Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 1440–1452
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Solution to an ice melting cylindrical problem

Abdellatif Boureghda

Laboratoire de Mathématiques, Informatique et Applications, Université de Haute Alsace, Faculté des Sciences et Techniques 4-6, rue des Frères Lumière 68093, Mulhouse Cedex, France.

Department of Mathematics, Ferhat Abbas University, Sétif 1, Algeria.

Communicated by M. Jleli

Abstract

We give a solution to an ice melting cylindrical problem using the "modified variable time step method", earlier suggested by the author. New numerical techniques are proposed for the one-dimensional melting problem. The numerical results are obtained for the position of the moving boundary, time and temperatures.

Keywords: Stefan problems, phase changes, moving boundary problems, partial differential equations, finite difference methods, heat equation.
2010 MSC: 35R35, 35R37, 80A22, 65M06, 65N06, 35K05.

1. Introduction

A variety of diffusion problems with moving boundaries are described in Ockendon and Hodgkins [23], where we observe that analytical solutions are possible for some problems and arise when there is a singularity in the region of solution or on the boundary. Fox in [14, 23] mentions the need for a short time analytical solution in the case of discontinuous agreement of initial and boundary conditions. But the general approach should be necessarily numerical. Existence of solution was proved by Evans [13], while the uniqueness was proved by Douglas [11]. Crank [9] discusses finite difference methods for the classical one-dimensional diffusion problem, which include variable space steps and variable time steps and a change of space variable to fix the moving boundary. Other methods for the numerical solution of one-dimensional problem with moving boundary problems are compared by Furzeland [16]. More references to this problem may be found in the thesis numerical methods for solving one-dimensional problems with a moving boundary [2], in the thesis moving boundary value problems [5] and in [15, 23]. Several numerical methods have been developed

Email address: `abdellatif.boureghda@uha.fr` (Abdellatif Boureghda)

Received 2015-06-26

to solve Stefan problems. Crank [10] provides a good introduction to the Stefan problems and presents an elaborate collection of numerical methods used for these problems. Murray and Landis [22] compared an adapted grid procedure with a fixed grid and showed that the adaptive grid method captures more accurately the interface position, whereas the fixed grid algorithm gives a more precise heat distribution in the whole domain. However, such problems are highly nonlinear and their exact solutions seem to be inconceivable except in some very simple cases. The exact solutions of the one-dimensional Stefan problems have been surveyed by Hill [17]. Asaithambi [1] used a coordinate transformation, a fixed time step and solved a system of initial-value problems for each time level. Both these methods where illustrated by solving a Stefan problem involving a Dirichlet condition at the fixed boundary. Kutluay and Esen [12] applied an enthalpy formulation based on suitable finite difference approximations to one-dimensional Stefan problem with a Neumann condition at the fixed boundary, suggested by Hoffmann [18]. Kutluay and Esen [21] present a numerical scheme based on an isotherm migration formulation for one-dimensional, one-phase Stefan problem with a time dependent Neumann condition on the fixed boundary and a constant Dirichlet condition on the moving boundary. Caldwell and Kwan [7], describe and compare several effective methods for the numerical solution of one-dimensional Stefan problems for simple geometries including plane, cylindrical and spherical ones. Kutluay [20] applied variable space grid and boundary immobilization techniques based on the explicit finite difference to the one-phase Stefan-like problems with forcing term. Caldwell and Kwan [8] developed starting solutions for the boundary immobilization method (BIM) for outward spherical and cylindrical solidifications. The positions of the moving boundary, obtained by the method, are compared with those obtained by the perturbation method. Javierre, Vuik, Vermolen and Zwaag [19], presented a critical comparison of suitability of several numerical methods, level set, moving grid and phase field model, to address two well known Stefan problems in phase transformation studies: melting of pure phase and diffusional solid state phase transformation in a binary system. More recently, a method dealing with the one-dimensional Stefan problem is developed in [3, 4, 6]. In the present paper we describe a numerical solution to an ice melting cylindrical problem using a new techniques in the modified variable time step method.

