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Ground state solutions for an asymptotically periodic and superlinear Schrödinger equation

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Abstract

We consider the semilinear Schrödinger equation

$$\begin{cases} - \triangle u + V(x)u = f(x, u), x \in R^N, \\ u \in H^1(R^N), \end{cases}$$

where $V(x)$ is asymptotically periodic and sign-changing, $f(x, u)$ is a superlinear, subcritical nonlinearity. Under asymptotically periodic $V(x)$ and a super-quadratic condition about $f(x, u)$. We prove that the above problem has a ground state solution which minimizes the corresponding energy among all nontrivial solutions.

Keywords: Schrödinger equation, ground state solutions, asymptotically periodic, sign-changing, super-quadratic condition.
2010 MSC: 46E20, 35J10.

1. Introduction and Preliminaries

Consider the following semilinear Schrodinger equation

$$\begin{cases} - \triangle u + V(x)u = f(x, u), \ x \in R^N, \\ u \in H^1(R^N), \end{cases} \tag{1.1}$$

where $V : R^N \to R$ and $f : R^N \times R \to R$ satisfy the following standard assumptions, respectively:
(V) $V \in C(R^N)$ is 1-periodic in each of $x_1, x_2, ..., x_N$, and

$$\sup[\sigma(-\Delta + V) \cap (-\infty, 0)] := \underline{\Lambda} < 0 < \overline{\Lambda} := inf[\sigma(-\Delta + V) \cap (0, \infty)]; \tag{1.2}$$

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(F1) $f \in C(R^N \times R)$ is 1-periodic in each of $x_1, x_2, ..., x_N$, $F(x,t) := \int_0^t f(x,s)ds \geq 0$;
(F2) $f(x,t) = o(|t|)$, as $|t| \to 0$, uniformly in $x \in R^N$;
(F3)$lim_{|t|\to\infty} \frac{|F(x,t)|}{|t|^2} = \infty, a.e. x \in R^N$;
(F4) there exists a $\theta_0 \in (0,1)$ such that

$$\frac{1-\theta^2}{2} f(x,t)t \geq \int_{\theta t}^{t} f(x,s)ds, \forall \theta \geq 0, (x,t) \in R^N \times R.$$

The existence of a nontrivial solution of (1.1) has been obtained in [5, 8, 12, 15, 22, 24] under some other standard assumptions of V and f. In some very recent papers, under the above conditions, Xianhua Tang [18, 19, 20, 21] proved the problem (1.1) has a ground state solution of Nehari-Pankov type.

If $V(x)$ is asymptotically periodic, not only that the functional Φ loses the Z^N-translation invariance, but also the operator $-\Delta + V$ has also discrete spectrum except for continuous spectrum. For the periodic problem, it is very crucial to show (PS)-sequence or (C)-sequence is bounded that the operator $-\Delta + V$ has only continuous spectrum [16]. For the above knowledge, there are no existence results for (1.1) when $V(x)$ is asymptotically periodic and sign-changing and $f(x,u)$ is asymptotically linear as $|u| \to \infty$. Motivated by the works [4, 7, 21], in this paper, we will use some tricks introduced in [9, 10] to overcome the difficulties caused by the dropping of periodicity of $V(x)$.

Before presenting our theorem, we make the following assumptions.
(V1) $V(x) = V_0(x) + V_1(x), V_0 \in C(R^N) \cap L^\infty(R^N)$ and

$$\sup[\sigma(-\Delta + V_0) \cap (-\infty, 0)] < 0 < \overline{\Lambda} := \inf[\sigma(-\Delta + V_0) \cap (0, \infty)], \tag{1.3}$$

$V_1 \in C(R^N)$ and $lim_{|x|\to\infty} V_1(X) = 0$;
(V2) V_0 is 1-periodic in each of $x_1, x_2, ..., x_N$, and

$$0 \leq -V_1(x) \leq \sup_{R^N}(-V_1) < \overline{\Lambda}, \forall x \in R^N.$$

