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Positive solutions for some Riemann-Liouville fractional boundary value problems

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Abstract

We study the existence and global asymptotic behavior of positive continuous solutions to the following nonlinear fractional boundary value problem

$$(P_\lambda)\ \begin{cases} D^\alpha u(t) = \lambda f(t,u(t)),\ t\in(0,1), \\ \lim\limits_{t\to 0^+} t^{2-\alpha}u(t)=\mu,\ u(1)=\nu, \end{cases}$$

where $1<\alpha\le 2$, D^α is the Riemann-Liouville fractional derivative, and λ,μ and ν are nonnegative constants such that $\mu+\nu>0$.

Our purpose is to give two existence results for the above problem, where $f(t,s)$ is a nonnegative continuous function on $(0,1)\times[0,\infty)$, nondecreasing with respect to the second variable and satisfying some appropriate integrability condition. Some examples are given to illustrate our existence results.

Keywords: Fractional differential equation, positive solutions, Green's function, perturbation arguments, Schäuder fixed point theorem.
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1. Introduction

We aim at proving two existence results of positive continuous solutions to fractional boundary value

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problems of the form

$$(P_\lambda)\left\{\begin{array}{l} D^\alpha u(t)=\lambda f(t,u(t)),\ t\in(0,1),\\ \lim\limits_{t\to0^+}t^{2-\alpha}u(t)=\mu,\ u(1)=\nu,\end{array}\right.$$

where $1<\alpha\le 2$, λ,μ and ν are nonnegative constants such that $\mu+\nu>0$. Here D^α is the Riemann-Liouville fractional derivative of order α defined by (see [16, 25, 26]),

$$D^\alpha u(t)=\left\{\begin{array}{ll} \frac{1}{\Gamma(2-\alpha)}\left(\frac{d}{dt}\right)^2\int_0^t(t-s)^{1-\alpha}u(s)\,ds, & \text{if } 1<\alpha<2,\\ u''(t), & \text{if } \alpha=2.\end{array}\right.$$

The function $f(t,s)$ is required to be nonnegative continuous function on $(0,1)\times[0,\infty)$, nondecreasing with respect to the second variable and satisfying some appropriate integrability condition.

It is known that fractional differential equations appear in various fields of science and engineering (see for example [7, 8, 10, 13, 16, 19, 21, 25–29] and references therein). Many researchers have considered various forms of fractional differential equations subject to different boundary conditions (see for instance [1–6, 9, 11, 12, 14, 15, 17, 18, 20, 22–24, 30] and the references therein).

Mâagli et al [18] by exploiting Karamata regular variation theory, proved the existence and uniqueness of a positive solution to the following sublinear singular fractional boundary value problem

$$\left\{\begin{array}{l} D^\alpha u(t)=-p(t)u^\sigma(t),\ t\in(0,1),\\ \lim\limits_{t\to0^+}t^{2-\alpha}u(t)=0,\ u(1)=0,\end{array}\right.$$

where $\sigma\in(-1,1)$ and p is a nonnegative continuous function satisfying some sharp estimates.

In the first part of this paper, we study the superlinear fractional boundary value problem

$$\left\{\begin{array}{l} D^\alpha u(t)=u(t)\varphi(t,u(t)),\ t\in(0,1),\ 1<\alpha\le2,\\ \lim\limits_{t\to0^+}t^{2-\alpha}u(t)=\mu,\ u(1)=\nu,\end{array}\right. \tag{1.1}$$

where μ,ν are nonnegative constants such that $\mu+\nu>0$ and $\varphi(t,s)$ is a nonnegative continuous function in $(0,1)\times[0,\infty)$ satisfying some adequate conditions. Note that the condition $\mu+\nu>0$ is essential to obtain positive solution. To simplify our statements, we denote by

(i) $\mathcal{B}^+((0,1))$ the set of nonnegative measurable functions on $(0,1)$.

(ii) $C(X)$ (resp. $C^+(X)$) the set of continuous (resp. nonnegative continuous) functions on a metric space X.

(iii) $C_{2-\alpha}([0,1])$, $(1<\alpha\le2)$ the set of all functions g such that $s\to s^{2-\alpha}g(s)$ is continuous on $[0,1]$.

Definition 1.1. Let $1<\alpha\le2$. We consider

$$\mathcal{K}_\alpha=\left\{q\in B^+((0,1)):\int_0^1 r^{\alpha-1}(1-r)^{\alpha-1}q(r)dr<\infty\right\}.$$

Throughout this paper, for $\alpha\in(1,2]$ and $t\in(0,1]$, we let

$$h_1(t):=t^{\alpha-2}(1-t),\ h_2(t):=t^{\alpha-1},$$

and $h_0(t):=\mu h_1(t)+\nu h_2(t)$, be the unique solution of the problem

$$(P_0)\left\{\begin{array}{l} D^\alpha u(t)=0,\ t\in(0,1),\\ \lim\limits_{t\to0^+}t^{2-\alpha}u(t)=\mu,\ u(1)=\nu.\end{array}\right.$$

Let $G(t,s)$ be the Green's function of the operator $u \to D^{\alpha}u$, with boundary conditions $\lim\limits_{t\to 0^+} t^{2-\alpha}u(t) = u(1) = 0$. From [18, Lemma 8], we have

$$\begin{aligned} G(t,s) &= \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & \text{if } 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & \text{if } 0 \le t \le s \le 1, \end{cases} \\ &= \frac{1}{\Gamma(\alpha)} \left(t^{\alpha-1}(1-s)^{\alpha-1} - \left((t-s)^+\right)^{\alpha-1} \right), \end{aligned} \tag{1.2}$$

where $t^+ = \max(t,0)$. For $q \in B^+((0,1))$, we put

$$\alpha_q := \sup_{t,s\in(0,1)} \int_0^1 \frac{G(t,r)G(r,s)}{G(t,s)} q(r)dr, \tag{1.3}$$

and we will prove that if $q \in \mathcal{K}_\alpha$, then $\alpha_q < \infty$.

