2n-1}), d(Bx_{2n-1}, SFz) \end{pmatrix}\\
&\leq \frac{1}{s^4}\phi\left(d(Az, Az), s^2d(z, z), sd(Az, z), sd(Az, z), sd(z, Az)\right)\\
&= \frac{1}{s^4}\phi\left(0, 0, sd(Az, z), sd(Az, z), sd(Az, z)\right)\\
&\leq \frac{1}{s^4}\psi\left(sd(Az, z)\right) \leq \frac{1}{s^2}\psi\left(sd(Az, z)\right).
\end{aligned}
$$

The above inequality becomes $sd(Az, z) \leq \psi\left(sd(Az, z)\right)$. Therefore, by the property of ψ, we get $sd(Az, z) = 0$, this is $Az = z$, and so $Az = SFz = z$. Similarly, we can prove that z is the unique common fixed point of A, B, S, T, F, and G in X, furthermore, z is also the unique common fixed point of the pairs of $\{A, SF\}$ and $\{B, TG\}$.

If TG is surjection, similarly, we can prove that z is the unique common fixed point of A, B, S, T, F, and G in X, z is also unique common fixed point of the pairs $\{A, SF\}$ and $\{B, TG\}$. □

As in the proof of Theorem 2.1 we have the following result.

Theorem 2.2. *Let A, B, S, T, F, and G be six self-mappings on a b-complete b-metric space (X, d), and the following conditions hold:*

(i) $A(X) \subset TG(X)$, $B(X) \subset SF(X)$;

(ii) $AF = FA$, $SF = FS$, $BG = GB$, $TG = GT$;

(iii) *For all $x, y \in X$,*

$$
d(Ax, By) \leq \frac{1}{s^4}\phi\begin{pmatrix} d(Ax, SFx), d(By, TGy), d(SFx, TGy), \\ d(Ax, TGy), d(By, SFx) \end{pmatrix},
$$

where $\phi \in \Phi_2$, $s \geq 1$ is the coefficient of (X, d).

If one of the following conditions is satisfied, then the mappings A, B, S, T, F, and G have a unique common fixed point z. And z is the unique common fixed point of the pairs $\{A, SF\}$ and $\{B, TG\}$.

1) *Either A or SF is continuous, $\{A, SF\}$ is compatible and $\{B, TG\}$ is weak compatible;*
2) *either B or TG is continuous, $\{B, TG\}$ is compatible and $\{A, SF\}$ is weak compatible;*
3) *either SF or TG is surjection, and $\{A, SF\}$ and $\{B, TG\}$ are weak compatible.*

Proof. Since the proof of Theorem 2.2 is very similar to that of Theorem 2.1, so we omit it. □

Remark 2.3. Theorems 2.1 and 2.2 improve and extend the corresponding results of Kang et al. [12] in its three aspects:

(1) the generalization from four mappings to six mappings;
(2) by using one continuous function as opposed to two;
(3) the two pairs are both compatible decrease to one pair is compatible and another is weak compatible;
(4) the X is a metric space is replaced by the X is a b-metric space.

In Theorems 2.1 and 2.2, if $F = G = I$ (I is identity mapping, the same below), we deduce the following results of common fixed point for four self-mappings.

Corollary 2.4. *Let $A, B, S,$ and T be four self-mappings on a b-complete b-metric space (X, d) and the following conditions hold:*

(i) $A(X) \subset T(X),\ B(X) \subset S(X)$;
(ii) *For all $x, y \in X$,*

$$d(Ax, By) \leq \frac{1}{s^4}\phi\left(d(Ax, Sx), d(By, Ty), d(Sx, Ty), d(Ax, Ty), d(By, Sx)\right), \tag{2.28}$$

where $\phi \in \Phi_1$, $s \geq 1$ is the coefficient of (X, d). If it satisfies one of the following condition, then A, B, S and T have a unique common fixed point z in X. Moreover, z is also a unique common fixed point of the pairs $\{A, S\}$ and $\{B, T\}$.

1) *Either A or S is continuous, $\{A, S\}$ is compatible, $\{B, T\}$ is weak compatible;*
2) *either B or T is continuous, $\{B, T\}$ is compatible, $\{A, S\}$ is weak compatible;*
3) *either S or T is surjection, and $\{A, S\}$ and $\{B, T\}$ are weak compatible.*

Corollary 2.5. *Let $A, B, S,$ and T be four self-mappings on a b-complete b-metric space (X, d), and the following conditions hold:*

(i) $A(X) \subset T(X), B(X) \subset S(X)$;
(ii) *For all $x, y \in X$,*

$$d(Ax, By) \leq \frac{1}{s^4}\phi\left(d(Ax, Sx), d(By, Ty), d(Sx, Ty), d(Ax, Ty), d(By, Sx)\right), \tag{2.29}$$

where $\phi \in \Phi_2$, and $s \geq 1$ is the coefficient of (X, d). If one of the following conditions is satisfied, then the mappings A, B, S and T have a unique common fixed point z. And z is the unique common fixed point of the pairs $\{A, S\}$ and $\{B, T\}$.