Let $\mathcal{A}_0 = -\Delta + V_0$. Then $\mathcal{A}_0$ is self-adjoint in $L^2(R^N)$ with domain $\mathcal{D}(\mathcal{A}_0) = H^2(R^N)$ (see [2], Theorem 4.26). Let $\{\varepsilon(\lambda) : -\infty < \lambda < +\infty\}$ and $|\mathcal{A}_0|$ be the spectral family and the absolute value of $\mathcal{A}_0$, respectively, and $|\mathcal{A}_0|^{1/2}$ be the square root of $|\mathcal{A}_0|$. Set $\mathcal{U} = id - \varepsilon(0) - \varepsilon(0-)$. Then $\mathcal{U}$ commutes with $\mathcal{A}_0$ (see [1], Theorem IV 3.3). Let

$$E = \mathcal{D}(|\mathcal{A}_0|^{1/2}), \qquad E^- = \varepsilon(0)E, \qquad E^+ = [id - \varepsilon(0)]E. \tag{1.4}$$

For any $u \in E$, it is easy to see that $u = u^- + u^+$, where

$$u^- := \varepsilon(0)u \in E^-, \qquad u^+ := [id - \varepsilon(0)]u \in E^+ \tag{1.5}$$

and

$$\mathcal{A}_0 u^- = -|\mathcal{A}_0|u^-, \qquad \mathcal{A}_0 u^+ = |\mathcal{A}_0|u^+ \quad \forall u \in E \cap \mathcal{D}(\mathcal{A}_0). \tag{1.6}$$

Define an inner product

$$(u,v) = (|\mathcal{A}_0|^{1/2}u, \mathcal{A}_0|^{1/2}v)_{L^2}, \quad u, v \in E \tag{1.7}$$

and the corresponding norm

$$\|u\| = \left\||\mathcal{A}_0|^{1/2}u\right\|_2, \quad u \in E, \tag{1.8}$$

where$(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(R^N)$ and $\|\cdot\|$ denotes the norm of $L^s(R^N)$. By (V1), $E = H^1(R^N)$ with equivalent norms. Therefore, E embeds continuously in $L^s(R^N)$ for all $2 \leq s \leq 2^*$. In addition, one has the decomposition $E = E^- \oplus E^+$ orthogonal with respect to both $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$.

Let

$$\Phi(u) = \frac{1}{2}\int_{R^N} (|\nabla u|^2 + V(x)u^2)dx - \int_{R^N} F(x,u)dx, \forall u \in E \tag{1.9}$$

and

$$\Phi_0(u) = \frac{1}{2}\int_{R^N}(|\nabla u|^2 + V_0(x)u^2)dx - \int_{R^N} F(x,u)dx \ \ \forall u \in E, \tag{1.10}$$

where $F(x,t) = \int_0^t f(x,s)ds$. Then $\Phi_0(u)$ is also of class $C^1(E,R)$, and

$$\langle \Phi_0'(u), v\rangle = \int_{R^N}(\nabla u \nabla v + V_0(x)uv)dx - \int_{R^N} f(x,u)vdx \quad \forall u,v \in E. \tag{1.11}$$

In view of (1.5), (1.8) and (1.10), we have

$$\Phi_0(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{R^N} F(x,u)dx \tag{1.12}$$

and

$$\langle \Phi_0'(u), u\rangle = \|u^+\|^2 - \|u^-\|^2) - \int_{R^N} f(x,u)udx \quad \forall u = u^- + u^+ \in E. \tag{1.13}$$

Now, we are in a position to state the main result of this paper.

Theorem 1.1. *Assume that V and f satisfy (V1), (V2), (F1), (F2), (F3) and (F4). Then problem* (1.1) *has a nontrivial solution $u_0 \in E$. such that $\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi > 0$, where*

$$\mathcal{N}^- = \{u \in E\backslash E^- : \langle \Phi'(u), u\rangle = \langle \Phi'(u), v\rangle = 0 \quad \forall v \in E^-\}. \tag{1.14}$$

The set $\mathcal{N}^-$ was first introduced by Pankov [13, 14]*, which is a subset of the Nehari manifold*

$$\mathcal{N} = \{u \in E\backslash\{0\} : \langle \Phi'(u), u\rangle = 0\}. \tag{1.15}$$

The rest of this paper is organized as follows. In Section 2, some preliminary results and the proofs of Theorem 1.1 are presented.

