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A monotone projection algorithm for solving fixed points of nonlinear mappings and equilibrium problems

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Abstract

In this paper, fixed points of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense and equilibrium problems are investigated based on a monotone projection algorithm. Strong convergence theorems are established in the framework of reflexive Banach spaces.

Keywords: Asymptotically quasi-ϕ-nonexpansive mapping, equilibrium problem, fixed point, generalized projection.
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1. Introduction and Preliminaries

Throughout this paper, we always assume that E is a real Banach space, C is a nonempty subset of E and E^* is the dual space of E. Let G be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem in the sense of Blum and Oettli [4]. Find $\bar{x} \in C$ such that

$$G(\bar{x} y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The problem was first introduced by Ky Fan [11]. In this paper, We use $Sol(G)$ to denote the solution set of equilibrium problem (1.1). That is,

$$Sol(G) = \{x \in C : G(x, y) \geq 0, \quad \forall y \in C\}.$$

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Let

$$G(x,y) := \langle Ax, y-x\rangle, \quad \forall x,y \in C,$$

where $A : C \to E^*$ is a given nonlinear mapping. Then $\bar{x} \in Sol(G)$ if and only if $\bar{x}$ is a solution of the following variational inequality. Find $\bar{x}$ such that

$$\langle A\bar{x}, y-\bar{x}\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

In order to study the solutions of problem (1.1), we assume that G satisfies the following restrictions:

(R-1) $\limsup_{t\downarrow 0} G(tz+(1-t)x, y) \leq G(x,y), \forall x,y,z \in C$;

(R-2) $G(x,y)+G(y,x) \leq 0, \forall x,y \in C$;

(R-3) for each $x \in C$, $y \mapsto G(x,y)$ is weakly lower semi-continuous and convex;

(R-4) $G(x,x) = 0, \forall x \in C$.

The equilibrium problem provides us a natural, novel and unified framework to study a wide class of problems arising in physics, economics, finance, transportation, network, elasticity and optimization. The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative. It has been shown that variational inequalities and mathematical programming problems can be viewed as a special realization of the abstract equilibrium problems. Equilibrium problems have numerous applications, including but not limited to problems in economics, game theory, finance, traffic analysis, circuit network analysis and mechanics; see, [3], [8], [12], [13], [25] and the references therein. Recently, equilibrium problem (1.1) has been extensively investigated based on fixed point algorithms; see [7, 9, 10], [14]-[17], [23]-[28] and the references therein.

Let E be a real Banach space and let E^* be the dual space of E. Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x^*\|^2 = \|x\|^2\}.$$

Let $B_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. E is said to be strictly convex if $\|x+y\| < 2$ for all $x, y \in B_E$ with $x \neq y$. It is said to be uniformly convex if for any $\epsilon \in (0,2]$ there exists $\delta > 0$ such that for any $x, y \in B_E$,

$$\|x-y\| \geq \epsilon \quad \text{implies} \quad \|x+y\| \leq 2-2\delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. E is said to be smooth or is said to have a Gâteaux differentiable norm iff $\lim_{s\to\infty}(\|sx+y\| - s\|x\|)$ exists for each $x, y \in B_E$. E is said to have a uniformly Gâteaux differentiable norm if for each $y \in B_E$, the limit is uniformly obtained for each $x \in B_E$. If the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm to weak* continuous on each bounded subset of E and single valued. It is also said to be uniformly smooth if and only if the above limit is attained uniformly for any $x, y \in B_E$. It is well known if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that E is uniformly smooth if and only if E^* is uniformly convex.

From now on, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. Recall that E is said to has the Kadec-Klee property (KKP) if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $\|x_n\| \to \|x\|$ and $x_n \rightharpoonup x$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is known if E is a uniformly convex Banach space, then E has the KKP.

Let E be a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x,y \in E.$$

In a Hilbert space H, $\sqrt{\phi(x,y)} \equiv \|x-y\|$. Let C be a closed convex subset of H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x-y\|$, for all $y \in C$. The operator

P_C is called the metric projection from H onto C. It is known that P_C is firmly nonexpansive. Recently, Alber [2] introduced a generalized projection operator Π_C, in a Banach space E, which is an analogue of the metric projection P_C in Hilbert spaces.

Recall that the generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x).$$

Existence and uniqueness of operator Π_C follows from the properties of functional $\phi(x, y)$ and strict monotonicity of mapping J. In the framework of Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E \tag{1.3}$$

and

$$\phi(x, y) - 2\langle x - z, Jz - Jy\rangle = \phi(x, z) + \phi(z, y), \quad \forall x, y, z \in E. \tag{1.4}$$

Let $T : C \to C$ be a mapping. Recall that a point $p \in C$ is said to be a fixed point of T if and only if $p = Tp$. In this paper, we use $Fix(T)$ to denote the fixed point set of T. Recall that a point p in C is said to be an asymptotic fixed point [21] of T if and only if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{Fix}(T)$. Recall that T is said to be asymptotically regular on C if and only if any bounded subset K of C,

$$\limsup_{n\to\infty} \sup_{x\in K} \{\|T^{n+1}x - T^n x\|\} = 0.$$

T is said to be relatively nonexpansive [5], [6] iff

$$\widetilde{Fix}(T) = Fix(T) \ne \emptyset, \quad \phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, \forall p \in Fix(T).$$

T is said to be relatively asymptotically nonexpansive [1] iff

$$\widetilde{Fix}(T) = Fix(T) \ne \emptyset, \quad \phi(p, T^n x) \le k_n \phi(p, x), \quad \forall x \in C, \forall p \in Fix(T), \forall n \ge 1,$$

where $\{k_n\} \subset [1, \infty)$ is a sequence such that $k_n \to 1$ as $n \to \infty$.