1) *Either A or SP is continuous, $\{A, S\}$ is compatible and $\{B, T\}$ is weak compatible;*
2) *either B or T is continuous, $\{B, T\}$ is compatible and $\{A, S\}$ is weak compatible;*
3) *either SP or T is surjection, and $\{A, S\}$ and $\{B, T\}$ are weak compatible.*

Remark 2.6. Corollaries 2.4 and 2.5 improve and extend Theorem 2.1 of Roshan et al. [18] in its three aspects:

(1) The contractive condition (1.1) is replaced by the new contractive condition defined by (2.28) and (2.29);
(2) by using one continuous function as opposed to two;
(3) the two pairs maps are both compatible decrease to one pair is compatible and another is weak compatible.

Remark 2.7. Theorems 2.1, 2.2 and Corollaries 2.4, 2.5 generalize and extend the corresponding results in Jungck [10], Diviccaro and Sessa [8], and Ding [7].

If there exists a function $\phi : [0,\infty)^5 \to [0,\infty)$ in Theorem 2.1 and Corollary 2.4 such that

$$\phi(t_1,t_2,t_3,t_4,t_5) = \frac{k}{s^4}\max\left\{t_1,t_2,t_3,\frac{1}{2}(t_4+t_5)\right\}$$

for all $(t_1,t_2,t_3,t_4,t_5) \in [0,\infty)^5$, $k \in (0,1)$ and $s \geq 1$, then we can obtain the following results.

Corollary 2.8. *Let A, B, S, T, F, and G be six self-mappings on a b-complete b-metric space (X,d) and the following conditions hold:*

(i) $A(X) \subset TG(X), B(X) \subset SF(X)$;
(ii) $AF = FA, SF = FS, BG = GB, TG = GT$;
(iii) *For all* $x, y \in X$,

$$d(Ax,By) \leq \frac{k}{s^4}\max\left\{d(Ax,SFx),d(By,TGy),d(SFx,TGy),\frac{d(Ax,TGy)+d(By,SFx)}{2}\right\}$$

where $k \in (0,1)$ and $s \geq 1$ is the coefficient of (X,d). If it satisfies one of the following condition, then A, B, S, T, F and G have a unique common fixed point z in X. Moreover, z is also a unique common fixed point of the pairs $\{A, SF\}$ and $\{B, TG\}$.

1) *Either A or SF is continuous, $\{A, SF\}$ is compatible and $\{B, TG\}$ is weak compatible;*
2) *either B or TG is continuous, $\{B, TG\}$ is compatible and $\{A, SF\}$ is weak compatible;*
3) *either SF or TG is surjection, and $\{A, SF\}$ and $\{B, TG\}$ are weak compatible.*

Corollary 2.9. *Let A, B, S, and T are four self mappings on a b-complete b-metric space (X,d), and the following conditions hold:*

(i) $A(X) \subset T(X), B(X) \subset S(X)$;
(ii) *For all* $x, y \in X$,

$$d(Ax,By) \leq \frac{k}{s^4}\max\left\{d(Ax,Sx),d(By,Ty),d(Sx,Ty),\frac{d(Ax,Ty)+d(By,Sx)}{2}\right\}, \tag{2.30}$$

where $k \in (0,1)$ and $s \geq 1$ is the coefficient of (X,d). If it satisfies one of the following condition, then A, B, S, and T have a unique common fixed point z in X. Moreover, z is also a unique common fixed point of the pairs $\{A, S\}$ and $\{B, T\}$.

1) *Either A or S is continuous, $\{A, S\}$ is compatible, $\{B, T\}$ is weak compatible;*
2) *Either B or T is continuous, $\{B, T\}$ is compatible, $\{A, S\}$ is weak compatible;*
3) *Either S or T is surjection, and $\{A, S\}$ and $\{B, T\}$ are weak compatible.*

Remark 2.10. Corollary 2.9 improve and extend the main results in Kang et al. [12] and Roshan et al. [18].

Corollary 2.11. *Let A, B, S, T, F, and G be six self-mappings on a b-complete b-metric space (X,d), and the following conditions hold:*

(i) $A(X) \subset TG(X), B(X) \subset SF(X)$;
(ii) $AF = FA, SF = FS, BG = GB, TG = GT$;
(iii) *For all* $x, y \in X$,

$$d(Ax,By) \leq \frac{1}{s^4}\left(c_1 d(Ax,SFx) + c_2 d(By,TGy) + c_3 d(SFx,TGy) + c_4 d(Ax,TGy) + c_5 d(By,SFx)\right),$$

where $c_1, c_2, c_3, c_4, c_5 \geq 0$ with $c_1 + c_2 + c_3 + 2\max\{c_4, c_5\} < 1$ and $s \geq 1$ is the coefficient of (X,d). If one of the following conditions is satisfied, then the mappings A, B, S, T, F and G have a unique common fixed point z, and z is the unique common fixed point of the pairs $\{A, SF\}$ and $\{B, TG\}$.