2. Main results

Let X be a real Hilbert space with $X = X^- \bigoplus X^+$ and $X^- \perp X^+$. For a functional $\varphi \in C^1(X,R)$, φ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in X one has $\varphi(u) \le \liminf_{n\to\infty} \varphi(u_n)$, and φ' is said to be weakly sequentially continuous if $\lim_{n\to\infty}\langle \varphi'(u_n), v\rangle = \langle \varphi'(u), v\rangle$ for each $v \in X$.

Lemma 2.1 ([3], [6], [7])**.** *Let X be a real Hilbert space with $X = X^- \bigoplus X^+$ and $X^- \perp X^+$, and let $\varphi \in C^1(X,R)$ of the form*

$$\varphi(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied:

(LS1) $\psi \in C^1(X,R)$ is bounded from below and weakly sequentially lower semi-continuous;

(LS2) ψ' is weakly sequentially continuous;

(LS3) there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that

$$k := \inf \varphi(S_\rho^+) > \sup \varphi(\partial Q),$$

where

$$S_\rho^+ = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{w + se : w \in X^1, s \ge 0, \|w + se\| \le r\}.$$

Then for some $c \in [k, \sup \Phi(Q)]$, there exists a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{2.1}$$

Such a sequence is called a Cerami sequence on the level c or a $(C)_c$ sequence.

We set

$$\Psi(u) = \int_{R^N}[-V_1(x)u^2 + F(x,u)]dx \quad \forall u \in E. \tag{2.2}$$

Lemma 2.2. *Suppose that (V1), (V2), (F1), (F2) and (F3) are satisfied. Then Ψ is nonnegative, weakly sequentially lower semi-continuous, and Ψ' is weakly sequentially continuous.*

Using Sobolev's embedding theorem, one can check easily the above lemma, so we omit the proof.

Lemma 2.3. *Suppose that (V1), (F1), (F2), (F3) and (F4) are satisfied. Then for $u \in \mathcal{N}^-$,*

$$\begin{aligned}\Phi(u) \geq \Phi(tu+w) + \frac{1}{2}\|w\|^2 - \frac{1}{2}\int_{R^N} V_1(x)w^2 dx + \frac{1-t^2}{2}\langle \Phi'(u), u\rangle \\ - t\langle \Phi'(u), w\rangle \quad \forall u \in E,\ t \geq 0,\ w \in E^-.\end{aligned} \tag{2.3}$$

Proof. For any $x \in R^N$ and $\tau \neq 0$, (F4) yields

$$\frac{1-t^2}{2}\tau f(x,\tau) \geq \int_{t\tau}^{\tau} f(x,s)ds, \quad t \geq 0. \tag{2.4}$$

It follows that

$$\Big(\frac{1-t^2}{2}\tau - t\tau\Big) f(x,\tau) \geq \int_{t\tau+\sigma}^{\tau} f(x,s)ds, \quad t \geq 0, \quad \sigma \in R. \tag{2.5}$$

To show (2.5), we consider four possible cases. By virtue of (2.4) and $sf(x,s) \geq 0$, one has

Case 1) $0 \leq t\tau + \sigma \leq \tau$ or $t\tau + \sigma \leq \tau \leq 0$,

$$\int_{t\tau+\sigma}^{\tau} f(x,s)ds \leq \frac{f(x,\tau)}{|\tau|}\int_{t\tau+\sigma}^{\tau} |s|ds \leq \Big(\frac{1-t^2}{2}\tau - t\tau\Big) f(x,\tau);$$

Case 2) $t\tau + \sigma \leq 0 \leq \tau$,

$$\int_{t\tau+\sigma}^{\tau} f(x,s)ds \leq \int_{0}^{\tau} f(x,s)ds \leq \frac{f(x,\tau)}{|\tau|}\int_{t\tau+\sigma}^{\tau} |s|ds \leq \Big(\frac{1-t^2}{2}\tau - t\tau\Big) f(x,\tau);$$

