Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 1424–1431
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Iterative algorithm for strongly continuous semigroup of Lipschitz pseudocontraction mappings

Liping Yang

School of Applied Mathematics, Guangdong University of Technology, Guangzhou 510520, Guangdong, China.

Communicated by C. Park

Abstract

In this paper, an implicit iterative process is considered for strongly continuous semigroup of Lipschitz pseudocontraction mappings. Weak and strong convergence theorems for common fixed points of strongly continuous semigroup of Lipschitz pseudocontraction mappings are established in a real Banach space.

Keywords: Semigroup of pseudocontraction mappings, uniformly convex Banach spaces, Opial's condition, variational inequality.
2010 MSC: 47H10, 47J25, 49J40, 65K10.

1. Introduction and Preliminaries

The study of fixed points of mappings has been at the center of vigorous research activity in the last years. The idea of considering fixed point iteration procedures comes from practical numerical computations. The class of pseudocontractive mappings in their relation with iteration procedures has been studied by several researchers under suitable conditions; for more details, see [17, 18, 20] and the references therein. The class of strongly pseudocontractive mappings has been studied by many researchers (see, [3, 12, 15]) under certain conditions.

Viscosity method provides an efficient approach to a large number of problems coming from different branches of Mathematical Analysis. Various applications of the viscosity methods can be found in optimal control theory, singular perturbations, minimal cost problem. Dewangan et al. [6] studied the strong convergence of viscosity iteration and modified viscosity iteration process for strongly continuous semigroup of uniformly Lipschitzian asymptotically pseudocontractive mappings.

Email address: yanglping2003@126.com (Liping Yang)

Received 2015-10-25

The variational inequality problem was first introduced by Hartman and Stampacchia [9]. Then, the variational inequality has achieved an increasing attention in many research fields, such as mathematical programming, constrained linear and nonlinear optimization, automatic control, manufacturing system design, signal and image processing and the complementarity problem in economics and pattern recognition (see [4, 7, 8] and the references therein).

Let E denote an arbitrary real Banach space and E^* denote the dual space of E. The normalized duality map $J : E \to 2^{E^*}$ is defined by

$$Jx := \{u^* \in E^* : \langle x, u^* \rangle = \|x\|^2; \|u^*\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between elements of E and E^*. First of all, we recall and define the concepts as follows:

Definition 1.1. Let E be a real Banach space and T be a mapping with domain $D(T)$ and range $R(T)$ in E.

(1) A mapping T is said to be nonexpansive, if $\|Tx - Ty\| \le \|x - y\|$ holds for all x, $y \in D(T)$.
(2) A mapping T is said to be strongly pseudocontractive, if for all x, $y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \le \beta \|x - y\|^2, \quad \text{for some } 0 < \beta < 1. \tag{1.1}$$

(3) A mapping T is said to be pseudocontractive if there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \le \|x - y\|^2 \tag{1.2}$$

for all x, $y \in D(T)$.

If I denotes the identity operator, then (1.2) is equivalent to the following [16]:

$$\|x - y\| \le \|x - y + s[(I - T)x - (I - T)y]\|, \quad \forall s > 0. \tag{1.3}$$

Closely related to the class of pseudocontractive mappings is strongly continuous semigroup of Lipschitz pseudocontractive mappings. Let E be a real Banach space, K be a nonempty subset of E. One parameter family $\mathscr{T} := \{ T(t) : t \in \mathbb{R}^+ \}$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers, is said to be strongly continuous semigroup of Lipschitz pseudocontraction mappings from K into itself if the following conditions are satisfied:

(1) $T(0)x = x$ for all $x \in K$;
(2) $T(s + t)x = T(s)T(t)x$ for all $x \in K$ and s, $t \in \mathbb{R}^+$;
(3) $T(t)$ is pseudocontractive for each $t \in \mathbb{R}^+$;
(4) for each $x \in K$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into K is continuous;
(5) for each $t > 0$, there exists a bounded measurable function $L(t) : (0, \infty) \to [0, \infty)$ such that

$$\|T(t)x - T(t)y\| \le L(t)\|x - y\|, \quad \forall x, y \in K.$$

If $L(t) = 1$ in (5), (3) is replaced by the following (3'): $T(t)$ is a nonexpansive mapping for each $t \in \mathbb{R}^+$, then $\mathscr{T}$ is said to be a strongly continuous semigroup of nonexpansive mapping on K. $\mathscr{T}$ is said to have a fixed point if there exists $y \in K$ such that $T(t)y = y$ for all $t \in \mathbb{R}^+$. We denote by $F(\mathscr{T})$, the set of fixed points of $\mathscr{T}$, i.e., $F(\mathscr{T}) := \cap_{t \in \mathbb{R}^+} F(T(t))$.

Recall that a mapping $f : K \to K$ is a contraction on K if there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \le \alpha \|x - y\|$, $x, y \in K$. We use Π_K to denote the collection of mappings f verifying the above inequality. That is, $\Pi_K = \{f : K \to K \mid f \text{ is a contraction with constant } \alpha\}$. Note that each $f \in \Pi_K$ has a unique fixed point in K.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Assume that A is strongly positive bounded linear operator on H, that is, there is a constant $\bar{\gamma} > 0$ with property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \quad \forall x \in H.$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where C is the fixed point set of a nonexpansive mapping T on H and b is given point in H.

Let $T : H \to H$ be a nonexpansive mapping and $f \in \Pi_H$. Recently, Marino and Xu [10] introduced, in Hilbert spaces, the following general iteration process

$$x_t = (I - tA)Tx_t + t\gamma f(x_t), \tag{1.4}$$

$t \in (0, 1)$ such that $t < \|A\|^{-1}$ and $0 < \gamma < \bar{\gamma}/\alpha$ and proved that the sequence $\{x_t\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in Fix(T),$$

which is the optimality condition for the minimization problem

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for $x \in H$) and $Fix(T) = \{x \in H : Tx = x\}$.

Construction of common fixed points of nonexpansive semigroup is an important subject (see, e.g., [2, 21]). This brings us to the following question.

Question. Can the following implicit iteration sequence

$$u_n = \alpha_n\gamma f(u_n) + (I - \alpha_n A)T(t_n)u_n,$$

provide the same result for the more general class of strongly continuous semigroups of Lipschitz pseudocontraction mappings in Banach spaces?

It is our purpose, in this paper to prove a convergence theorem for a strongly continuous semigroup of Lipschitz pseudocontraction mappings in Banach spaces. More precisely, Let K be a nonempty closed convex subset of a uniformly convex Banach space which has uniformly Gâteaux differentiable norm such that $K+K \subset K$. Let $\mathscr{T} := \{\, T(t) : \, t \in \mathbb{R}^+ \,\}$ be a strongly continuous semigroup of Lipschitz pseudocontraction mappings from K into itself such that $F(\mathscr{T}) \neq \emptyset$. Then, for all $n \geq 1$, the implicit iteration sequence

$$u_n = \alpha_n\gamma f(u_n) + (I - \alpha_n A)T(t_n)u_n,$$

converges strongly to a point of $F(\mathscr{T})$. This provides an affirmative answer to the above Question.

Now we recall the well-known following concepts and results.

A Banach space E is said to be strictly convex if $\|x\| = \|y\| = 1$ for $x \neq y$ implies $\frac{1}{2}\|x + y\| < 1$. In a strictly convex Banach space E, we have if $\|x\| = \|y\| = \|tx + (1 - t)y\|$, for $t \in (0, 1)$ and x, $y \in E$, then $x = y$.