1) *Either A or SF is continuous, $\{A, SF\}$ is compatible and $\{B, TG\}$ is weak compatible;*
2) *either B or TG is continuous, $\{B, TG\}$ is compatible and $\{A, SF\}$ is weak compatible;*
3) *either SF or TG is surjection, and $\{A, SF\}$ and $\{B, TG\}$ are weak compatible.*

Proof. It suffices to take $\phi(t_1, t_2, t_3, t_4, t_5) = c_1t_1 + c_2t_2 + c_3t_3 + c_4t_4 + c_5t_5$ in Theorem 2.1. □

In Corollary 2.11, if we take $F = G = I$, we deduce the following result of common fixed point for four self-mappings.

Corollary 2.12. *Let $A, B, S,$ and T be four self-mappings on a b-complete b-metric space (X, d) and the following conditions hold:*

(i) *$A(X) \subset T(X), B(X) \subset S(X)$;*
(ii) *For all $x, y \in X$,*

$$d(Ax, By) \leq \frac{1}{s^4}\left(c_1d(Ax,Sx)+c_2d(By,Ty)+c_3d(Sx,Ty)+c_4d(Ax,Ty)+c_5d(By,Sx)\right), \tag{2.31}$$

where $c_1, c_2, c_3, c_4, c_5 \geq 0$ with $c_1 + c_2 + c_3 + 2\max\{c_4, c_5\} < 1$ and $s \geq 1$ is the coefficient of (X, d). If one of the following conditions is satisfied, then the mappings A, B, S and T have a unique common fixed point z, and z is the unique common fixed point of the pairs $\{A, S\}$ and $\{B, T\}$.

1) *Either A or S is continuous, $\{A, S\}$ is compatible and $\{B, T\}$ is weak compatible;*
2) *either B or T is continuous, $\{B, T\}$ is compatible and $\{A, S\}$ is weak compatible;*
3) *either S or T is surjection, and $\{A, S\}$ and $\{B, T\}$ are weak compatible.*

Remark 2.13. Let $\alpha, \beta \geq 0$ and $\alpha + \beta = 2$, then

$$c_1 + c_2 + c_3 + \alpha c_4 + \beta c_5 \leq c_1 + c_2 + c_3 + (\alpha + \beta)\max\{c_4, c_5\} = c_1 + c_2 + c_3 + 2\max\{c_4, c_5\}.$$

Therefore, if the condition $c_1+c_2+c_3+2\max\{c_4, c_5\} < 1$ is replaced by the condition $c_1+c_2+c_3+\alpha c_4+\beta c_5 < 1$ in Corollary 2.12, then the conclusion of corollary 2.12 is still holds. Hence, Corollary 2.12 improves and extends Theorem 2.7 of Roshan et al. [18] in its three aspects:

(1) the contractive condition is replaced by the new contractive condition defined by (2.31);
(2) by using one continuous function as opposed to two;
(3) the two pairs maps are both compatible decrease to one pair is compatible and another is weak compatible.

Remark 2.14. If we take: (1) $A = B$; (2) $S = T$; (3) $S = T = I$; (4) $A = B$ and $S = T$; (5) $A = B$ and $S = T = I$ in Corollaries 2.4, 2.5, 2.9 and 2.12, then several new results can be obtained, and here we omit them.

Now we introduce some examples to support our new result.

Example 2.15. Let $X = [0, 2]$, and (X, d) be a b- metric space defined by $d(x, y) = (x - y)^2$ for all x, y in X. Suppose that A, B, S and T be four self-mappings defined by

$$Ax = \frac{7}{4} \quad \forall x \in [0, 2]; \qquad Bx = \begin{cases} \frac{9}{4}, & x \in [0, 1], \\ \frac{7}{4}, & x \in (1, 2], \end{cases}$$

$$Sx = \begin{cases} 1, & x \in [0, 1], \\ \frac{7}{4}, & x \in (1, 2), \\ \frac{9}{4}, & x = 2, \end{cases} \qquad Tx = \begin{cases} \frac{1}{8}, & x \in [0, 1], \\ \frac{7}{4}, & x \in (1, 2), \\ 1, & x = 2. \end{cases}$$

Note that A is continuous in X, and B, S and T are not continuous functions in X.

It is easy to show that (X, d) is b-complete b-metric space, $A(X) \subset T(X)$, $B(X) \subset S(X)$ and $s = 2$ is the coefficient of (X, d).