Next, we require a combination of the following assumptions.

(H$_1$) $\varphi \in C^+((0,1)\times[0,\infty))$.

(H$_2$) There exists a function $q \in \mathcal{K}_\alpha \cap C^+((0,1))$ with $\alpha_q \le \frac{1}{2}$ such that, for all $t \in (0,1)$, the function $s \longrightarrow s\left(q(t) - \varphi(t, sh_0(t))\right)$ is nondecreasing on $[0,1]$.

(H$_3$) For all $t \in (0,1)$, the function $s \to s\varphi(t,s)$ is nondecreasing on $[0,\infty)$.

Our approach is as follows: For a given function $q \in \mathcal{K}_\alpha \cap C^+((0,1))$ with $\alpha_q \le \frac{1}{2}$, we will first prove that the operator $u \to D^\alpha u - q(t)u$, with boundary conditions $\lim\limits_{t\to 0^+} t^{2-\alpha}u(t) = u(1) = 0$ has a positive Green function $\mathcal{G}(t,s)$.

By exploiting properties of $\mathcal{G}(t,s)$ and using a perturbation argument, we prove the following result.

Theorem 1.2. *Assume that hypotheses* (H$_1$)-(H$_2$) *are satisfied. Then problem* (1.1) *has a positive solution* u *in* $C_{2-\alpha}([0,1])$ *satisfying for all* $t \in (0,1]$,

$$mh_0(t) \le u(t) \le h_0(t), \tag{1.4}$$

where $m \in (0,1]$. *Moreover, if hypothesis* (H_3) *is also satisfied, then this solution is unique.*

Corollary 1.3. *Let* $g : [0,\infty) \to [0,\infty)$ *be a* C^1*-function such that the map* $s \to \theta(s) = sg(s)$ *is nondecreasing on* $[0,\infty)$. *Let* $p \in C^+((0,1))$ *such that the function* $t \to \widetilde{p}(t) := p(t) \max\limits_{0\le\xi\le h_0(t)} \theta'(\xi) \in \mathcal{K}_\alpha$. *Then for* $\lambda \in [0, \frac{1}{2\alpha_{\widetilde{p}}})$, *the following problem*

$$\begin{cases} D^\alpha u(t) = \lambda p(t)u(t)g(u(t)), \quad t \in (0,1),\ 1 < \alpha \le 2, \\ \lim\limits_{t\to 0^+} t^{2-\alpha}u(t) = \mu,\ u(1) = \nu, \end{cases}$$

has a unique positive solution u *in* $C_{2-\alpha}([0,1])$ *satisfying for all* $t \in (0,1]$,

$$(1 - \lambda\alpha_{\widetilde{p}})h_0(t) \le u(t) \le h_0(t).$$

As typical example of nonlinearity satisfying (H$_1$)-(H$_3$), we quote $\varphi(t,s) = \lambda p(t)s^\sigma$ for $\sigma \ge 0$, $p \in C^+((0,1))$ such that

$$\int_0^1 s^{(\alpha-1)+(\alpha-2)\sigma}(1-s)^{\alpha-1}p(s)ds < \infty,$$

and $q(t) = \lambda\widetilde{p}(t) := \lambda(\sigma+1)p(t)\left(h_0(t)\right)^\sigma \in \mathcal{K}_\alpha$, with $\lambda \in [0, \frac{1}{2\alpha_{\widetilde{p}}})$.

In the second part of this paper, we study the fractional boundary value problem

$$\begin{cases} D^\alpha u(t) = \lambda f(t,u(t)),\ t \in (0,1),\ 1 < \alpha \le 2, \\ \lim\limits_{t\to 0^+} t^{2-\alpha}u(t) = \mu,\ u(1) = \nu, \end{cases} \tag{1.5}$$

where $\lambda \ge 0, \mu, \nu$ are positive constants and $f(t,s)$ satisfies the following conditions:

($\mathbf{H}_4$) $(t,s) \to f(t,s) \in C^{+}((0,1)\times[0,\infty))$ which is nondecreasing with respect to the second variable.

($\mathbf{H}_5$) The function $t \to \frac{1}{h_0(t)} f(t, h_0(t))$ belongs to the class $\mathcal{K}_\alpha$.

Using the Schäuder fixed point theorem, we prove the following result.