Let E be a Banach space with dimension $E \geq 2$. The modulus of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : \|x\| = \|y\| = 1, \, \|x - y\| = \varepsilon\right\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

Let $S(E) = \{ x \in E : \|x\| = 1 \}$. The space E is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. For any $x, y \in E(x \neq 0)$, we denote this limit by (x, y). The norm $\|\cdot\|$ of E is said to be Fréchet differentiable if for all $x \in S(E)$, the limit (x, y) exists uniformly for all $y \in S(E)$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in S(E)$ the limit (x, y) is attained uniformly for $x \in S(E)$.

We need the following notation: $x_n \rightharpoonup x$ denotes a sequence $\{x_n\}$ converges weakly to x.

A Banach space E is said to satisfy Opial's condition [14] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \quad \forall y \in E \quad (y \neq x).$$

A mapping $T : K \to K$ is said to be demiclosed at zero if, for each sequence $\{x_n\}$ in K, $x_n \rightharpoonup x$ and $Tx_n \to 0$ strongly imply $Tx = 0$.

Let μ be a continuous linear functional on l^∞ and $(a_0, a_1, \dots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \dots))$. Recall a Banach limit μ is a bounded functional on l^∞ such that

$$\|\mu\| = \mu_n(1) = 1, \quad \liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n,$$
$$\mu_n(a_{n+r}) = \mu_n(a_n)$$

for any fixed positive integer r and for all $(a_0, a_1, \dots) \in l^\infty$.

Recall that an operator A is strongly positive on a smooth Banach space E if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J(x) \rangle \geq \bar{\gamma} \|x\|^2, \quad \|aI - bA\| = \sup_{\|x\| \leq 1} |\langle (aI - bA)x, J(x) \rangle|, \tag{1.5}$$

where $x \in E$ and $a \in [0, 1]$, $b \in [-1, 1]$, I is the identity mapping and J is the normalized duality mapping.

The following Lemma is useful in the sequel.

Lemma 1.2. *Assume that A is a strongly positive linear bounded operator on a smooth Banach space E with coefficient $\bar{\gamma} > 0$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho \bar{\gamma}$.*

Proof. The proof follows as in the proof of Lemma 2.5 of [10]. □

Lemma 1.3 ([5])**.** *Let E be a Banach space. Let K be a nonempty closed and convex subset of E and let $T : K \to K$ be a continuous and strong pseudocontraction mapping. Then T has a unique fixed point in K.*

Lemma 1.4 ([13])**.** *Let E be a real normed linear space and J be the normalized duality map on E. Then for any given x, $y \in E$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 1.5 ([22])**.** *Let E be a real reflexive Banach space which satisfies Opial's condition. Let K be a nonempty closed convex subset of E and $T : K \to K$ be a continuous pseudocontractive mapping. Then $I - T$ is demiclosed at zero.*

Lemma 1.6 ([19])**.** *Let $r > 0$. Then a real Banach space E is uniformly convex if and only if there exists a continuous and strictly increasing convex function $g_1 : \mathbb{R}^+ \to \mathbb{R}^+$ with $g_1(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda) y\|^2 \leq \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda) g_1(\|x - y\|)$$

for all x, $y \in B_r$, $\lambda \in [0, 1]$, where $B_r = \{ x \in E : \|x\| \leq r \}$.

2. Main results

Theorem 2.1. *Let K be a nonempty closed convex subset of a real smooth Banach space E such that $K+K \subset K$. Let $\mathscr{T} := \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of Lipschitz pseudocontraction mappings from K into itself such that $F(\mathscr{T}) \neq \emptyset$. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$, $f \in \Pi_K$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $\{u_n\}$ be a sequence defined by*

$$u_n = \alpha_n \gamma f(u_n) + (I - \alpha_n A) T(t_n) u_n \tag{2.1}$$

for all $n \geq 1$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(\alpha_n/t_n) = 0$, $0 < \alpha_n < 1$ for all $n \geq 1$. Then $\lim_{n\to\infty}\|u_n - T(t)u_n\| = 0$ for any $t \in \mathbb{R}^+$.