By the definition of the functions of A and S, only for $\{x_n\} \subset (1, 2)$, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t\left(= \frac{7}{4}\right).$$

At this time

$$\lim_{n\to\infty} d(ASx_n, SA_nx) = d\left(\frac{7}{4}, \frac{7}{4}\right) = 0,$$

this implies that the pair $\{A, S\}$ is compatible.

By the definition of the functions of B and T, only for $x \in (1, 2)$, $Bx = Tx = \frac{7}{4}$, at this time

$$BTx = B(\frac{7}{4}) = \frac{7}{4} = T(\frac{7}{4}) = TBx,$$

so $BTx = TBx$, which implies that the pair $\{B, T\}$ is weakly compatible.

Now, we will show that the functions A, B, S and T are satisfying the condition (2.28) of Corollary 2.4 with $k \in [\frac{256}{289}, 1)$ and control function $\phi(t_1, t_2, t_3, t_4, t_5) = k \max\left\{t_1, t_2, t_3, \frac{t_4+t_5}{2}\right\}$. For this purpose, we consider the following five cases:

Case 1. $x, y \in [0, 1]$. In this case, we have

$$d(Ax, By) = d\left(\frac{7}{4}, \frac{9}{4}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

and

$$\begin{aligned}
&\phi\left(d(Ax, Sx), d(By, Ty), d(Sx, Ty), d(Ax, Ty), d(By, Sx)\right) \\
&\quad = \phi\left(d\left(\frac{7}{4}, 1\right), d\left(\frac{9}{4}, \frac{1}{8}\right), d\left(1, \frac{1}{8}\right), d\left(\frac{7}{4}, \frac{1}{8}\right), d\left(\frac{9}{4}, 1\right)\right) \\
&\quad = \phi\left(\left(\frac{3}{4}\right)^2, \left(\frac{17}{8}\right)^2, \left(\frac{7}{8}\right)^2, \left(\frac{13}{8}\right)^2, \left(\frac{5}{4}\right)^2\right) \\
&\quad = k \max\left\{\left(\frac{3}{4}\right)^2, \left(\frac{17}{8}\right)^2, \left(\frac{7}{8}\right)^2, \frac{\left(\frac{13}{8}\right)^2 + \left(\frac{5}{4}\right)^2}{2}\right\} \\
&\quad = k \cdot \frac{289}{64}.
\end{aligned}$$

Therefore, we give that

$$\begin{aligned}
d(Ax, By) = \frac{1}{4} = \frac{1}{2^4} \cdot \frac{256}{289} \cdot \frac{289}{64} &\leq \frac{1}{2^4} \cdot k \cdot \frac{289}{64} \\
&= \frac{1}{s^4}\phi\left(d(Ax, Sx), d(By, Ty), d(Sx, Ty), d(Ax, Ty), d(By, Sx)\right).
\end{aligned}$$

Case 2. $x \in [0, 1], y \in (1, 2]$. Obviously, we have

$$d(Ax, By) = d\left(\frac{7}{4}, \frac{7}{4}\right) = 0 \leq \frac{1}{s^4}\phi\left(d(Ax, Sx), d(By, Ty), d(Sx, Ty), d(Ax, Ty), d(By, Sx)\right).$$

Case 3. $x \in (1, 2), y \in [0, 1]$. In this case, we obtain

$$d(Ax, By) = d\left(\frac{7}{4}, \frac{9}{4}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

and

$$\begin{aligned}
&\phi\left(d(Ax,Sx),d(By,Ty),d(Sx,Ty),d(Ax,Ty),d(By,Sx)\right)\\
&=\phi\left(d\left(\frac{7}{4},\frac{7}{4}\right),d\left(\frac{9}{4},\frac{1}{8}\right),d\left(\frac{7}{4},\frac{1}{8}\right),d\left(\frac{7}{4},\frac{1}{8}\right),d\left(\frac{9}{4},\frac{7}{4}\right)\right)\\
&=\phi\left(0^2,\left(\frac{17}{8}\right)^2,\left(\frac{13}{8}\right)^2,\left(\frac{13}{8}\right)^2,\left(\frac{1}{2}\right)^2\right)\\
&=k\max\left\{0^2,\left(\frac{17}{8}\right)^2,\left(\frac{13}{8}\right)^2,\frac{\left(\frac{13}{8}\right)^2+\left(\frac{1}{2}\right)^2}{2}\right\}\\
&=k\cdot\frac{289}{64}.
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
d(Ax,By)=\frac{1}{4}=\frac{1}{2^4}\cdot\frac{256}{289}\cdot\frac{289}{64}&\le\frac{1}{2^4}\cdot k\cdot\frac{289}{64}\\
&=\frac{1}{s^4}\phi\left(d(Ax,Sx),d(By,Ty),d(Sx,Ty),d(Ax,Ty),d(By,Sx)\right).
\end{aligned}$$