Case 3) $0 \leq \tau \leq t\tau + \sigma$ or $\tau \leq t\tau + \sigma \leq 0$,

$$\int_{\tau}^{t\tau+\sigma} f(x,s)ds \geq \frac{f(x,\tau)}{|\tau|}\int_{\tau}^{t\tau+\sigma} |s|ds \geq -\Big(\frac{1-t^2}{2}\tau - t\tau\Big) f(x,\tau);$$

Case 4) $\tau \leq 0 \leq t\tau + \sigma$,

$$\int_{\tau}^{t\tau+\sigma} f(x,s)ds \geq \int_{\tau}^{0} f(x,s)ds \geq \frac{f(x,\tau)}{|\tau|}\int_{\tau}^{0} |s|ds \geq -\Big(\frac{1-t^2}{2}\tau - t\tau\Big) f(x,\tau).$$

The above four cases show that (2.5) holds.

We let $b : E \times E \to R$ denote the symmetric bilinear from given by

$$b(u,v) = \int_{R^N} (\triangledown u \triangledown v + V(x)uv)dx \qquad \forall u, v \in E. \tag{2.6}$$

By virtue of (1.9) and (2.6), one has

$$\Phi(u) = \frac{1}{2}b(u,u) - \int_{R^N} F(x,u)dx \qquad \forall u \in E \tag{2.7}$$

and

$$\langle \Phi'(u), v\rangle = b(u,v) - \int_{R^N} f(x,u)vdx \qquad \forall u, v \in E. \tag{2.8}$$

Thus, by (1.9), (1.11), (2.6), (2.7) and (2.8), one has

$$\begin{aligned}
\Phi(u)-\Phi(tu+w) &= \frac{1}{2}[b(u,u)-b(tu+w,tu+w)]+\int_{R^N}[F(x,tu+w)-F(x,u)]dx \\
&= \frac{1-t^2}{2}b(u,u)-tb(u,w)-\frac{1}{2}b(w,w)+\int_{R^N}[F(x,tu+w)-F(x,u)]dx \\
&= -\frac{1}{2}b(w,w)+\frac{1-t^2}{2}\langle\Phi'(u),u\rangle-t\langle\Phi'(u),w\rangle \\
&\quad+\int_{R^N}[\frac{1-t^2}{2}f(x,u)u-tf(x,u)w-\int_{tu+w}^{u}f(x,s)ds]dx \\
&= \frac{1}{2}\|w\|_0^2-\frac{1}{2}\int_{R^N}V_1(x)w^2dx+\frac{1-t^2}{2}\langle\Phi'(u),u\rangle-t\langle\Phi'(u),w\rangle \\
&\quad+\int_{R^N}[\frac{1-t^2}{2}f(x,u)u-tf(x,u)w-\int_{tu+w}^{u}f(x,s)ds]dx \\
&\geq \frac{1}{2}\|w\|_0^2-\frac{1}{2}\int_{R^N}V_1(x)w^2dx+\frac{1-t^2}{2}\langle\Phi'(u),u\rangle-t\langle\Phi'(u),w\rangle \quad \forall t\geq 0, \quad w\in E^-.
\end{aligned}$$

This shows that (2.3) holds. □

Lemma 2.4. *Suppose that (V1), (F1), (F2) and (F4) are satisfied. Then there exists $\rho>0$ such that*

$$m:=\inf_{\mathcal{N}^-}\Phi\geq\kappa:=inf\{\Phi(u):u\in E^+,\|u\|=\rho\}>0. \tag{2.9}$$

Lemma 2.4 can be proved in the same way as ([17], Lemmas 2.4).

Lemma 2.5. *Suppose that (V1), (F1), (F2) and (F3) are satisfied. Let $e\in E^+$ with $\|e\|=1$. Then there is a $r_0>0$ such that $\sup\Phi(\partial Q)\leq 0$, where*