Theorem 1.4. *Assume that hypotheses* (H_4)-(H_5) *are satisfied. Then there exists a constant $\lambda_0 > 0$, such that for each $\lambda \in [0, \lambda_0)$, problem* (1.5) *has a positive solution u in $C_{2-\alpha}([0,1])$ satisfying*

$$(1-\frac{\lambda}{\lambda_0})h_0(t) \le u(t) \le h_0(t), \quad \textit{for all } t \in (0,1].$$

Our paper is organized as follows. In Section 2, we prove that for all $t, r, s \in (0,1)$,

$$\frac{G(t,r)G(r,s)}{G(t,s)} \le \frac{1}{(\alpha-1)\Gamma(\alpha)} r^{\alpha-1}(1-r)^{\alpha-1}.$$

This implies that for each $q \in \mathcal{K}_\alpha$, $\alpha_q < \infty$. In Section 3, for a given function $q \in \mathcal{K}_\alpha$ with $\alpha_q \le \frac{1}{2}$, we construct the Green's function $\mathcal{G}(t,s)$ of the operator $u \to D^\alpha u - q(t)u$, with boundary conditions $\lim\limits_{t\to 0^+} t^{2-\alpha}u(t) = u(1) = 0$ and we establish some of its properties including the following:

$$(1-\alpha_q)\, G(t,s) \le \mathcal{G}(t,s) \le G(t,s), \text{ for all } (t,s) \in [0,1]\times[0,1].$$

Also we establish the following resolvent equation

$$V\psi = V_q\psi + V_q(qV\psi) = V_q\psi + V(qV_q\psi), \text{ for all } \psi \in \mathcal{B}^{+}((0,1)),$$

where V and V_q are defined on $\mathcal{B}^{+}((0,1))$ by

$$V\psi(t) := \int_0^1 G(t,s)\,\psi(s)ds \text{ and } V_q\psi(t) := \int_0^1 \mathcal{G}(t,s)\,\psi(s)ds, \; t \in [0,1].$$

Using a perturbation argument, we establish Theorem 1.2. In Section 4, we prove Theorem 1.4 by means of the Schäuder fixed point theorem.

2. Estimates on the Green function

The following properties on $G(t,s)$ given by (1.2) are established in [18].

Proposition 2.1. *Let $1 < \alpha \le 2$ and $\psi \in B^{+}((0,1))$. On $(0,1)\times(0,1)$, one has*

(i)
$$(\alpha-1)\, H(t,s) \le \Gamma(\alpha)\, G(t,s) \le H(t,s),$$

where $H(t,s) := t^{\alpha-2}(1-s)^{\alpha-2}(t\wedge s)(1-t\vee s)$ *with* $t\wedge s = \min(t,s)$ *and* $t \vee s = \max(t,s)$.

(ii) $(\alpha-1)\, t^{\alpha-1}(1-t)s(1-s)^{\alpha-1} \le \Gamma(\alpha)\, G(t,s) \le t^{\alpha-2}s(1-s)^{\alpha-1}$.

(iii) $G(t,s) = G(1-s, 1-t)$.

The next proposition is also established in [18].

Proposition 2.2. *Let $1 < \alpha \le 2$ and $\psi \in B^{+}((0,1))$, then*

(i) *The function* $t \to V\psi(t) \in C_{2-\alpha}([0,1]) \iff \int_0^1 r(1-r)^{\alpha-1}\psi(r)dr < \infty$.

(ii) *If the function* $s \to s(1-s)^{\alpha-1}\psi(s)$ *is continuous and integrable on* $(0,1)$, *then* $V\psi$ *is the unique solution in* $C_{2-\alpha}([0,1])$ *of the following problem*

$$\begin{cases} D^{\alpha}u(t) = -\psi(t), & t \in (0,1), \\ \lim\limits_{t\to 0^+} t^{2-\alpha}u(t) = 0, \ u(1) = 0. \end{cases}$$

Proposition 2.3. *For each* $t, r, s \in (0,1)$, *we have*

$$\frac{G(t,r)G(r,s)}{G(t,s)} \leq \frac{1}{(\alpha-1)\Gamma(\alpha)} r^{\alpha-1}(1-r)^{\alpha-1}. \tag{2.1}$$

Proof. Using Proposition 2.1 (i), for each $t, r, s \in (0,1)$, we have

$$\frac{G(t,r)G(r,s)}{G(t,s)} \leq \frac{1}{(\alpha-1)\Gamma(\alpha)} r^{\alpha-2}(1-r)^{\alpha-2}F(t,r,s),$$

where

$$F(t,r,s) := \frac{(t\wedge r)(1-t\vee r)(r\wedge s)(1-r\vee s)}{(t\wedge s)(1-t\vee s)}.$$

To prove (2.1), it is enough to show that

$$F(t,r,s) \leq r(1-r).$$

By symmetry, we may assume that $t \leq s$. Then we obtain

$$\begin{aligned} F(t,r,s) &= \frac{(t\wedge r)(1-t\vee r)(r\wedge s)(1-r\vee s)}{t(1-s)} \\ &\leq (r\wedge s)(1-t\vee r) \\ &\leq r(1-r). \end{aligned}$$

This proves our result. □

Proposition 2.4. *Let* q *be a function in* $\mathcal{K}_\alpha$, *then*

(i)

$$\alpha_q \leq \frac{1}{(\alpha-1)\Gamma(\alpha)} \int_0^1 r^{\alpha-1}(1-r)^{\alpha-1}q(r)dr < \infty, \tag{2.2}$$

where α_q *is given by* (1.3).

(ii) *On* $(0,1]$, *one has*

$$\int_0^1 G(t,s)h_1(s)q(s)ds \leq \alpha_q h_1(t). \tag{2.3}$$

(iii) *On* $(0,1]$, *one has*

$$\int_0^1 G(t,s)h_2(s)q(s)ds \leq \alpha_q h_2(t). \tag{2.4}$$

In particular, for all $t \in (0,1]$, *we have*

$$\int_0^1 G(t,s)h_0(s)q(s)ds \leq \alpha_q h_0(t). \tag{2.5}$$

Proof. Let $q \in \mathcal{K}_\alpha$.

(i) The inequality in (2.2) follows from (1.3) and (2.1).