Proof. First, we show that $\{u_n\}$ is well defined. Let

$$Tx := \alpha_n \gamma f(x) + (I - \alpha_n A) T(t_n) x, \quad \forall x \in K, \quad \forall n \geq 1.$$

Since

$$\begin{aligned}\langle Tx - Ty, j(x-y)\rangle &= \alpha_n \gamma \langle f(x) - f(y), j(x-y)\rangle \\ &\quad + \langle (I - \alpha_n A)(T(t_n)x - T(t_n)y), j(x-y)\rangle \\ &\leq \alpha_n \gamma \|f(x) - f(y)\|\|x-y\| + (1 - \alpha_n \bar{\gamma})\|x-y\|^2 \\ &\leq (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x-y\|^2,\end{aligned}$$

we know that T is strongly pseudocontractive and strongly continuous. It follows from Lemma 1.3 that T has a unique fixed point u_n for each $n \geq 1$ such that

$$u_n = \alpha_n \gamma f(u_n) + (I - \alpha_n A) T(t_n) u_n.$$

That is, the sequence $\{u_n\}$ is well defined.

Next, we show that $\{u_n\}$ is bounded. Indeed, fixing $p \in F(\mathscr{T})$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \langle \alpha_n(\gamma f(u_n) - Ap) + (I - \alpha_n A)(T(t_n)u_n - p), j(u_n - p)\rangle \\ &= \alpha_n \gamma \langle f(u_n) - f(p), j(u_n - p)\rangle + \alpha_n \langle \gamma f(p) - Ap, j(u_n - p)\rangle \\ &\quad + \langle (I - \alpha_n A)(T(t_n)u_n - p), j(u_n - p)\rangle \\ &\leq \alpha_n \gamma \|f(u_n) - f(p)\|\|u_n - p\| + \alpha_n \langle \gamma f(p) - Ap, j(u_n - p)\rangle \\ &\quad + \|I - \alpha_n A\|\|u_n - p\|^2 \\ &\leq (1 - \alpha_n(\bar{\gamma} - \alpha\gamma))\|u_n - p\|^2 + \alpha_n \langle \gamma f(p) - Ap, j(u_n - p)\rangle.\end{aligned}$$

This implies that

$$\|u_n - p\|^2 \leq \frac{1}{\bar{\gamma} - \alpha\gamma}\langle \gamma f(p) - Ap, j(u_n - p)\rangle. \tag{2.2}$$

Thus we have

$$\|u_n - p\| \leq \frac{1}{\bar{\gamma} - \alpha\gamma}\|\gamma f(p) - Ap\|.$$

This shows that $\{u_n\}$ and hence $\{f(u_n)\}$, $\{T(t_n)u_n\}$ are bounded. Additionally, from (2.1) we have $\|u_n - T(t_n)u_n\| = \alpha_n\|f(u_n) - AT(t_n)u_n\| \to 0$ as $n \to \infty$.

Put $M = \sup_{t\geq 0} L(t)$. From the assumption, we have that $M < \infty$. For each given $t > 0$,

$$\|u_n - T(t)u_n\| \leq \sum_{k=0}^{[t/t_n]-1} \|T((k+1)t_n)u_n - T(kt_n)u_n\| + \|T(t)u_n - T([t/t_n]t_n)u_n\|$$

$$
\begin{aligned}
&\leq [t/t_n]L(kt_n)\|T(t_n)u_n - u_n\| \\
&\quad + L([t/t_n]t_n)\|T(t - [t/t_n]t_n)u_n - u_n\| \\
&\leq M[t/t_n]\alpha_n\|\gamma f(u_n) - AT(t_n)u_n\| + M\|T(t - [t/t_n]t_n)u_n - u_n\| \\
&\leq t(\alpha_n/t_n)M\|\gamma f(u_n) - AT(t_n)u_n\| \\
&\quad + M\max\{ \|T(s)u_n - u_n\| : 0 \leq s \leq t_n \}
\end{aligned}
$$

for each $n \geq 1$, where $[t/t_n]$ is a nonnegative integer not greater than t/t_n. Since $\lim_{n\to\infty}(\alpha_n/t_n) = 0$ and $T(\cdot)x : \mathbb{R}^+ \to K$ is continuous for any $x \in K$, it follows from $\lim_{n\to\infty}\|u_n - T(t_n)u_n\| = 0$ that $\|u_n - T(t)u_n\| \to 0$ as $n \to \infty$. This completes the proof. □