Case 4. $x=2, y\in[0,1]$. In this case, we have

$$d(Ax,By)=d\left(\frac{7}{4},\frac{9}{4}\right)=\frac{1}{4}$$

and

$$\begin{aligned}
&\phi\left(d(Ax,Sx),d(By,Ty),d(Sx,Ty),d(Ax,Ty),d(By,Sx)\right)\\
&=\phi\left(d\left(\frac{7}{4},\frac{9}{4}\right),d\left(\frac{9}{4},\frac{1}{8}\right),d\left(\frac{9}{4},\frac{1}{8}\right),d\left(\frac{7}{4},\frac{1}{8}\right),d\left(\frac{9}{4},\frac{9}{4}\right)\right)\\
&=\phi\left(\left(\frac{1}{2}\right)^2,\left(\frac{17}{8}\right)^2,\left(\frac{17}{8}\right)^2,\left(\frac{13}{8}\right)^2,0^2\right)\\
&=k\max\left\{\left(\frac{1}{2}\right)^2,\left(\frac{17}{8}\right)^2,\left(\frac{17}{8}\right)^2,\frac{\left(\frac{13}{8}\right)^2+0}{2}\right\}\\
&=k\cdot\frac{289}{64}.
\end{aligned}$$

Thus, we get that

$$\begin{aligned}
d(Ax,By)=\frac{1}{4}=\frac{1}{2^4}\cdot\frac{256}{289}\cdot\frac{289}{64}&\le\frac{1}{2^4}\cdot k\cdot\frac{289}{64}\\
&=\frac{1}{s^4}\phi\left(d(Ax,Sx),d(By,Ty),d(Sx,Ty),d(Ax,Ty),d(By,Sx)\right).
\end{aligned}$$

Case 5. $x,y\in(1,2]$. Clearly, we have

$$d(Ax,By)=d\left(\frac{7}{4},\frac{7}{4}\right)=0\le\frac{1}{s^4}\phi\left(d(Ax,Sx),d(By,Ty),d(Sx,Ty),d(Ax,Ty),d(By,Sx)\right).$$

Then in all the above cases, the mappings A, B, S and T are satisfying the condition (2.28) of the Corollary 2.4 with $k\in[\frac{256}{289},1)$ and $\phi(t_1,t_2,t_3,t_4,t_5)=k\max\left\{t_1,t_2,t_3,\frac{t_4+t_5}{2}\right\}$. So that all the conditions of Corollary 2.4 are satisfied. Clearly, $\frac{7}{4}$ is the unique common fixed point for all of the mappings A, B, S and T.

Example 2.16. Let $X = [0, +\infty)$ and (X, d) be $b-$metric space on X given by $d(x, y) = (x - y)^2$ for all $x, y \in X$. Define self-maps A, B, S, and T on X by

$$Ax = \ln\left(1 + \frac{x}{2}\right),\ Bx = \ln\left(1 + \frac{x}{4}\right),\ Sx = e^{4x} - 1,\ Tx = e^{2x} - 1,\ \forall x \in X.$$

Obviously, (X, d) is b-complete b-metric space with the coefficient $s = 2$, and

$$A(X) = B(X) = S(X) = T(X) = [0, +\infty).$$

Since

$$(Sx - Ax)^2 = \left((e^{4x} - 1) - \ln\left(1 + \frac{x}{2}\right)\right)^2 = 0 \ \Leftrightarrow\ x = 0,$$

then for all $\{x_n\} \subset X$ satisfying $x_n \to 0$, we have $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n (= 0)$. At this time, we have

$$\lim_{n\to\infty} d(ASx_n, ASx_n) = 0.$$

Otherwise, $\lim_{n\to\infty} Ax_n \neq \lim_{n\to\infty} Sx_n$. Therefore, the pair $\{A, S\}$ is compatible.

By the definition of the functions of B and T, only for $x = 0$, we get $Bx = Tx(= 0)$. At this time $BTx = TBx(= 0)$. Otherwise, $B(x) \neq T(x)$. Hence, the pair $\{B, T\}$ is weak compatible.

Next we show that the maps A, B, S, and T are satisfying the condition (2.30) of Corollary 2.9 with $k = \frac{1}{4}$. In fact

$$\begin{aligned}
d(Ax, By) = (Ax - By)^2 &= \left\{\ln\left(1 + \frac{x}{2}\right) - \ln\left(1 + \frac{y}{4}\right)\right\}^2 \\
&\leq \left(\frac{x}{2} - \frac{y}{4}\right)^2 = \frac{1}{64}(4x - 2y)^2 \leq \frac{1}{64}(e^{4x} - e^{2y})^2 \\
&\leq \frac{1}{64}\left\{(e^{4x} - 1) - (e^{2y} - 1)\right\}^2 \\
&= \frac{1}{64}(Sx - Ty)^2 = \frac{1}{2^4} \cdot \frac{1}{4} d(Sx, Ty) \\
&\leq \frac{1}{2^4} \cdot \frac{1}{4} \max\left\{d(Ax, Sx), d(By, Ty), d(Sx, Ty), \frac{d(Ax, Ty) + d(By, Sx)}{2}\right\}.
\end{aligned}$$

Therefore, in all the above cases, the mappings A, B, S, and T are satisfying all the conditions of the Corollary 2.9. Obviously, 0 is the unique common fixed point for all of the mappings A, B, S, and T.