2. Cylindrical problem

The nondimensional form of the heat conduction equation in three dimensions is $u_t = \nabla^2 u$ which, in cylindrical polar coordinates (x, θ, z) is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{x\partial x} + \frac{\partial^2 u}{x^2\partial\theta^2} + \frac{\partial^2 u}{\partial z^2}. \tag{2.1}$$

For simplicity, assuming that u is independent of z, this reduces to the two-dimensional equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{x\partial x} + \frac{\partial^2 u}{x^2\partial\theta^2}. \tag{2.2}$$

Finally, assuming that u is independent of θ, this leads to the simplest equation with cylindrical symmetry

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{x\partial x}, \tag{2.3}$$

where x is the radial space variable. There may be difficulties at the point $x = 0$, which we shall consider.

2.1. Method of solution

In the variable time step method we subdivided the whole region into a fixed space grid for all times, but choose each time step that the moving boundary coincides with a grid line in space at each time level. Let us suppose that during time t_j, the boundary has moved a distance $j.\delta x$ in successive time steps of δt_r, $r = 0, 1, 2, \ldots, j-1$, δt_r being the time taken by the boundary to move a distance δx from its position $x_r = r.\delta x$ to $x_{+1} = (r+1)\delta x$.

We assume that the values $u(x_i, t_j)$ are known. We wish to find δt_j where $t_{j+1} = t_j + \delta t_j$ such that the boundary moves a distance δx during that time.

This method has been proposed with the aim of avoiding the increased complication and loss of accuracy associated with unequal space intervals near the moving boundary. Finally $u(x_i, t_{j+1})$ may be computed by standard finite difference methods. The δt_j is improved using one of the following four different formula for calculating the time, proved in [6].

$$\delta t_j = \frac{(\delta x)^2}{u_{j,j+1}}; \tag{2.4}$$

$$\delta t_j = \frac{2\,(\delta x)^2}{u_{j-1,j} + u_{j,j+1}}; \tag{2.5}$$

$$\delta t_j = \frac{(\delta x)^2}{-1 + \sqrt{1 + 2u_{j,j+1}}}; \tag{2.6}$$

$$\delta t_j = \frac{2\,(\delta x)^2}{-2 + \sqrt{1 + 2u_{j-1,j}} + \sqrt{1 + 2u_{j,j+1}}}. \tag{2.7}$$

2.2. Finite difference representation of the equation (2.3)

There is an apparent difficulty in representing the term $\frac{\partial u}{x \partial x}$ at $x = 0$, but this is eliminated by noting that there is symmetry about the line $x = 0$ i.e., $u(x) = u(-x)$. Hence, we can assume an expansion about $x = 0$,

$$u = u_0 + a_2 x^2 + a_4 x^4 + \cdots ,$$

i.e.,

$$\frac{\partial u}{\partial x} = 2a_2 x + 4a_4 x^3 + \cdots ,$$
$$\frac{\partial^2 u}{\partial x^2} = 2a_2 + 12a_4 x^2 + \cdots .$$

Thus at $x = 0$, $\frac{\partial u}{x \partial x} = \frac{\partial^2 u}{\partial x^2}$ and the equation is

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}. \tag{2.8}$$

The backward finite difference representation is

$$u_{0,t} - u_{0,t-\delta t} = 2k(u_{1,t} - 2u_{0,t} + u_{-1,t}) \tag{2.9}$$

and

$$u_{1,t} = u_{-1,t},$$

where

$$k = \frac{\delta t}{(\delta x)^2}.$$

This leads to

$$u_{0,t} - u_{0,t-\delta t} = 4k(u_{1,t} - u_{0,t}) \tag{2.10}$$

or

$$(1 + 4k)u_{0,t} - 4ku_{1,t} = u_{0,t-\delta t}.$$

2.3. Singularity

Consider

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{x \partial x} = \frac{\partial u}{\partial t}. \tag{2.11}$$

This has a singularity when $x = 0$, provided $\frac{du}{dx} \neq 0$ at $x = 0$ taking the case

$$\frac{\partial u}{\partial t} = c \text{ (constant)},$$

$$\frac{d^2u}{dx^2} + \frac{du}{xdx} = c, \tag{2.12}$$

$$x^{-1}\frac{d}{dx}\left[x\left(\frac{du}{dx}\right)\right] = c, \tag{2.13}$$

$$\frac{d}{dx}\left[x\left(\frac{du}{dx}\right)\right] = cx, \tag{2.14}$$

with solution

$$u = \frac{cx^2}{4} + b\ln(x) + a, \quad b \neq 0, \tag{2.15}$$

where a, b are constants. Hence, $u \longrightarrow \infty$ as $x \longrightarrow 0$ and this can cause a difficulty, with inaccurate approximations for small x.