Theorem 2.2. *Let E be a reflexive and smooth Banach space which satisfies the Opial's condition and K a nonempty closed convex subset of E such that $K + K \subset K$. Let $\mathscr{T} := \{ T(t) : t \in \mathbb{R}^+ \}$ be a strongly continuous semigroup of Lipschitz pseudocontraction mappings from K into itself such that $F(\mathscr{T}) \neq \emptyset$. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$, $f \in \Pi_K$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Suppose that $\{u_n\}$ is a sequence defined by* (2.1) *and*

(i) $0 < \alpha_n < 1$ *for all* $n \geq 1$*;*

(ii) $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(\alpha_n/t_n) = 0$.

Then the sequence $\{u_n\}$ converges weakly to a common fixed point of the semigroup $\mathscr{T}$.

Proof. Let $\omega_w(\{u_n\}) := \{ u : \exists u_{n_j} \rightharpoonup u \}$ denote the weak limit set of $\{u_n\}$. Since E is reflexive and $\{u_n\}$ is bounded, it follows from Lemma 1.5 that $\omega_w(\{u_n\}) \subset F(\mathscr{T})$. Additionally, since the space E satisfies Opial's condition, we conclude that $\omega_w(\{u_n\})$ is singleton. This completes the proof. □

Theorem 2.3. *Let K be a nonempty closed convex subset of a uniformly convex Banach space which has uniformly Gâteaux differentiable norm such that $K + K \subset K$. Let $\mathscr{T} := \{ T(t) : t \in \mathbb{R}^+ \}$ be a strongly continuous semigroup of Lipschitz pseudocontraction mappings from K into itself such that $F(\mathscr{T}) \neq \emptyset$. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$, $f \in \Pi_K$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Suppose that $\{u_n\}$ is a sequence defined by* (2.1) *and*

(i) $0 < \alpha_n < 1$ *for all* $n \geq 1$*;*

(ii) $\lim_{n\to\infty} t_n = \lim_{n\to\infty}(\alpha_n/t_n) = 0$.

Then the sequence $\{u_n\}$ converges strongly to a point p of $F(\mathscr{T})$ which solves the variational inequality:

$$\langle (A - \gamma f)p, j(p - z)\rangle \leq 0, \quad z \in F(\mathscr{T}). \tag{2.3}$$

Proof. Define a mapping $g(x) := 2x - T(t)x$, then the mapping g has a nonexpansive inverse, denoted by h, which maps K into K. Additionally, it follows from Theorem 6 of [11] that $F(h) = F(\mathscr{T})$. From $\|u_n - T(t)u_n\| \to 0$ as $n \to \infty$, we have that $\|u_n - h(u_n)\| \to 0$ as $n \to \infty$. Define a mapping $\phi : K \to \mathbb{R}$ by

$$\phi(y) := \mu_n\|u_n - y\|^2, \quad \forall y \in K,$$

where μ_n is a Banach limit. Since E is reflexive and ϕ is continuous, convex and $\phi(x) \to \infty$ as $\|x\| \to \infty$, we get that ϕ attains its infimum over K (see, e.g., [1]). Hence

$$C := \{ x^* \in K : \phi(x^*) = \min_{x\in K} \phi(x) \}$$

is a nonempty bounded closed convex subset of K. Next, we shall show that C is singleton. Since C and $\{u_n\}$ are bounded, there exists $r > 0$ such that C, $\{u_n\} \subset B_r$ for all $n \geq 1$. Since E is uniformly convex, it follows from Lemma 1.6 that there exists a continuous and strictly increasing convex function $g_1 : \mathbb{R}^+ \to \mathbb{R}^+$ with $g_1(0) = 0$ such that, for any p_1, $p_2 \in C$,

$$\|u_n - \frac{p_1 + p_2}{2}\|^2 \leq \frac{1}{2}\|u_n - p_1\|^2 + \frac{1}{2}\|u_n - p_2\|^2 - \frac{1}{4}g_1(\|p_1 - p_2\|).$$