Example 2.17. Let $X = [0, 1]$, and (X, d) be a b-metric space defined by $d(x, y) = (x - y)^2$ for all x, y in X. Suppose that A, B, S, and T be four self-mappings defined by

$$Ax = \begin{cases} 1, & x \in [0, \frac{1}{2}], \\ \frac{15}{16}, & x \in (\frac{1}{2}, 1], \end{cases} \qquad Bx = \begin{cases} \frac{14}{15}, & x \in [0, \frac{1}{2}], \\ \frac{15}{16}, & x \in (\frac{1}{2}, 1], \end{cases}$$

$$Sx = x; \qquad Tx = \begin{cases} 1, & x \in [0, \frac{1}{4}), \\ \frac{1}{5}, & x \in [\frac{1}{4}, \frac{1}{2}] \\ \frac{15}{16}, & x \in (\frac{1}{2}, 1]. \end{cases}$$

We know that S is continuous in X, and A, B and T are not continuous mappings in X.

It is easy to see that (X, d) is b-complete b-metric space, $s = 2$ is the coefficient of (X, d), and $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

By the definition of the mappings of A and S, only for $\{x_n\} \subset (\frac{1}{2}, 1]$, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t\left(= \frac{15}{16}\right).$$

At this time

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = \lim_{n\to\infty} d\left(Ax_n, S\left(\frac{15}{16}\right)\right) = d\left(\frac{15}{16}, \frac{15}{16}\right) = 0,$$

so we can get the pair $\{A, S\}$ is compatible.

By the definition of the mappings of B and T, only for $x \in (\frac{1}{2}, 1]$, $Bx = Tx = \frac{15}{16}$, at this time $BTx = B(\frac{15}{16}) = \frac{15}{16} = T(\frac{15}{16}) = TBx$, so $BTx = TBx$, thus we can obtain the pair $\{B, T\}$ is weakly compatible.

Now we prove that the mappings A, B, S and T are satisfying the condition (2.31) of Corollary 2.12 with $c_1 = c_2 = c_3 = \frac{1}{6}, c_4 = c_5 = \frac{1}{10}$. So we consider the following cases:

Case 1. For all $x, y \in [0, \frac{1}{2}]$, we show that

$$d(Ax, By) = d\left(1, \frac{14}{15}\right) = \left(\frac{1}{15}\right)^2 = \frac{1}{225},$$

then, we divide the study in two subcases.

(i) If $y \in [0, \frac{1}{4})$, thus we have

$$\begin{aligned}
&c_1 d(Ax, Sx) + c_2 d(By, Ty) + c_3 d(Sx, Ty) + c_4 d(Ax, Ty) + c_5 d(By, Sx)\\
&= \frac{1}{6} \cdot d(1, x) + \frac{1}{6} \cdot d\left(\frac{14}{15}, 1\right) + \frac{1}{6} \cdot d(x, 1) + \frac{1}{10} \cdot d(1, 1) + \frac{1}{10} \cdot d\left(\frac{14}{15}, x\right)\\
&= \frac{1}{6} \cdot (1-x)^2 + \frac{1}{6} \cdot \left(\frac{14}{15} - 1\right)^2 + \frac{1}{6} \cdot (x-1)^2 + \frac{1}{10} \cdot 0^2 + \frac{1}{10} \cdot \left(\frac{14}{15} - x\right)^2\\
&= \frac{1}{3} \cdot (1-x)^2 + \frac{1}{6} \cdot \left(\frac{1}{15}\right)^2 + \frac{1}{10} \cdot \left(\frac{14}{15} - x\right)^2\\
&= \frac{13}{30}x^2 - \frac{64}{75}x + \frac{2843}{6750}\\
&\geq \frac{13}{30} \cdot \left(\frac{1}{2}\right)^2 - \frac{64}{75} \cdot \left(\frac{1}{2}\right) + \frac{2843}{6750}\\
&= \frac{2777}{27000}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
d(Ax, By) = \frac{1}{225} = \frac{1}{2^4} \cdot \frac{16}{225} = \frac{1}{2^4} \cdot \frac{1920}{27000} &< \frac{1}{2^4} \cdot \frac{2777}{27000}\\
&\leq \frac{1}{s^4}[c_1 d(Ax, Sx) + c_2 d(By, Ty) + c_3 d(Sx, Ty) + c_4 d(Ax, Ty) + c_5 d(By, Sx)].
\end{aligned}$$