(ii) Since for each $t, s \in (0,1)$, we have $\lim_{r\to 0} \frac{G(s,r)}{G(t,r)} = \frac{h_1(s)}{h_1(t)}$, then we deduce by Fatou's lemma and (1.3), that
$$\int_0^1 G(t,s)\frac{h_1(s)}{h_1(t)}q(s)ds \le \liminf_{r\to 0}\int_0^1 G(t,s)\frac{G(s,r)}{G(t,r)}q(s)ds \le \alpha_q.$$
This gives
$$\int_0^1 G(t,s)h_1(s)q(s)ds \le \alpha_q h_1(t), \quad \text{for } t \in (0,1].$$

(iii) Since $\lim_{r\to 1} \frac{G(s,r)}{G(t,r)} = \frac{h_2(s)}{h_2(t)}$, inequality (2.4) follows by similar arguments.

Finally, by combining (2.3), (2.4) we obtain (2.5). □

3. First existence result

Let $q \in \mathcal{K}_\alpha$ and $\mathcal{G} : [0,1]\times[0,1] \to \mathbb{R}$, be defined by
$$\mathcal{G}\,(t,s) = \sum_{k=0}^{\infty} (-1)^k\, G_k(t,s),$$
provided that the series converges, where $G_0(t,s) = G(t,s)$ and
$$G_k(t,s) = \int_0^1 G(t,r)G_{k-1}(r,s)q(r)dr, \quad k \ge 1. \tag{3.1}$$
The following properties on $G_k(t,s)$ hold.

Lemma 3.1. *Let $q \in \mathcal{K}_\alpha$ with $\alpha_q < 1$. For each $k \in \mathbb{N}$ and all $(t,s) \in [0,1]\times[0,1]$, we have*

(i) $G_k(t,s) \le \alpha_q^k G(t,s)$. *So, $\mathcal{G}\,(t,s)$ is well-defined in* $[0,1]\times[0,1]$.

(ii)
$$l_k t^{\alpha-1}(1-t)\, s\,(1-s)^{\alpha-1} \le G_k(t,s) \le r_k t^{\alpha-2} s\,(1-s)^{\alpha-1}, \tag{3.2}$$
where
$$l_k = \frac{(\alpha-1)^{k+1}}{(\Gamma(\alpha))^{k+1}}(\int_0^1 r^\alpha(1-r)^\alpha q(r)dr)^k,$$
$$r_k = \frac{1}{(\Gamma(\alpha))^{k+1}}(\int_0^1 r^{\alpha-1}(1-r)^{\alpha-1} q(r)dr)^k.$$

(iii) $G_{k+1}(t,s) = \int_0^1 G_k(t,r)G(r,s)q(r)dr$ *for each* $k \in \mathbb{N}$.

(iv) $\int_0^1 \mathcal{G}\,(t,r)\,G(r,s)q(r)dr = \int_0^1 G\,(t,r)\,\mathcal{G}(r,s)q(r)dr$.

Proof.

(i) We proceed by the induction. The property is trivial for $k = 0$.

Using (3.1) and (1.3), we obtain

$$G_{k+1}(t,s) \le \alpha_q^k \int_0^1 G(t,r)G(r,s)q(r)dr \le \alpha_q^{k+1} G(t,s).$$

So, the inequality in (i) holds for all $k \in \mathbb{N}$. Now, since $G_k(t,s) \le \alpha_q^k G(t,s)$, it follows that $\mathcal{G}(t,s)$ is well-defined in $[0,1] \times [0,1]$.

(ii) The inequalities in (3.2) follow from Proposition 2.1 (ii), (3.1) and simple induction.

(iii) Assume that for a given integer $k \ge 1$ and $(t,s) \in [0,1] \times [0,1]$, we have

$$G_k(t,s) = \int_0^1 G_{k-1}(t,r)G(r,s)q(r)dr.$$

Using (3.1) and Fubini-Tonelli theorem, we obtain

$$\begin{aligned} G_{k+1}(t,s) &= \int_0^1 G(t,r) \left(\int_0^1 G_{k-1}(r,\xi)G(\xi,s)q(\xi)d\xi \right) q(r)dr \\ &= \int_0^1 \left(\int_0^1 G(t,r)G_{k-1}(r,\xi)q(r)dr \right) G(\xi,s)q(\xi)d\xi \\ &= \int_0^1 G_k(t,\xi)G(\xi,s)q(\xi)d\xi. \end{aligned}$$

(iv) Let $k \ge 0$ and $t,r,s \in [0,1]$. By Lemma 3.1 (i) we have

$$0 \le G_k(t,r)G(r,s)q(r) \le \alpha_q^k G(t,r)G(r,s)q(r).$$

Hence the series $\sum\limits_{k\ge 0} \int_0^1 G_k(t,r)G(r,s)q(r)dr$ converges.