$$Q=\{se+w:w\in E^-,s\geq 0,\|se+w\|\leq r_0\}. \tag{2.10}$$

Proof. (F1) yields that $F(x,t)\geq 0$ for $(x,t)\in R^N\times R$, so we have $\Phi(u)\leq 0$ for $u\in E^-$. Next, it is sufficient to show that $\Phi(u)\to-\infty$ as $u\in E^-\oplus Re, \|u\|\to\infty$. Arguing indirectly, assume that for some sequence $\{w_n+s_ne\}\subset E^-\oplus Re$ with $\|w_n+s_ne\|\to\infty$, there is $M>0$ such that $\Phi(w_n+s_ne)\geq -M$ for all $n\in N$. Set $v_n=(w_n+s_ne)/\|w_n+s_ne\|=v_n^-+t_ne$, then $\|v_n^-+t_ne\|=1$. Passing to a subsequence, we may assume that $v_n\rightharpoonup v$ in E, then $v_n\to v$ *a.e.* on R^N, $v_n^-\to v^-$ in E, $t_n\to\bar{t}$, and

$$-\frac{M}{\|w_n+s_ne\|^2}\leq\frac{\Phi(w_n+s_ne)}{\|w_n+s_ne\|^2}=\frac{t_n^2}{2}-\frac{1}{2}\|v_n^-\|^2-\int_{R^N}\frac{F(x,w_n+s_ne)}{\|w_n+s_ne\|^2}dx. \tag{2.11}$$

If $\bar{t}=0$, then it follows from (2.11) that

$$0\leq\frac{1}{2}\|v_n^-\|^2+\int_{R^N}\frac{F(x,w_n+s_ne)}{\|w_n+s_ne\|^2}dx\leq\frac{t_n^2}{2}+\frac{M}{\|w_n+s_ne\|^2}\to 0,$$

which yields $\|v_n^-\|\to 0$, and so $1=\|v_n\|\to 0$, is a contradiction.

If $\bar{t}\neq 0$, then $v\neq 0$, it follows from (2.11), (F3) and Fatou's lemma that

$$\begin{aligned}
0&\leq\limsup_{n\to\infty}[\frac{t_n^2}{2}-\frac{1}{2}\|v_n^-\|^2-\int_{R^N}\frac{F(x,w_n+s_ne)}{\|w_n+s_ne\|^2}dx] \\
&=\limsup_{n\to\infty}[\frac{t_n^2}{2}-\frac{1}{2}\|v_n^-\|^2-\int_{R^N}\frac{F(x,w_n+s_ne)}{(w_n+s_ne)^2}v_n^2dx] \\
&\leq\frac{1}{2}\lim_{n\to\infty}t_n^2-\liminf_{n\to\infty}\int_{R^N}\frac{F(x,w_n+s_ne)}{(w_n+s_ne)^2}v_n^2dx] \\
&\leq\frac{\bar{t}^2}{2}-\int_{R^N}\liminf_{n\to\infty}\frac{F(x,w_n+s_ne)}{(w_n+s_ne)^2}v_n^2dx] \\
&=-\infty,
\end{aligned}$$

is a contradiction. □

Lemma 2.6. *Suppose that (V1), (F1), (F2), (F3) and (F4) are satisfied. Then there exist a constant $c \geq \kappa$ and a sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c \quad and \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{2.12}$$

Proof. Lemma 2.6 is a direct corollary of Lemmas 2.1, 2.2, 2.4 and 2.5. □

Lemma 2.7. *Suppose that (V1), (F1), (F2), (F3) and (F4) are satisfied. Then any sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c \quad and \quad \langle \Phi'(u_n), u_n^{\pm} \rangle \to 0, \tag{2.13}$$

is bounded in E.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\|_0 \to \infty$. Let $v_n = u_n/\|u_n\|_0$. Then $1 = \|v_n\|_0^2$. By Sobolev imbedding theorem, there exists a constant $C_4 > 0$ such that $\|v_n\|_2 \leq C_4$. Passing to a subsequence, we have $v_n \rightharpoonup \overline{v}$ in E. There are two possible cases: i)$\overline{v} = 0$ and ii)$\overline{v} \neq 0$.