(ii) If $y \in [\frac{1}{4}, \frac{1}{2}]$, then we obtain

$$\begin{aligned}
&c_1 d(Ax, Sx) + c_2 d(By, Ty) + c_3 d(Sx, Ty) + c_4 d(Ax, Ty) + c_5 d(By, Sx)\\
&= \frac{1}{6} \cdot d(1, x) + \frac{1}{6} \cdot d\left(\frac{14}{15}, \frac{1}{5}\right) + \frac{1}{6} \cdot d\left(x, \frac{1}{5}\right) + \frac{1}{10} \cdot d\left(1, \frac{1}{5}\right) + \frac{1}{10} \cdot d\left(\frac{14}{15}, x\right)\\
&= \frac{1}{6} \cdot (1-x)^2 + \frac{1}{6} \cdot \left(\frac{14}{15} - \frac{1}{5}\right)^2 + \frac{1}{6} \cdot \left(x - \frac{1}{5}\right)^2 + \frac{1}{10} \cdot \left(1 - \frac{1}{5}\right)^2 + \frac{1}{10} \cdot \left(\frac{14}{15} - x\right)^2\\
&= \frac{13}{30} \cdot x^2 - \frac{44}{75} \cdot x + \frac{559}{1350}\\
&\geq \frac{13}{30} \cdot \left(\frac{1}{2}\right)^2 - \frac{44}{75} \cdot \left(\frac{1}{2}\right) + \frac{559}{1350}\\
&= \frac{1237}{5400}.
\end{aligned}$$

Hence we have

$$d(Ax,By)=\frac{1}{225}=\frac{1}{2^4}\cdot\frac{16}{225}=\frac{1}{2^4}\cdot\frac{384}{5400}<\frac{1}{2^4}\cdot\frac{1237}{5400}$$
$$=\frac{1}{s^4}[c_1d(Ax,Sx)+c_2d(By,Ty)+c_3d(Sx,Ty)+c_4d(Ax,Ty)+c_5d(By,Sx)].$$

Case 2. For all $x\in[0,\frac{1}{2}], y\in(\frac{1}{2},1]$, we get

$$d(Ax,By)=d\left(1,\frac{15}{16}\right)^2=\left(\frac{1}{16}\right)^2=\frac{1}{256},$$

and

$$\begin{aligned}
&c_1d(Ax,Sx)+c_2d(By,Ty)+c_3d(Sx,Ty)+c_4d(Ax,Ty)+c_5d(By,Sx)\\
&=\frac{1}{6}\cdot d\left(1,x\right)+\frac{1}{6}\cdot d\left(\frac{15}{16},\frac{15}{16}\right)+\frac{1}{6}\cdot d\left(x,\frac{15}{16}\right)+\frac{1}{10}\cdot d\left(1,\frac{15}{16}\right)+\frac{1}{10}\cdot d\left(\frac{15}{16},x\right)\\
&=\frac{1}{6}\cdot(1-x)^2+\frac{1}{6}\cdot 0^2+\frac{1}{6}\cdot\left(x-\frac{15}{16}\right)^2+\frac{1}{10}\cdot\left(1-\frac{15}{16}\right)^2+\frac{1}{10}\cdot\left(\frac{15}{16}-x\right)^2\\
&=\frac{13}{30}\cdot x^2-\frac{5}{6}\cdot x+\frac{3083}{7680}\\
&\geq\frac{13}{30}\cdot\left(\frac{1}{2}\right)^2-\frac{5}{6}\cdot\left(\frac{1}{2}\right)+\frac{3083}{7680}\\
&=\frac{143}{1536}.
\end{aligned}$$

Hence, we deduce that

$$d(Ax,By)=\frac{1}{256}=\frac{1}{2^4}\cdot\frac{16}{256}=\frac{1}{2^4}\cdot\frac{96}{1536}<\frac{1}{2^4}\cdot\frac{143}{1536}$$
$$\leq\frac{1}{s^4}[c_1d(Ax,Sx)+c_2d(By,Ty)+c_3d(Sx,Ty)+c_4d(Ax,Ty)+c_5d(By,Sx)].$$

Case 3. For all $x\in(\frac{1}{2},1], y\in[0,\frac{1}{2}]$, we show that

$$d(Ax,By)=d\left(\frac{15}{16},\frac{14}{15}\right)=\left(\frac{1}{240}\right)^2=\frac{1}{57600}.$$

Next, we divide the study in two subcases.