So, we deduce by the dominated convergence theorem and Lemma 3.1 (iii) that

$$\begin{aligned} \int_0^1 \mathcal{G}(t,r)\, G(r,s)q(r)dr &= \sum_{k=0}^{\infty} \int_0^1 (-1)^k\, G_k(t,r)G(r,s)q(r)dr \\ &= \sum_{k=0}^{\infty} \int_0^1 (-1)^k\, G(t,r)G_k(r,s)q(r)dr \\ &= \int_0^1 G(t,r)\, \mathcal{G}(r,s)q(r)dr. \end{aligned}$$

□

Proposition 3.2. *Let $q \in \mathcal{K}_\alpha$ with $\alpha_q < 1$. Then the function $(t,s) \to \mathcal{G}(t,s)$ is in $C([0,1] \times [0,1])$.*

Proof. Using Lemma 3.1 and Proposition 2.1, we have for all $k \ge 0$, $G_k \in C([0,1] \times [0,1])$ and

$$G_k(t,s) \le \alpha_q^k G(t,s) \le \frac{1}{\Gamma(\alpha)} \alpha_q^k.$$

Therefore, the function $(t,s) \to \mathcal{G}(t,s)$ belongs to $C([0,1] \times [0,1])$. □

Lemma 3.3. *Let $q \in \mathcal{K}_\alpha$ with $\alpha_q \le \frac{1}{2}$. Then for all $(t,s) \in [0,1] \times [0,1]$, we have*

$$(1-\alpha_q)\, G(t,s) \le \mathcal{G}(t,s) \le G(t,s). \tag{3.3}$$

Proof. Since $\alpha_q \leq \frac{1}{2}$, we deduce from Lemma 3.1 (i), that

$$|\mathcal{G}(t,s)| \leq \sum_{k=0}^{\infty} (\alpha_q)^k G(t,s) = \frac{1}{1-\alpha_q} G(t,s). \tag{3.4}$$

Now, from the expression of $\mathcal{G}$, we have

$$\mathcal{G}(t,s) = G(t,s) - \sum_{k=0}^{\infty} (-1)^k G_{k+1}(t,s). \tag{3.5}$$

Since the series $\sum\limits_{k\geq 0} \int_0^1 G(t,r)G_k(r,s)q(r)dr$ is convergent, we deduce by (3.5) and (3.1) that

$$\begin{aligned}\mathcal{G}(t,s) &= G(t,s) - \sum_{k=0}^{\infty} (-1)^k \int_0^1 G(t,r)G_k(r,s)q(r)dr \\ &= G(t,s) - \int_0^1 G(t,r)(\sum_{k=0}^{\infty} (-1)^k G_k(r,s))q(r)dr;\end{aligned}$$

that is,

$$\mathcal{G}(t,s) = G(t,s) - V(q\mathcal{G}(.,s))(t). \tag{3.6}$$

Using (3.4) and Lemma 3.1 (i) (with $k=1$), we obtain

$$V(q\mathcal{G}(.,s))(t) \leq \frac{1}{1-\alpha_q} V(qG(.,s))(t) = \frac{1}{1-\alpha_q} G_1(t,s) \leq \frac{\alpha_q}{1-\alpha_q} G(t,s).$$

This implies by (3.6) that

$$\mathcal{G}(t,s) \geq G(t,s) - \frac{\alpha_q}{1-\alpha_q} G(t,s) = \frac{1-2\alpha_q}{1-\alpha_q} G(t,s) \geq 0.$$

Hence $\mathcal{G}(t,s) \leq G(t,s)$ and by (3.6) and Lemma 3.1 (i) (with $k=1$), we have

$$\mathcal{G}(t,s) \geq G(t,s) - V(qG(.,s))(t) \geq (1-\alpha_q) G(t,s).$$

□

Corollary 3.4. *Let $q \in \mathcal{K}_\alpha$ with $\alpha_q \leq \frac{1}{2}$ and $\psi \in \mathcal{B}^+((0,1))$. Then*

$$V_q\psi \in C_{2-\alpha}([0,1]) \Longleftrightarrow \int_0^1 s(1-s)^{\alpha-1}\psi(s)\,ds < \infty.$$

Lemma 3.5. *Let $q \in \mathcal{K}_\alpha$ with $\alpha_q \leq \frac{1}{2}$ and $\psi \in \mathcal{B}^+((0,1))$. Then for all $t \in [0,1]$*

$$V\psi(t) = V_q\psi(t) + V_q(qV\psi)(t) = V_q\psi(t) + V(qV_q\psi)(t). \tag{3.7}$$

In particular, if $V(q\psi) < \infty$, we have

$$(I - V_q(q.))(I + V(q.))\psi = (I + V(q.))(I - V_q(q.))\psi = \psi, \tag{3.8}$$

where $V(q.)\psi := V(q\psi)$.

Proof. Using (3.6), we have

$$G(t,s) = \mathcal{G}(t,s) + V(q\mathcal{G}(.,s))(t), \text{ for all } (t,s) \in [0,1]\times[0,1].$$

Hence for $\psi \in \mathcal{B}^+((0,1))$, we obtain

$$\begin{aligned} V\psi(t) &= \int_0^1 \left(\mathcal{G}(t,s) + V(q\mathcal{G}(.,s))(t)\right)\psi(s)ds \\ &= V_q\psi(t) + V(qV_q\psi)(t). \end{aligned}$$