Case i) $\overline{v} = 0, i.e.\ v_n \rightharpoonup 0$ in E. Then $v_n^+ \to 0$ and $v_n^- \to 0$ in $L^s_{loc}(R^N)$, $2 \leq s < 2^*$ and $v_n^+ \to 0$ and $v_n^- \to 0$ $a.e.$ on R^N. By (V1) and (V2), it is easy to show that

$$\lim_{n\to\infty} \int_{R^N} V_1(x)(v_n^+)^2 dx = \lim_{n\to\infty} \int_{R^N} V_1(x)(v_n^-)^2 dx = 0. \tag{2.14}$$

If

$$\delta := \limsup_{n\to\infty} \sup_{y\in R^N} \int_{B_1(y)} |v_n^+|^2 dx = 0,$$

then by Lion's concentration compactness principle [11] or ([23], Lemma 1.21), $v_n^+ \to 0$ in $L^s(R^N)$ for $2 < s < 2^*$. Fix $R > [2(1 + c_*)]^{1/2}$. By virtue of (F0) and (F1), for $\epsilon = 1/4(RC_4)^2 > 0$, there exists $C_\epsilon > 0$ such that (1.12) holds. Hence, it follows that

$$\begin{aligned} \limsup_{n\to\infty} \int_{R^N} F(x, Rv_n^+) dx &\leq \limsup_{n\to\infty} [\epsilon R^2 \|v_n^+\|_2^2 + C_\epsilon R^P \|v_n^+\|_P^P] \\ &\leq \epsilon (RC_4)^2 = \frac{1}{4}. \end{aligned} \tag{2.15}$$

Let $t_n = R/\|u_n\|_0$. Hence, by virtue of (2.10) and (2.11), one can get that

$$\begin{aligned} c_* + o(1) &= \Phi(u_n) \\ &\geq \frac{t_n^2}{2}\|u_n\|_0^2 - \int_{R^N} F(x, t_n u_n^+) dx + \frac{1 - t_n^2}{2} \langle \Phi'(u_n), u_n \rangle \\ &\quad + t_n^2 \langle \Phi'(u_n), u_n^- \rangle + \frac{t_n^2}{2} \int_{R^N} V_1(x)[(u_n^+)^2 - (u_n^-)^2] dx \\ &= \frac{R^2}{2}\|v_n\|_0^2 - \int_{R^N} F(x, Rv_n^+) dx + (\frac{1}{2} - \frac{R^2}{2\|u_n\|_0^2}) \langle \Phi'(u_n), u_n \rangle \\ &\quad + \frac{R^2}{\|u_n\|^2} \langle \Phi'(u_n), u_n^- \rangle + \frac{R^2}{2} \int_{R^N} V_1(x)[(v_n^+)^2 - (v_n^-)^2] dx \\ &\geq \frac{R^2}{2} - \int_{R^N} F(x, Rv_n^+) dx + o(1) \\ &\geq \frac{R^2}{2} - \frac{1}{4} + o(1) > c_* + \frac{3}{4} + o(1). \end{aligned}$$

This contradiction shows that $\delta > 0$.

Passing to a subsequence, we may assume the existence of $\kappa_n \in Z^N$ such that $\int_{B_{1+\sqrt{N}}(\kappa_n)} |v_n^+|^2 dx > \frac{\delta}{2}$. Let $w_n(x) = v_n(x+\kappa_n)$. Since $V_0(x)$ is 1-periodic in each of $x_1, x_2, ..., x_N$. Then

$$\int_{B_{1+\sqrt{N}}(0)} |w_n^+|^2 dx > \frac{\delta}{2}. \tag{2.16}$$

Now we define $\tilde{u_n}(x) = u_n(x+\kappa_n)$, then $\tilde{u_n}/\|u_n\|_0 = w_n$ and $\|w_n\|_0 = \|v_n\|_0 = 1$. Passing to a subsequence, we have $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^s_{loc}(R^N)$, $2 \le s < 2^*$ and $w_n \to w$ a.e. on R^N. Obviously we have $w \neq 0$. Hence, it follows from (2.17), (F4) and Fatou's lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c_* + o(1)}{\|u_n\|_0^2} = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|_0^2} \\
&= \lim_{n\to\infty} \Big[\frac{1}{2}(\|v_n^+\|_0^2 - \|v_n^-\|_0^2) + \frac{1}{2}\int_{R^N} V_1(x)[(v_n^+)^2 - (v_n^-)^2]dx \\
&\quad - \int_{R^N} \frac{F(x+\kappa_n, \tilde{u_n})}{\tilde{u_n}^2} w_n^2 dx\Big] \\
&\le \frac{1}{2} - \liminf_{n\to\infty} \int_{R^N} \frac{F(x+\kappa_n, \tilde{u_n})}{\tilde{u_n}^2} w_n^2 dx \le \frac{1}{2} - \int_{R^N} \liminf_{n\to\infty} \frac{F(x+\kappa_n, \tilde{u_n})}{\tilde{u_n}^2} w_n^2 dx \\
&= -\infty,
\end{aligned}$$

which is a contradiction.