3. Two cylindrical problems

We consider a unit cylinder at a temperature greater than zero and the ice either outside or inside the cylinder. Thus, we have two distinct problems.

3.1. Ice outside cylinder

The equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{x\partial x} = \frac{\partial u}{\partial t} \tag{3.1}$$

with initial conditions

$$(t = 0), \qquad u = 0, \qquad x > 1$$

and boundary conditions:

- on the fixed boundary $(x = 1)$

$$\frac{\partial u}{\partial x} = -1, \qquad t > 0;$$

- on the moving boundary $(x = s(t) > 1)$

$$u = 0,$$

$$\frac{dx}{dt} = -\frac{\partial u}{\partial x}. \tag{3.2}$$

In the solution to a cylindrical problem, the same condition employed in the cartesian problem solved in [6] was used, hence the following implicit formula can be utilized

$$\left(\frac{1}{\delta t}\right)\nabla_t u_{0,t} = \left[\frac{1}{(\delta x)^2}\right]\delta_x^2 u_{0,t} + (x\delta_x)^{-1}\mu_x\delta_x u_{0,t}, \tag{3.3}$$

i.e.,

$$u_{0,t} - u_{0,0} = \left[\frac{\delta t}{(\delta x)^2}\right]\left\{\delta_x^2 + \frac{\delta_x}{x}\mu_x\delta_x\right\}u_{0,t}, \tag{3.4}$$

or

$$u_{0,t} - u_{0,0} = k\left\{u_{1,t} - 2u_{0,t} + u_{-1,t} + \left(\frac{\delta x}{2x}\right)(u_{1,t} - u_{-1,t})\right\}, \tag{3.5}$$

$$(1+2k)u_{0,t} - k\left(1 + \frac{\delta_x}{2x}\right)u_{1,t} - k\left(1 - \frac{\delta_x}{2x}\right)u_{-1,t} = u_{0,0}, \tag{3.6}$$

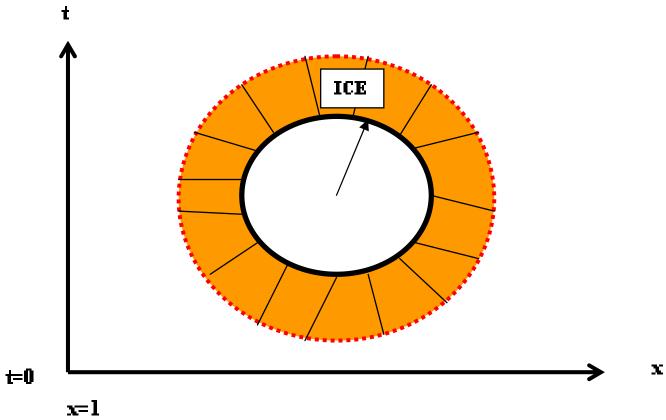

Figure 1: Ice outside cylinder.

i.e.,

$$(1+2k)u_{i,j+1} - k\left(1+\frac{\delta_x}{2x}\right)u_{i+1,j+1} - k\left(1-\frac{\delta_x}{2x}\right)u_{i-1,j+1} = u_{i,j}, \quad x \neq 0. \tag{3.7}$$

For melting ice outside the unit cylinder ($1 \leq x$, see Figure 1).

The point $0,0$ corresponds to $t=0, \quad x=1$.

The point i,j corresponds to $x = 1 + i\,\delta x$, $t = j\,\delta t$. The equation is

$$(1+2k^n)\,u^n_{i,j+1} - k^n\left(1+\frac{0.5}{\frac{1}{\delta x}+i}\right)u^n_{i+1,j+1} - k^n\left(1-\frac{0.5}{\frac{1}{\delta x}+i}\right)u^n_{i-1,j+1} = u^n_{i,j}, \tag{3.8}$$

where $k^n = \dfrac{\delta t^n_j}{(\delta x)^2}$, $i = 1,2,3,\ldots,j$ and n denotes n^{th} iteration

$$u_{0,j+1} = u_{1,j+1} + \delta x \tag{3.9}$$

on the boundary

$$u = 0$$

and so

$$u_{0,0} = 0, \quad u_{1,1} = 0,$$

and etc (see Figure 2). Also

$$u_{0,1} = \delta x.$$

3.2. Ice inside cylinder

The equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{x\partial x} = \frac{\partial u}{\partial t} \tag{3.10}$$

with initial conditions

$$(t=0), \qquad u=0, \qquad x<1$$

and boundary conditions:

- on the fixed boundary $(x=1)$

$$\frac{\partial u}{\partial x} = -1, \qquad t>0.$$

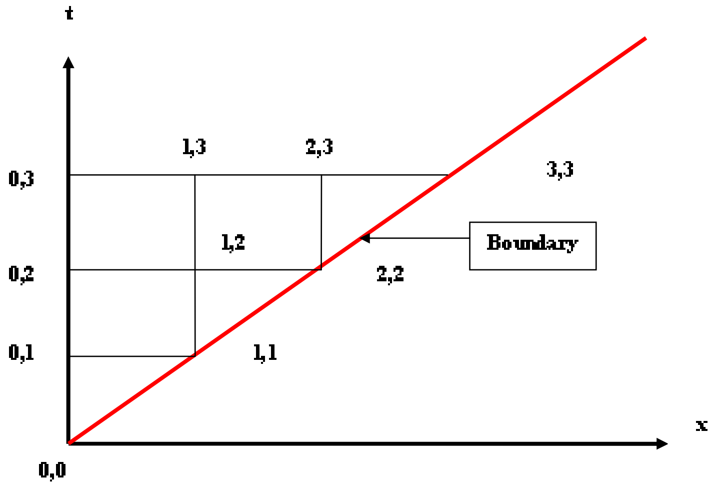

Figure 2: Grid system showing position of moving boundary.

- on the moving boundary ($x = s(t) < 1$)

$$u = 0,$$
$$\frac{dx}{dt} = -\frac{\partial u}{\partial x}. \tag{3.11}$$

For melting ice inside the unit cylinder (see Figure 3) we have:

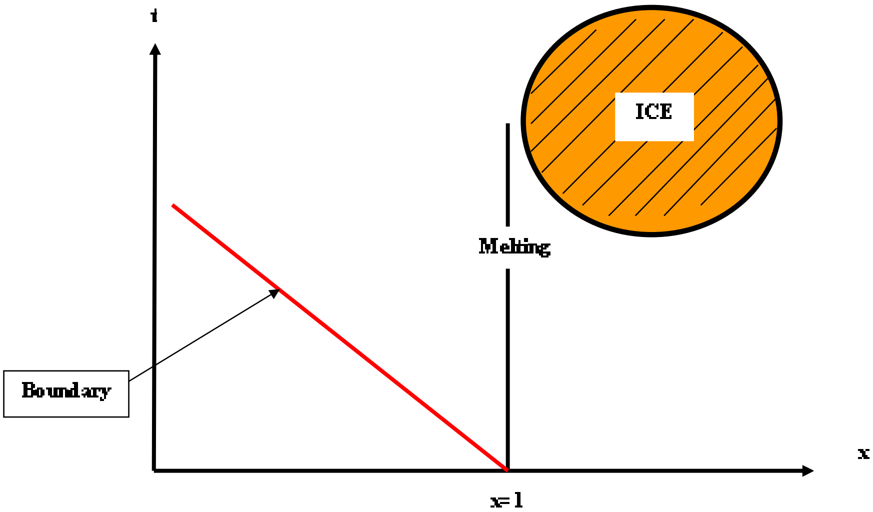

Figure 3: Ice inside cylinder.

the point $0, 0$ corresponds to $x = 1, \quad t = 0$.
the point i, j corresponds to $x = 1 - i\delta x, t = j\delta t$.
The equation is

$$-k^n\left(1+\frac{0.5}{\frac{1}{\delta x}-i}\right)u^n_{i-1,j+1}+(1+2k^n)\,u^n_{i,j+1}-k^n\left(1-\frac{0.5}{\frac{1}{\delta x}-i}\right)u^n_{i+1,j+1}=u^n_{i,j}, \tag{3.12}$$

$i \neq 0$, $\frac{1}{\delta x}$ changes slope when $i = \frac{1}{\delta x}$, i.e., $j = \frac{1}{\delta x} - 1$, the same conditions for the initial value as for outside the cylinder.