Using Lemma 3.1 (iv) and Fubini-Tonelli theorem, we obtain for $\psi \in \mathcal{B}^+((0,1))$ and $t \in [0,1]$

$$\int_0^1 \int_0^1 \mathcal{G}(t,r)\,G(r,s)q(r)\psi(s)drds = \int_0^1 \int_0^1 G(t,r)\,\mathcal{G}(r,s)q(r)\psi(s)drds;$$

that is,

$$V_q(qV\psi)(t) = V(qV_q\psi)(t).$$

So we obtain

$$V\psi(t) = V_q\psi(t) + V(qV_q\psi)(t) = V_q\psi(t) + V_q(qV\psi)(t).$$

□

Proposition 3.6. *Let $q \in \mathcal{K}_\alpha \cap C^+((0,1))$ with $\alpha_q \le \frac{1}{2}$ and $\psi \in \mathcal{B}^+((0,1))$ such that $s \to s(1-s)^{\alpha-1}\psi(s) \in C((0,1)) \cap L^1((0,1))$. Then $V_q\psi$ is the unique nonnegative solution in $C_{2-\alpha}([0,1])$ of*

$$\begin{cases} D^\alpha u(t) - q(t)u(t) = -\psi(t), \ t \in (0,1), \ 1 < \alpha \le 2, \\ \lim\limits_{t\to 0^+} t^{2-\alpha}u(t) = 0, \ u(1) = 0, \end{cases} \tag{3.9}$$

satisfying

$$(1-\alpha_q)\,V\psi \le u \le V\psi. \tag{3.10}$$

Proof. By Corollary 3.4 we deduce that the function $t \to q(t)V_q\psi(t) \in C^+((0,1))$. Using (3.7) and Proposition 2.1 (ii), we obtain

$$V_q\psi(t) \le V\psi(t) \le \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-2}s(1-s)^{\alpha-1}\psi(s)ds = Mt^{\alpha-2}. \tag{3.11}$$

This implies that

$$\int_0^1 s(1-s)^{\alpha-1}q(s)V_q\psi(s)ds \le M \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}q(s)ds < \infty.$$

Therefore, by Proposition 2.2 (ii), the function $u = V_q\psi = V\psi - V(qV_q\psi)$ satisfies the equation

$$\begin{cases} D^\alpha u(t) = -\psi(t) + q(t)u(t), \quad t \in (0,1), \\ \lim\limits_{t\to 0^+} t^{2-\alpha}u(t) = 0, \ u(1) = 0. \end{cases}$$

By integration of inequalities (3.3), we obtain (3.10).

Next, we prove the uniqueness. Assume that $v \in C_{2-\alpha}([0,1])$ is another solution of problem (3.9) satisfying (3.10). Put $\widetilde{v} := v + V(qv)$. Since the function $s \to s(1-s)^{\alpha-1}q(s)v(s) \in C((0,1)) \cap L^1((0,1))$, by Proposition 2.2 (ii) we deduce that

$$\begin{cases} D^\alpha \widetilde{v}(t) = -\psi(t), \quad t \in (0,1), \\ \lim\limits_{t\to 0^+} t^{2-\alpha}\widetilde{v}(t) = 0, \ \widetilde{v}(1) = 0. \end{cases}$$

Again from Proposition 2.2 (ii), we conclude that

$$\widetilde{v} := v + V(qv) = V\psi.$$

So

$$(I + V(q.))((v - u)^+) = (I + V(q.))((v - u)^-),$$

where $(v - u)^+ = \max(v - u, 0)$ and $(v - u)^- = \max(u - v, 0)$.

By using (3.10), (3.11) and Proposition 2.4, we have

$$V(q\,|v - u|) \le 2MV(q[h_1 + h_2]) \le 2M\alpha_q(h_1 + h_2) < \infty.$$

Therefore, by applying (3.8), we obtain $u = v$. □

Proof of Theorem 1.2. Let $\mu \ge 0$ and $\nu \ge 0$ with $\mu + \nu > 0$ and recall that

$$h_0(t) := \mu h_1(t) + \nu h_2(t).$$

Let $q \in \mathcal{K}_\alpha \cap C^+((0,1))$ as in (H$_2$). Consider

$$\Lambda := \left\{u \in \mathcal{B}^+((0,1)) : (1 - \alpha_q)\, h_0 \le u \le h_0\right\},$$

and define the operator T on Λ by

$$Tu = h_0 - V_q\,(qh_0) + V_q((q - \varphi\,(.,u))\,u).$$

Using (3.7) and (2.5) we have

$$V_q(qh_0) \le V\,(qh_0) \le \alpha_q h_0 \le h_0. \tag{3.12}$$

Hence by (H$_2$), we get

$$0 \le \varphi(.,u) \le q \text{ for all } u \in \Lambda. \tag{3.13}$$

Next we prove that $T\Lambda \subseteq \Lambda$. Using (3.13) and (3.12), we obtain for all $u \in \Lambda$ that

$$Tu \le h_0 - V_q\,(qh_0) + V_q(qu) \le h_0,$$

$$Tu \ge h_0 - V_q\,(qh_0) \ge (1 - \alpha_q)\,h_0.$$

On the other hand, from (H$_2$), we deduce that the operator T is nondecreasing on Λ.

Now, let $\{u_k\}$ be the sequence defined by $u_0 = (1 - \alpha_q)\,h_0$ and $u_{k+1} = Tu_k$ for $k \in \mathbb{N}$. Since T is nondecreasing on Λ and $T\Lambda \subseteq \Lambda$, we obtain

$$(1 - \alpha_q)\,h_0 = u_0 \le u_1 \le ... \le u_k \le u_{k+1} \le h_0.$$

Hence by dominated convergence theorem and (H$_1$) - (H$_2$), the sequence $\{u_k\}$ converges to a function $u \in \Lambda$ satisfying

$$u = (I - V_q\,(q.))\,h_0 + V_q((q - \varphi\,(.,u))\,u);$$

that is,

$$(I - V_q\,(q.))\,u = (I - V_q\,(q.))\,h_0 - V_q\,(u\varphi\,(.,u))\,. \tag{3.14}$$

Applying the operator $(I + V\,(q.))$ on the both sides of (3.14) and using (3.7) and (3.8), we obtain

$$u = h_0 - V\,(u\varphi\,(.,u))\,. \tag{3.15}$$

Let us prove that u is a solution. Using (3.13), there exists a constant $c > 0$ such that

$$s(1 - s)^{\alpha-1}u(s)\varphi\,(s, u(s)) \le s(1 - s)^{\alpha-1}h_0(s)q(s) \le cs^{\alpha-1}(1 - s)^{\alpha-1}q(s). \tag{3.16}$$

So by Proposition 2.2 (i) the function $t \to V\,(u\varphi\,(.,u))\,(t)$ is in $C_{\alpha-2}([0,1])$ and by (3.15), u belongs to $C_{\alpha-2}([0,1])$.