Case ii) $\overline{v} \neq 0$. In this case, we can also deduce a contradiction by a standard argument.

Case i)and ii) show that $\{u_n\}$ is bounded in E. □

Proof of Theorem 1.1. Applying Lemmas 2.6 and 2.7, we deduce that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (2.8). Passing to a subsequence, we have $u_n \rightharpoonup \overline{u}$ in E. Next, we prove $\overline{u} \neq 0$.

Arguing by contradiction, suppose that $\overline{u} = 0$, *i.e.* $u_n \rightharpoonup \overline{0}$ in E, and so $u_n \to 0$ in $L^s_{loc}(R^N)$, $2 \le s < 2^*$ and $u_n \to 0$ a.e. on R^N. By (V1) and (V2), it is easy to show that

$$\lim_{n\to\infty} \int_{R^N} V_1(x) u_n^2 dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{R^N} V_1(x) u_n v dx = 0 \quad \forall v \in E. \tag{2.17}$$

Note that

$$\Phi_0(u) = \Phi(u) - \frac{1}{2}\int_{R^N} V_1(x) u^2 dx \quad \forall u \in E \tag{2.18}$$

and

$$\langle \Phi_0'(u), v\rangle = \langle \Phi'(u), v\rangle - \int_{R^N} V_1(x) uv dx \quad \forall u, v \in E. \tag{2.19}$$

From (2.10)-(2.13), one can get that

$$\Phi_0(u_n) \to c_* \quad \text{and} \quad \|\Phi_0'(u_n)\|(1+\|u_n\|) \to 0. \tag{2.20}$$

Passing to a subsequence, we may assume the existence of $\kappa_n \in Z^N$ such that $\int_{B_{1+\sqrt{N}}(\kappa_n)} |u_n|^2 dx > \frac{\delta}{2}$ for some $\delta > 0$. Let $v_n(x) = u_n(x+\kappa_n)$. Then $\|v_n\|_0 = \|u_n\|_0$ and

$$\int_{B_{1+\sqrt{N}}(0)} |v_n|^2 dx > \frac{\delta}{2}. \tag{2.21}$$

Passing to a subsequence, we have $v_n \rightharpoonup \overline{v}$ in E, $v_n \to \overline{v}$ in $L^s_{loc}(R^N)$, $2 \le s < 2^*$ and $v_n \to \overline{v}$ *a.e.* on R^N. Obviously, (2.21) implies that $\overline{v} \neq 0$. Since $V_0(x)$ and $f(x,u)$ are periodic in x, then by (2.14), we have

$$\Phi_0(v_n) \to c_* \quad \text{and} \quad \|\Phi_0'(v_n)\|(1+\|v_n\|) \to 0. \tag{2.22}$$

By a standard argument, one has $\Phi'(\overline{v}) = 0$. This shows that $\overline{v} \in \mathcal{N}^-$ and so $\Phi(\overline{v}) \geq m$. On the other hand, by using (2.22), (F4) and Fatou's lemma, we have

$$\begin{aligned} m \geq c_* &= \lim_{n\to\infty} \big[\Phi(v_n) - \frac{1}{2}\langle \Phi'(v_n), v_n\rangle\big] = \lim_{n\to\infty} \int_{R^N} \big[\frac{1}{2} f(x, v_n)v_n - F(x, v_n)\big] dx \\ &\geq \int_{R^N} \lim_{n\to\infty} \big[\frac{1}{2} f(x, v_n)v_n - F(x, v_n)\big] dx \\ &= \Phi(\overline{v}) - \frac{1}{2}\langle \Phi'(\overline{v}), \overline{v}\rangle = \Phi(\overline{v}). \end{aligned}$$