3.3. Other numerical methods

Another technique which has been used to solve the cylindrical problem is to apply Douglas formula, see [5, 6], with central differences for the space derivatives and forward difference for the time derivative. The finite difference replacement of (2.11) may then be written as:

– outside cylinder

$$\left[-0.5\left(k^n-\frac{1}{6}\right)+\frac{0.5k^n}{\frac{1}{\delta x}+i}\right]u^n_{i,j-1}+\left(k^n+\frac{5}{6}\right)u^n_{i,j+1}+\left[-0.5\left(k^n-\frac{1}{6}\right)-\frac{0.5k^n}{\frac{1}{\delta x}+i}\right]u^n_{i+1,j+1}$$
$$=0.5\left(k^n+\frac{1}{6}\right)u^n_{i-1,j}-\left(k^n-\frac{5}{6}\right)u^n_{i,j}+0.5\left(k^n+\frac{1}{6}\right)u^n_{i+1,j}; \quad (3.13)$$

– inside cylinder

$$\left[-0.5\left(k^n-\frac{1}{6}\right)-\frac{0.5k^n}{\frac{1}{\delta x}-i}\right]u^n_{i-1,j+1}+\left(k^n+\frac{5}{6}\right)u^n_{i,j+1}+\left[-0.5\left(k^n-\frac{1}{6}\right)+\frac{0.5k^n}{\frac{1}{\delta x}-i}\right]u^n_{i+1,j+1}$$
$$=0.5\left(k^n+\frac{1}{6}\right)u^n_{i-1,j}-\left(k^n-\frac{5}{6}\right)u^n_{i,j}+0.5\left(k^n+\frac{1}{6}\right)u^n_{i+1,j}. \quad (3.14)$$

4. Numerical solution to cylindrical problems

The computation and the permitted error are the same as in the Cartesian problem. Each computation has been made until the boundary $s = 1$ is reached.

An error of 0.5% has been allowed in the calculation of δt i.e., we iterate until two successive values of δt differ by not more than 0.5%. The iterations are required with $\delta x = 0.01$ and $\delta x = 0.1$. Various results are given in Tables 1–8 using this method and a new replacement of (2.11) is given as in (3.7), (3.8), (3.12), (3.13) and (3.14). Furthermore, in all our methods we solve the problems by using one of the four formula (2.4)–(2.7) for δt. Figure 4 shows the position of the moving boundary, as calculated by this method and $\delta x = 0.01$. Figure 5 shows the temperature on fixed surface $x = 0$, increases with time as expected and $\delta x = 0.01$.

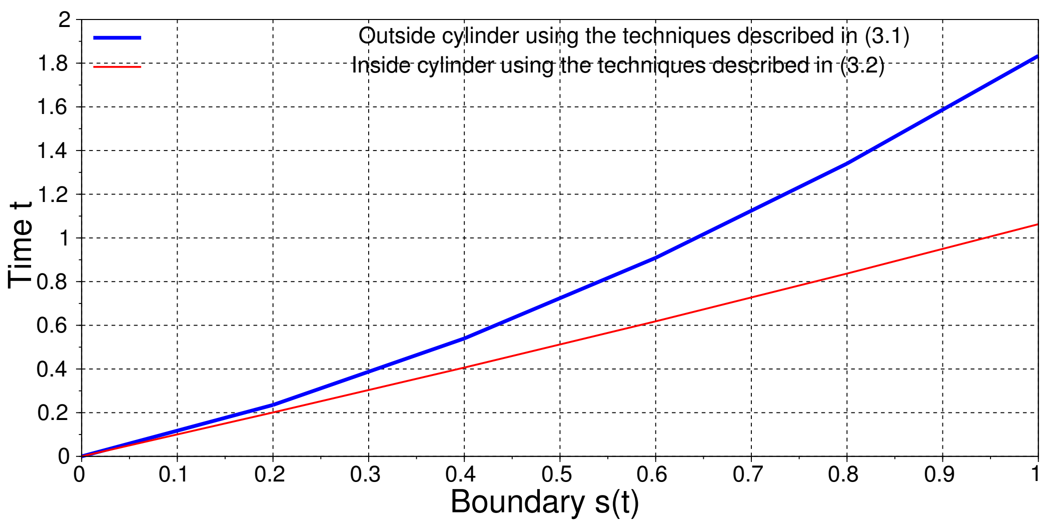

Figure 4: Position of moving boundary versus time.