Taking Banach limit μ_n on the above inequality, it follows that

$$\frac{1}{4}g_1(\|p_1 - p_2\|) \leq \tfrac{1}{2}\mu_n\|u_n - p_1\|^2 + \tfrac{1}{2}\mu_n\|u_n - p_2\|^2 - \mu_n\|u_n - \tfrac{p_1+p_2}{2}\|^2 \leq 0.$$

This implies $p_1 = p_2$ and so C is a singleton. Now, we show that $T(t)$ has a fixed point in C. Indeed, since $\lim_{n\to\infty}\|u_n - h(u_n)\| = 0$, for all $q \in C$, we get that

$$\phi(h(q)) = \mu_n\|u_n - h(q)\|^2 = \mu_n\|h(u_n) - h(q)\|^2 \leq \mu_n\|u_n - q\|^2 = \phi(q),$$

we have that C is $h-$invariant and hence $h(q) = q$. Therefore, $T(t)q = q$. That is, $q \in F(\mathscr{T})$. It follows from (2.2) that

$$\mu_n\|u_n - q\|^2 \leq \mu_n \frac{1}{\bar{\gamma} - \alpha\gamma}\langle \gamma f(q) - Aq, j(u_n - q)\rangle.$$

Now observing that $u_{n_j} \rightharpoonup q$ implies $j(u_{n_j} - q) \rightharpoonup 0$, we conclude from the above inequality that $\mu_n\|u_n - q\|^2 \to 0$ as $j \to \infty$. Hence, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\{u_{n_j}\}$ converges strongly to q. Next, we prove that q solves the variational inequality (2.3). For any $z \in F(\mathscr{T})$, we have that

$$\begin{aligned}\langle (I - T(t_n))u_n - (I - T(t_n))z, j(u_n - z)\rangle &= \langle u_n - z, j(u_n - z)\rangle \\ &\quad - \langle T(t_n)u_n - T(t_n)z, j(u_n - z)\rangle \\ &\geq \|u_n - z\|^2 - \|u_n - z\|^2 = 0. \end{aligned} \tag{2.4}$$

It follows from (2.1) that we can derive

$$(A - \gamma f)u_n = -\frac{1}{\alpha_n}(I - T(t_n))u_n + A(I - T(t_n))u_n.$$

Then

$$\begin{aligned}\langle (A - \gamma f)u_n, j(u_n - z)\rangle &= -\frac{1}{\alpha_n}\langle (I - T(t_n))u_n - (I - T(t_n))z, j(u_n - z)\rangle \\ &\quad + \langle A(I - T(t_n))u_n, j(u_n - z)\rangle \\ &\leq \langle A(I - T(t_n))u_n, j(u_n - z)\rangle. \end{aligned} \tag{2.5}$$

It follows from Lemma 1.5 and (2.5) that we obtain

$$\langle (A - \gamma f)q, j(q - z)\rangle \leq 0.$$

So, $q \in F(\mathscr{T})$ is a solution of the variational inequality (2.3), and hence $q = p$ by the uniqueness. In a summary, we have shown that each cluster point of $\{u_n\}$ equals p. Thus, $u_n \to p$ as $n \to \infty$. This completes the proof. □

Remark 2.4. Theorem 2.3 improves Theorem 3.2 of Marino and Xu [10] in the sense that our theorem is applicable in uniformly convex Banach spaces for the more general class of strongly continuous continuous semigroup of Lipschitz pseudocontraction mappings.