Since by (H_1) and (3.16), the function $s \to s(1-s)^{\alpha-1}u(s)\varphi(s,u(s)) \in C((0,1)) \cap L^1((0,1))$, then by Proposition 2.2 (ii) u is a solution of problem (1.1).

Finally, we prove the uniqueness. To this end, let $v \in C_{\alpha-2}([0,1])$ be another solution to problem (1.1) satisfying (1.4). Since $v \le h_0$, we deduce by (3.16) that

$$0 \le v(s)\varphi(s,v(s)) \le h_0(s)q(s) \le cs^{\alpha-2}q(s).$$

So the function $s \to s(1-s)^{\alpha-1}v(s)\varphi(s,v(s)) \in C((0,1)) \cap L^1((0,1))$. Put $\widetilde{v} := v + V(v\varphi(.,v))$, then by Proposition 2.2 (ii), we have

$$\begin{cases} D^\alpha \widetilde{v}(t) = 0, \quad t \in (0,1), \\ \lim\limits_{t\to 0^+} t^{2-\alpha}\widetilde{v}(t) = \mu, \ \widetilde{v}(1) = \nu. \end{cases}$$

Hence

$$v = h_0 - V(v\varphi(.,v)). \tag{3.17}$$

Let $h : (0,1) \to \mathbb{R}$, be defined by

$$h(t) = \begin{cases} \frac{v(t)\varphi(t,v(t)) - u(t)\varphi(t,u(t))}{v(t)-u(t)} & \text{if } v(t) \neq u(t), \\ 0 & \text{if } v(t) = u(t). \end{cases}$$

From (H_3) we have $h \in \mathcal{B}^+((0,1))$ and by (3.15) and (3.17), we obtain

$$(I + V(h.))((v-u)^+) = (I + V(h.))((v-u)^-),$$

where $(v-u)^+ = \max(v-u,0)$ and $(v-u)^- = \max(u-v,0)$. Using (H_2), we have $h \le q$ and by (2.5) we deduce that

$$V(h\,|v-u|) \le 2V(qh_0) \le 2\alpha_q h_0 < \infty.$$

So $u = v$ by (3.8). □

Proof of Corollary 1.3. We obtain the results by applying Theorem 1.2 with $\varphi(t,s) = \lambda p(t)g(s)$ and $q(t) := \lambda\widetilde{p}(t)$. □

Example 3.7. Let $1 < \alpha \le 2$ and $\mu \ge 0$, $\nu \ge 0$ with $\mu + \nu > 0$. Let $\sigma \ge 0$, $\gamma \ge 0$ and $p \in C^+((0,1)$ such that

$$\int_0^1 s^{(\alpha-1)+(\alpha-2)(\sigma+\gamma)}(1-s)^{\alpha-1}p(s)ds < \infty.$$

Let $\theta(s) = s^{\sigma+1}\log(1+s^\gamma)$ and $\widetilde{p}(s) := p(s)\max\limits_{0\le\xi\le h_0(s)}\theta'(\xi)$. Since $\widetilde{p} \in \mathcal{K}_\alpha$, then for $\lambda \in [0, \frac{1}{2\alpha_{\widetilde{p}}})$, the problem

$$\begin{cases} D^\alpha u(t) = \lambda p(t)u^{\sigma+1}(t)\log(1+u^\gamma(t)), \quad t \in (0,1), \\ \\ \lim\limits_{t\to 0^+} t^{2-\alpha}u(t) = \mu, \ u(1) = \nu, \end{cases}$$

has a unique positive solution u in $C_{2-\alpha}([0,1])$ satisfying

$$(1 - \lambda\alpha_{\widetilde{p}})h_0(t) \le u(t) \le h_0(t), \text{ for all } t \in (0,1].$$

4. Second existence result

Assume that hypotheses (H_4)-(H_5) are satisfied. Let $\mu, \nu > 0$ and recall that $h_0(t) := \mu t^{\alpha-2}(1-t) + \nu t^{\alpha-1}$, for $t \in (0,1]$. Observe that for $t \in (0,1]$,

$$\min(\mu,\nu)t^{\alpha-2} \le h_0(t) \le \max(\mu,\nu)t^{\alpha-2}. \tag{4.1}$$

The next lemma will be used in the proof of Theorem 1.4.

Lemma 4.1. *Let q be a function in $\mathcal{K}_\alpha$, then the family of functions*

$$\Lambda_q = \{\frac{1}{h_0(t)}\int_0^1 G(t,s)h_0(s)\rho(s)ds,\ \ |\rho| \le q\}$$

is uniformly bounded and equicontinuous in $[0,1]$. *Consequently,* Λ_q *is relatively compact in* $C([0,1])$.