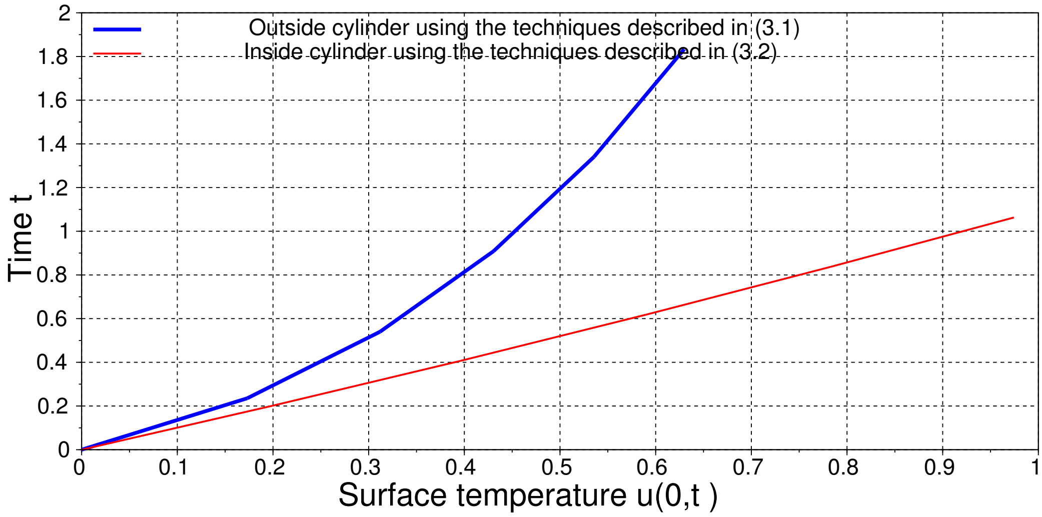

Figure 5: Temperature at fixed surface versus time.

5. Conclusion

We presented a modified variable time step method for solving the one-dimensional melting problem. As a conclusion, the standard finite difference method is accurate and efficient to solve one-dimensional melting problem in internal and external cylindrical geometry. This method is far superior in his ability to produce highly accurate $s(t)$ moving boundary or $u(x, t)$ values of the temperature distribution of the cylindrical problem. Note that the results obtained by the present method are excellent.

Tables 1–4 show the values of the temperature distribution of the cylindrical problem (outside cylinder) using the techniques described in 3.1 and 3.3 for the indicated values of t obtained by the present method. Tables 5–8 show the values of the temperature distribution of the cylindrical problem (inside cylinder) using the techniques described in 3.2 and 3.3 for the indicated values of t obtained by the present method.

$s(t)$	δt	t	n	$u(0,t)$
0.2	0.013	0.2354	2	0.1731
	0.013	0.2338	2	0.1731
	0.014	0.2363	2	0.1731
	0.013	0.2347	2	0.1731
0.4	0.017	0.5398	2	0.3118
	0.017	0.5368	2	0.3117
	0.017	0.5416	2	0.3119
	0.017	0.5385	2	0.3118
0.6	0.020	0.9089	2	0.4306
	0.020	0.9045	2	0.4305
	0.020	0.9114	2	0.4307
	0.020	0.9071	2	0.4305
0.8	0.023	1.3405	2	0.5355
	0.023	1.3349	2	0.5353
	0.023	1.3439	2	0.5355
	0.023	1.3383	2	0.5354
1.0	0.026	1.8334	2	0.6298
	0.026	1.8267	2	0.6296
	0.026	1.8376	2	0.6298
	0.026	1.8308	2	0.6297

Table 1: Comparison of time step δt_j, t time, n iteration number and the surface temperature $u(0, t)$ using the techniques described in 3.1 in solving cylindrical problem (outside cylinder). The upper entry corresponds to δt_j calculated by (2.4), the second by (2.5), the third by (2.6) and the lower by (2.7), $\delta x = 0.01$.