Acknowledgement

The author wishes to thank the editor and the referee for valuable suggestions. This work is supported by the Humanity and Social Science Planning Foundation of Ministry of Education of China (Grand No. 14YJAZH095), the National Natural Science Foundation of China (Grand No. 61374081), the Natural Science Foundation of Guangdong Province (S2013010013034, 2015A030313485).

References

[1] V. Barbu, T. Precupanu, *Convexity and optimization in Banach spaces*, Editura Academiei, Bucharest, (1978). 2

[2] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems, **20** (2004), 103–120. 1

[3] C. E. Chidume, *Global iteration schemes for strongly pseudo-contractive maps*, Proc. Amer. Math. Soc., **126** (1998), 2641–2649. 1

[4] A. Cochocki, R. Unbehauen, *Neural Networks for Optimization and Signal Processing*, Wiley, New York, (1993). 1

[5] K. Deimling, *Zeros of accretive operators*, Manuscripta Math., **13** (1974), 365–374. 1.3

[6] R. Dewangan, B. S. Thakur, M. Postolache, *Strong convergence of asymptotically pseudocontractive semigroup by viscosity iteration*, Appl. Math. Comput., **248** (2014), 160–168. 1

[7] F. Facchinei, J. Pang, *Finite-Dimensional Variational Inequalities and Nonlinear Complementarity Problems*, Springer-Verlag, New York, (2003). 1

[8] P. Harker, J. Pang, *Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications*, Math. Program., **48** (1990), 161–220. 1

[9] P. Hartman, G. Stampacchia, *On some nonlinear elliptic differential functional equations*, Acta Math., **115** (1996), 271–310. 1

[10] G. Marino, H. X. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **318** (2006), 43–52. 1, 1, 2.4

[11] R. H. Martin, *Differential equations on closed subsets of a Banach space*, Trans. Amer. Math. Soc., **179** (1973), 399–414. 2

[12] C. Morales, *Pseudocontractive mappings and Leray Schauder boundary condition*, Comment. Math. Univ. Carolin., **20** (1979), 745–746. 1

[13] C. H. Morales, J. S. Jung, *Convergence of paths for pseudocontractive mappings in Banach spaces*, Proc. Amer. Math. Soc., **128** (2000), 3411–3419. 1.4

[14] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73** (1967), 591–597. 1

[15] M. O. Osilike, *Iterative solutions of nonlinear $\phi-$strongly accretive operator equations in arbitrary Banach spaces*, Nonlinear Anal. Ser. A: Theory Methods, **36** (1999), 1–9. 1

[16] W. Takahashi, *Nonlinear Functional Analysis–Fixed Point Theory and Its Applications*, Yokohama Publishers Inc., Yokohama, (2000). 1

[17] B. B. Thakur, R. Dewangan, M. Postolache, *Strong convergence of new iteration process for a strongly continuous semigroup of asymptotically pseudocontractive mappings*, Numer. Funct. Anal. Optim., **34** (2013), 1418–1431. 1

[18] B. S. Thakur, R. Dewangan, M. Postolache, *General composite implicit iteration process for a finite family of asymptotically pseudocontractive mappings*, Fixed Point Theory Appl., **2014** (2014), 15 pages. 1

[19] H. K. Xu, R. G. Ori, *An implicit iteration process for nonexpansive mappings*, Numer. Funct. Anal. Optim., **22** (2001), 767–773. 1.6

[20] Y. Yao, M. Postolache, S. M. Kang, *Strong convergence of approximated iterations for asymptotically pseudocontractive mappings*, Fixed Point Theory Appl., **2014** (2014), 13 pages. 1

[21] D. Youla, *On deterministic convergence of iterations of related projection operators*, J. Vis. Commun. Image Represent., **1** (1990), 12–20. 1

[22] S. S. Zhang, *Convergence theorem of common fixed points for Lipschitzian pseudo-contraction semigroups in Banach spaces*, Appl. Math. Mech., **30** (2009), 145–152. 1.5