Proof. From Proposition 2.4, we deduce that for ρ such that $|\rho| \le q$ and $t \in (0,1]$, we have

$$\left|\frac{1}{h_0(t)}\int_0^1 G(t,s)h_0(s)\rho(s)ds\right| \le \frac{1}{h_0(t)}\int_0^1 G(t,s)h_0(s)q(s)ds \le \alpha_q < \infty.$$

So the family Λ_q is uniformly bounded.

On the other hand, by Proposition 2.1 (ii) and (4.1), for $(t,s) \in (0,1] \times [0,1]$, we have

$$\left|\frac{G(t,s)}{h_0(t)}h_0(s)q(s)\right| \le \frac{\max(\mu,\nu)}{\min(\mu,\nu)\Gamma(\alpha)} s^{\alpha-1}(1-s)^{\alpha-1}q(s). \tag{4.2}$$

Since the function $(t,s) \to \dfrac{G(t,s)}{h_0(t)} \in C\left([0,1]\times[0,1]\right)$ and $q \in \mathcal{K}_\alpha$, we deduce by (4.2) and the dominated convergence theorem that the family Λ_q is equicontinuous in $[0,1]$. Therefore, by Ascoli's theorem, the family Λ_q becomes relatively compact in $C([0,1])$. □

Proof of Theorem 1.4. Assume that hypotheses (H_4)-(H_5) are satisfied. So by (H_5) the function $s \to q(s) := \frac{1}{h_0(s)} f(s, h_0(s)) \in \mathcal{K}_\alpha$. Put

$$\lambda_0 := \inf_{t\in(0,1)} \frac{h_0(t)}{V(f(.,h_0))(t)}. \tag{4.3}$$

From (2.5) we have

$$V(f(.,h_0)) = V(h_0 q) \le \alpha_q h_0.$$

Therefore, $\lambda_0 \ge \frac{1}{\alpha_q} > 0$.

Let $\lambda \in [0,\lambda_0)$ and S be the nonempty closed bounded convex set given by

$$S = \{v \in C([0,1]):\ (1 - \frac{\lambda}{\lambda_0}) \le v \le 1\}.$$

We define the operator L on S by

$$Lv(t) = 1 - \frac{\lambda}{h_0(t)}\int_0^1 G(t,s) f\left(s, v(s)h_0(s)\right) ds. \tag{4.4}$$

Using (H_4), (H_5) and Lemma 4.1, we deduce that the family

$$\{\frac{1}{h_0(t)}\int_0^1 G(t,s) f\left(s, v(s)h_0(s)\right) ds,\quad v \in S\},$$

is relatively compact in $C([0,1])$ and therefore $L(S)$ becomes relatively compact in $C([0,1])$.
On the other hand, since f is a nonnegative function, it is clear from (4.4), (H_4) and (4.3) that $L(S) \subseteq S$.

Next, we prove the continuity of the operator L in S in the supremum norm. Let $\{v_k\}$ be a sequence in S which converges uniformly to a function v in S. Then we have

$$|Lv_k(t) - Lv(t)| \le \lambda \int_0^1 \frac{G(t,s)}{h_0(t)} \left|f\left(s, v(s)h_0(s)\right) - f\left(s, v_k(s)h_0(s)\right)\right| ds.$$

From the monotonicity of f, we have

$$|f(s, v(s)h_0(s)) - f(s, v_k(s)h_0(s))| \leq 2h_0(s)q(s).$$

So we conclude by the continuity of f, (4.2) and the dominated convergence theorem, that

$$\forall t \in [0,1],\ Lv_k(t) \to Lv(t) \text{ as } k \to \infty.$$

Since $L(S)$ is relatively compact in $C([0,1])$, we obtain the uniform convergence, namely

$$\|Lv_k - Lv\|_\infty \to 0 \quad \text{as } k \to \infty.$$

Thus we have proved that L is a compact operator mapping from S to itself. Hence by the Schäuder's fixed point theorem, there exists $v \in S$ such that

$$v(t) = 1 - \frac{\lambda}{h_0(t)} \int_0^1 G(t,s) f(s, v(s)h_0(s))\, ds.$$

Let $u(t) = v(t)h_0(t)$. Then u is a positive function in $C_{2-\alpha}([0,1])$, satisfying for each $t \in (0,1)$

$$u(t) = h_0(t) - \lambda \int_0^1 G(t,s) f(s, u(s))\, ds. \tag{4.5}$$

Finally, since by (H4) and (H_5) the map $s \to s(1-s)^{\alpha-1} f(s, u(s)) \in C((0,1)) \cap L^1((0,1))$, we deduce by (4.5) and Proposition 2.2 (ii) that u is a required solution. □

Example 4.2. Let $1 < \alpha \leq 2$, $\sigma \geq 0$ and $p \in C^+((0,1)$ such that

$$\int_0^1 s^{(\alpha-1)+(\alpha-2)(\sigma-1)}(1-s)^{\alpha-1} p(s) ds < \infty.$$

Let $\mu, \nu > 0$. Then by Theorem 1.4, there exists a constant $\lambda_0 > 0$ such that for each $\lambda \in [0, \lambda_0)$, problem

$$\begin{cases} D^\alpha u(t) = \lambda p(t) u^\sigma,\ t \in (0,1), \\ \lim\limits_{t \to 0^+} t^{2-\alpha} u(t) = \mu,\ u(1) = \nu, \end{cases}$$

has a positive solution u in $C_{2-\alpha}([0,1])$ satisfying

$$(1 - \frac{\lambda}{\lambda_0}) h_0(t) \leq u(t) \leq h_0(t) \text{ for all } t \in (0,1].$$

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