$s(t)$	δt	t	n	$u(0,t)$
0.2	0.118	0.2184	3	0.1845
	0.109	0.2087	2	0.1839
	0.123	0.2229	3	0.1847
	0.113	0.2134	3	0.1842
0.4	0.152	0.5055	3	0.3286
	0.144	0.4802	3	0.3275
	0.156	0.5185	3	0.3292
	0.148	0.4934	3	0.3281
0.6	0.183	0.8562	4	0.4516
	0.176	0.8171	3	0.4502
	0.187	0.8774	3	0.4523
	0.180	0.8385	3	0.4510
0.8	0.213	1.2680	4	0.5602
	0.207	1.2162	3	0.5586
	0.217	1.2973	3	0.5610
	0.211	1.2455	3	0.5595
1.0	0.243	1.7396	3	0.6577
	0.237	1.6758	3	0.6561
	0.247	1.7769	3	0.6587
	0.241	1.7127	3	0.6572

Table 2: Comparison of time step δt_j, t time, n iteration number and the surface temperature $u(0,t)$ using the techniques described in 3.1 in solving cylindrical problem (outside cylinder). The upper entry corresponds to δt_j calculated by (2.4), the second by (2.5), the third by (2.6) and the lower by (2.7)), $\delta x = 0.1$.

$s(t)$	δt	t	n	$u(0,t)$
0.2	0.013	0.2362	2	0.1732
	0.013	0.2345	2	0.1732
	0.014	0.2371	2	0.1732
	0.113	0.2354	2	0.1732
0.4	0.017	0.5412	2	0.3120
	0.017	0.5382	2	0.3119
	0.017	0.5430	2	0.3121
	0.017	0.5399	2	0.3120
0.6	0.020	0.9109	2	0.4309
	0.020	0.9066	2	0.4307
	0.020	0.9135	2	0.4310
	0.020	0.9091	2	0.4308
0.8	0.023	1.3431	2	0.5358
	0.023	1.3376	2	0.5356
	0.023	1.3465	2	0.5359
	0.023	1.3409	2	0.5357
1.0	0.026	1.8367	2	0.6301
	0.026	1.8299	2	0.6299
	0.026	1.8408	2	0.6302
	0.026	1.8341	2	0.6300

Table 3: Comparison of time step δt_j, t time, n iteration number and the surface temperature $u(0,t)$ using the techniques described in 3.3 in solving cylindrical problem (outside cylinder). The upper entry corresponds to δt_j calculated by (2.4), the second by (2.5), the third by (2.6) and the lower by (2.7), $\delta x = 0.01$.

$s(t)$	δt	t	n	$u(0,t)$
	0.125	0.2252	4	0.1799
	0.112	0.2120	3	0.1786
0.2	0.129	0.2295	4	0.1802
	0.117	0.2165	3	0.1731
	0.154	0.5174	4	0.3283
	0.147	0.4910	3	0.3267
0.4	0.158	0.5301	4	0.3288
	0.151	0.5039	3	0.3274
	0.186	0.8744	4	0.4531
	0.180	0.8346	3	0.4518
0.6	0.190	0.8952	4	0.4539
	0.184	0.8556	3	0.4526
	0.216	1.2920	4	0.5624
	0.210	1.2397	3	0.5609
0.8	0.220	1.3207	4	0.5633
	0.214	1.2685	3	0.5618
	0.246	1.7695	4	0.6605
	0.240	1.7054	3	0.6589
1.0	0.250	1.8060	4	0.6614
	0.244	1.7418	3	0.6599

Table 4: Comparison of time step δt_j, t time, n iteration number and the surface temperature $u(0,t)$ using the techniques described in 3.3 in solving cylindrical problem (outside cylinder). The upper entry corresponds to δt_j calculated by (2.4), the second by (2.5), the third by (2.6) and the lower by (2.7), $\delta x = 0.1$.

$s(t)$	δt	t	n	$u(0,t)$
	0.010	0.2010	1	0.1992
	0.010	0.2009	1	0.1992
0.2	0.010	0.2019	1	0.1993
	0.010	0.2018	1	0.1992
	0.010	0.4065	1	0.3958
	0.010	0.4063	1	0.3957
0.4	0.010	0.4081	1	0.3960
	0.010	0.4079	1	0.3959
	0.011	0.6182	1	0.5900
	0.011	0.6179	1	0.5900
0.6	0.011	0.6205	1	0.5904
	0.011	0.6202	1	0.5904
	0.011	0.8369	1	0.7827
	0.011	0.8364	1	0.7826
0.8	0.011	0.8399	1	0.7833
	0.011	0.8395	1	0.7832
	0.011	1.0629	1	0.9744
	0.011	1.0624	1	0.9742
1.0	0.012	1.0665	1	0.9751
	0.012	1.0660	1	0.9750

Table 5: Comparison of time step δt_j, t time, n iteration number and the surface temperature $u(0,t)$ using the techniques described in 3.2 in solving cylindrical problem (inside cylinder). The upper entry corresponds to δt_j calculated by (2.4), the second by (2.5), the third by (2.6) and the lower by (2.7), $\delta x = 0.01$.