(i) If $y\in[0,\frac{1}{4})$, then we have

$$\begin{aligned}
&c_1d(Ax,Sx)+c_2d(By,Ty)+c_3d(Sx,Ty)+c_4d(Ax,Ty)+c_5d(By,Sx)\\
&=\frac{1}{6}\cdot d\left(\frac{15}{16},x\right)+\frac{1}{6}\cdot d\left(\frac{14}{15},1\right)+\frac{1}{6}\cdot d\left(x,1\right)+\frac{1}{10}\cdot d\left(\frac{15}{16},1\right)+\frac{1}{10}\cdot d\left(\frac{14}{15},x\right)\\
&=\frac{1}{6}\cdot\left(\frac{15}{16}-x\right)^2+\frac{1}{6}\cdot\left(\frac{14}{15}-1\right)^2+\frac{1}{6}\cdot(x-1)^2+\frac{1}{10}\cdot\left(\frac{15}{16}-1\right)^2+\frac{1}{10}\cdot\left(\frac{14}{15}-x\right)^2\\
&=\frac{13}{30}\cdot x^2-\frac{333}{400}\cdot x+\frac{86701}{216000}\\
&\geq\frac{4\cdot\frac{13}{30}\cdot\frac{86701}{216000}-\left(\frac{333}{400}\right)^2}{4\cdot\frac{13}{30}}\\
&=\frac{6979}{4492800}.
\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}d(Ax,By) &= \frac{1}{57600} = \frac{1}{2^4}\cdot\frac{16}{57600} = \frac{1}{2^4}\cdot\frac{1248}{4492800} < \frac{1}{2^4}\cdot\frac{6979}{4492800}\\ &\le \frac{1}{s^4}[c_1d(Ax,Sx)+c_2d(By,Ty)+c_3d(Sx,Ty)+c_4d(Ax,Ty)+c_5d(By,Sx)].\end{aligned}$$

(ii) If $y\in[\frac{1}{4},\frac{1}{2}]$, then we have

$$\begin{aligned}&c_1d(Ax,Sx)+c_2d(By,Ty)+c_3d(Sx,Ty)+c_4d(Ax,Ty)+c_5d(By,Sx)\\ &= \frac{1}{6}\cdot d\left(\frac{15}{16},x\right)+\frac{1}{6}\cdot d\left(\frac{14}{15},\frac{1}{5}\right)+\frac{1}{6}\cdot d\left(x,\frac{1}{5}\right)+\frac{1}{10}\cdot d\left(\frac{15}{16},\frac{1}{5}\right)+\frac{1}{10}\cdot d\left(\frac{14}{15},x\right)\\ &= \frac{1}{6}\cdot\left(\frac{15}{16}-x\right)^2+\frac{1}{6}\cdot\left(\frac{14}{15}-\frac{1}{5}\right)^2+\frac{1}{6}\cdot\left(x-\frac{1}{5}\right)^2+\frac{1}{10}\cdot\left(\frac{15}{16}-\frac{1}{5}\right)^2+\frac{1}{10}\cdot\left(\frac{14}{15}-x\right)^2\\ &= \frac{13}{30}\cdot x^2-\frac{679}{1200}\cdot x+\frac{16601}{43200}\\ &\ge \frac{4\cdot\frac{13}{30}\cdot\frac{16601}{43200}-\left(\frac{679}{1200}\right)^2}{4\cdot\frac{13}{30}}\\ &= \frac{4483151}{22464000}.\end{aligned}$$

Hence we have

$$\begin{aligned}d(Ax,By) &= \frac{1}{57600} = \frac{1}{2^4}\cdot\frac{16}{57600} = \frac{1}{2^4}\cdot\frac{6240}{22464000} < \frac{1}{2^4}\cdot\frac{4483151}{22464000}\\ &\le \frac{1}{s^4}[c_1d(Ax,Sx)+c_2d(By,Ty)+c_3d(Sx,Ty)+c_4d(Ax,Ty)+c_5d(By,Sx)].\end{aligned}$$

Case 4. For all $x,y\in(\frac{1}{2},1]$, we show that

$$\begin{aligned}d(Ax,By) &= d\left(\frac{15}{16},\frac{15}{16}\right)=0\\ &\le \frac{1}{s^4}[c_1d(Ax,Sx)+c_2d(By,Ty)+c_3d(Sx,Ty)+c_4d(Ax,Ty)+c_5d(By,Sx)].\end{aligned}$$

Then in all the above cases, the mappings A, B, S and T are satisfying the condition (2.31) of Corollary 2.12 with $\phi(t_1,t_2,t_3,t_4,t_5)=\frac{1}{6}t_1+\frac{1}{6}t_2+\frac{1}{6}t_3+\frac{1}{10}t_4+\frac{1}{10}t_5$. So that all the conditions of Corollary 2.12 are satisfied. Obviously, $\frac{15}{16}$ is the unique common fixed point for all of the mappings A, B, S and T.

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