This shows that $\Phi(\overline{v}) \leq m$ and so $\Phi(\overline{v}) = m = \inf_{\mathcal{N}^-} \Phi > 0$. □

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References

[1] D. E. Edmunds, W. D. Evans, *Spectral Theory and Differential Operators*, Oxford Clarendon Press, New York, (1987). 1
[2] Y. Egorov, V. Kondratiev, *On Spectral Theory of Elliptic Operators*, Birkhäuser Verlag, Basel, (1996). 1
[3] Y. Ding, *Variational Methods for Strongly Indefinite Problems*, World Scientific Publishing Co., Hackensack, (2007). 2.1
[4] Y. Ding, C. Lee, *Multiple solutions of Schrödinger equations with indefinite linear part and super or asymptotically linear terms*, J. Differential Equations, **222** (2006), 137–163. 1
[5] Y. Ding, A. Szulkin, *Bound states for semilinear Schrödinger equations with sign-changing potential*, Calc. Var. Partial Differential Equations, **29** (2007), 397–419. 1
[6] W. Kryszewski, A. Suzlkin, *Generalized linking theorem with an application to a semilinear Schrödinger equation*, Adv. Differential Equations, **3** (1998), 441–472. 2.1
[7] G. B. Li, A. Szulkin, *An asymptotically periodic Schrödinger equation with indefinite linear part*, Commun Contemp Math., **4** (2002), 763–776. 1, 2.1
[8] Y. Li, Z. Q. Wang, J. Zeng, *Ground states of nonlinear Schrödinger equations with potentials*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **23** (2006), 829–837. 1
[9] X. Lin, X. H. Tang, *Semiclassical solutions of perturbed p-Laplacian equations with critical nonlinearity*, J. Math. Anal. Appl., **413** (2014), 438–449. 1
[10] X. Lin, X. H. Tang, *Nehari-type ground state solutions for superlinear asymptotically periodic Schrödinger equation*, Abstr. Appl. Anal., **2014** (2014), 7 pages. 1
[11] P. L. Lions, *The concentration-compactness principle in the calculus of variations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **1** (1984), 223–283. 2
[12] S. Liu, *On superlinear Schrödinger equations with periodic potential*, Calc. Var. Partial Differential Equations, **45** (2012), 1–9. 1
[13] A. Pankov, *Periodic nonlinear Schrödinger equations with application to photonic crystals*, Milan. J. Math., **73** (2005), 259–287. 1.1
[14] A. Pankov, *Periodic nonlinear Schrödinger equation with application to photonic crystals*, Milan. J. Math., **73** (2005), 259–287. 1.1
[15] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys., **43** (1992), 270–291. 1
[16] M. Schechter, *Superlinear Schrödinger operators*, J. Funct Anal., **262** (2012), 2677–2694. 1
[17] A. Szulkin, T. Weth, *Ground state solutions for some indefinite variational problems*, J. Funct. Anal., **257** (2009), 3802–3822. 2
[18] X. H. Tang, *Infinitely many solutions for semilinear Schrödinger equations with sign-changing potential and nonlinearity*, J. Math. Anal. Appl., **401** (2013), 407–415. 1
[19] X. H. Tang, *New conditions on nonlinearity for a periodic Schrödinger equation having zero as spectrum*, J. Math. Anal. Appl., **413** (2014), 392–410. 1
[20] X. H. Tang, *New super-quadratic conditions on ground state solutions for superlinear Schrödinger equation*, Adv. Nonlinear Stud., **14** (2014), 361–373. 1
[21] X. H. Tang, *Non-Nehari manifold method for asymptotically periodic Schrödinger equations*, Sci China Math., **58** (2015), 715–728. 1
[22] C. Troestler, M. Willem, *Nontrivial solution of a semilinear Schrödinger equation*, Comm. Partial Differential Equations, **21** (1996), 1431–1449. 1
[23] M. Willem, *Minimax Theorems*, Brikhauser Boston, Boston, (1996). 2
[24] M. Yang, *Ground state solutions for a periodic periodic Schrödinger equation with superlinear nonlinearities*, Nonlinear. Anal., **72** (2010), 2620–2627. 1