$s(t)$	δt	t	n	$u(0,t)$
	0.101	0.2008	2	0.1992
	0.100	0.2004	1	0.1991
0.2	0.105	0.2052	3	0.1996
	0.105	0.2050	2	0.1996
	0.103	0.4061	2	0.3950
	0.103	0.4047	2	0.3947
0.4	0.107	0.4184	2	0.3967
	0.107	0.4174	2	0.3964
	0.107	0.6179	3	0.5874
	0.106	0.6151	2	0.5871
0.6	0.110	0.6372	3	0.5910
	0.110	0.6350	2	0.5906
	0.110	0.8367	3	0.7779
	0.110	0.8324	2	0.7774
0.8	0.114	0.8625	3	0.7832
	0.113	0.8589	2	0.7827
	0.114	1.0629	3	0.9671
	0.113	1.0571	2	0.9664
1.0	0.117	1.0948	3	0.9741
	0.116	1.0898	2	0.9735

Table 6: Comparison of time step δt_j, t time, n iteration number and the surface temperature $u(0,t)$ using the techniques described in 3.2 in solving cylindrical problem (inside cylinder). The upper entry corresponds to δt_j calculated by (2.4), the second by (2.5), the third by (2.6) and the lower by (2.7), $\delta x = 0.1$.

$s(t)$	δt	t	n	$u(0,t)$
	0.010	0.1978	1	0.2009
	0.010	0.1979	1	0.2009
0.2	0.010	0.1987	1	0.2010
	0.010	0.1988	1	0.2010
	0.009	0.3888	1	0.4051
	0.009	0.3891	1	0.4051
0.4	0.009	0.3904	1	0.4053
	0.009	0.3907	1	0.4054
	0.008	0.5661	2	0.6104
	0.008	0.5667	2	0.6107
0.6	0.008	0.5684	2	0.6110
	0.008	0.5690	2	0.6113
	0.007	0.7196	2	0.8061
	0.007	0.7207	2	0.8070
0.8	0.007	0.7225	2	0.8071
	0.007	0.7236	2	0.8081
	0.002	0.8178	4	0.9451
	0.002	0.8205	3	0.9483
1.0	0.002	0.8212	4	0.9467
	0.002	0.8240	3	0.9500

Table 7: Comparison of time step δt_j, t time, n iteration number and the surface temperature $u(0,t)$ using the techniques described in 3.3 in solving cylindrical problem (inside cylinder). The upper entry corresponds to δt_j calculated by (2.4), the second by (2.5), the third by (2.6) and the lower by (2.7), $\delta x = 0.01$.

$s(t)$	δt	t	n	$u(0,t)$
0.2	0.092	0.1917	3	0.2090
	0.095	0.1954	2	0.2096
	0.096	0.1958	3	0.2099
	0.100	0.1997	1	0.2107
0.4	0.091	0.3764	4	0.4106
	0.092	0.3806	2	0.4126
	0.094	0.3879	4	0.4133
	0.096	0.3926	2	0.4155
0.6	0.083	0.5470	5	0.6151
	0.084	0.5533	3	0.6175
	0.086	0.5651	4	0.6200
	0.088	0.5719	3	0.6230
0.8	0.071	0.6953	6	0.8120
	0.073	0.7055	4	0.8182
	0.073	0.7189	5	0.8219
	0.075	0.7298	3	0.8284
1.0	0.045	0.8011	8	0.9669
	0.049	0.8180	5	0.9834
	0.047	0.8293	7	0.9831
	0.052	0.8477	4	0.9989

Table 8: Comparison of time step δt_j, t time, n iteration number and the surface temperature $u(0,t)$ using the techniques described in 3.3 in solving cylindrical problem (inside cylinder). The upper entry corresponds to δt_j calculated by (2.4), the second by (2.5), the third by (2.6) and the lower by (2.7), $\delta x = 0.1$.

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