Remark 1.1. The class of relatively asymptotically nonexpansive mappings, which covers the class of relatively nonexpansive mappings, was first considered in [1]; see the references therein.

T is said to be quasi-ϕ-nonexpansive [18] iff

$$Fix(T) \ne \emptyset, \quad \phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, \forall p \in Fix(T).$$

T is said to be asymptotically quasi-ϕ-nonexpansive [19] iff there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$Fix(T) \ne \emptyset, \quad \phi(p, T^n x) \le k_n \phi(p, x), \quad \forall x \in C, \forall p \in Fix(T), \forall n \ge 1.$$

Remark 1.2. The class of asymptotically quasi-ϕ-nonexpansive mappings, which covers the class of quasi-ϕ-nonexpansive mappings [18], was considered in Qin, Cho and Kang [19]. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive do not require restriction $Fix(T) = \widetilde{Fix}(T)$.

T is said to be asymptotically quasi-ϕ-nonexpansive in the intermediate sense [20] iff $Fix(T) \ne \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p\in Fix(T), x\in C} \big(\phi(p, T^n x) - \phi(p, x)\big) \le 0.$$

Putting $\xi_n = \max\{0, \sup_{p\in Fix(T), x\in C} \big(\phi(p, T^n x) - \phi(p, x)\big)\}$, we see $\xi_n \to 0$ as $n \to \infty$.

Remark 1.3. The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense, which is reduced to the class of asymptotically quasi-nonexpansive mappings in the intermediate sense in the framework of Hilbert spaces, was introduced by Qin and Wang [20].

In order to present our main results, we also need the following lemmas.

Lemma 1.4 ([4]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed convex subset of E. Assume that G is a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (R-1), (R-2), (R-3) and (R-4). Let $r > 0$ and $x \in E$. There exists $z \in C$ such that*

$$rG(z, y) - \langle z - y, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

Lemma 1.5 ([2]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed convex subset of E. Assume $x \in E$. Then*

$$\phi(y, \Pi_C x) \leq \phi(y, x) - \phi(\Pi_C x, x), \quad \forall y \in C.$$

Lemma 1.6 ([2]). *Let C be a closed and convex subset of a smooth Banach space E, and let $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \quad \forall y \in C.$$

The following result was implicitly proved in Qin and Wang [20].

Lemma 1.7. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the KKP and let C be a convex and closed subset of E. If T is a closed asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense on C, then $Fix(T)$ is closed and convex.*

The following result can be found in Blum and Oettli [4], Takahashi and Zembayashi [22].

Lemma 1.8. *Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed convex subset of E. Assume that G is a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (R-1), (R-2), (R-3) and (R-4). Let $r > 0$ and $x \in E$. Define a mapping $S_r : E \to C$ by*

$$S_r x = \{z \in C : rG(z, y) + \langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\}.$$

Then the following conclusions hold:

(1) *S_r is single-valued;*

(2) *S_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle S_r x - S_r y, JS_r x - JS_r y\rangle \leq \langle S_r x - S_r y, Jx - Jy\rangle;$$

(3) *S_r is quasi-ϕ-nonexpansive;*

(4) *$\phi(q, S_r x) + \phi(S_r x, x) \leq \phi(q, x), \forall q \in Fix(S_r)$;*

(5) *$Fix(S_r) = Sol(G)$;*

(6) *$Sol(G)$ is closed and convex.*

2. Main results

Theorem 2.1. *Let E be a reflexive strictly convex and smooth Banach space such that both E and E^* have the KKP and C a convex and closed subset of E. Let Λ be an index set and G_i a bifunction with (R-1), (R-2), (R-3) and (R-4). Let $T_i : C \to C$ be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense for every $i \in \Lambda$. Assume that T_i is closed and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \\ x_1 = \Pi_{C_1} x_0, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \phi(z, x_n) + \xi_{n,i} \geq \phi(z, u_{(n,i)})\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\xi_{(n,i)} = \max\{0, \sup_{p\in Fix(T_i), x\in C} \big(\phi(p, T_i^n x) - \phi(p, x)\big)\}$, $\{u_{(n,i)}\}$ is a sequence in C_n such that $r_{(n,i)} G_i(u_{(n,i)}, y) \geq \langle u_{(n,i)} - y, Ju_{(n,i)} - JT_i^n x_n\rangle$, $y \in C_n$, and $\{r_{(n,i)}\}$ is a real sequence in $[a_i, \infty)$, where $\{a_i\}$ is a positive real number sequence, for every $i \in \Lambda$. Then sequence $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)} x_1$.

Proof. From Lemma 1.7 and Lemma 1.8, one sees that $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)$ is closed and convex so that the generalization projection onto the set is well defined.

Next, we show C_n is closed and convex. To show C_n is closed and convex, it suffices to show that, for each fixed but arbitrary $i \in \Lambda$, $C_{(n,i)}$ is closed and convex. This can be proved by induction on n. It is obvious that $C_{(1,i)} = C$ is closed and convex. Assume that $C_{(m,i)}$ is closed and convex for some $m \geq 1$. Letting $z_1, z_2 \in C_{(m+1,i)}$, we see that $z_1, z_2 \in C_{(m,i)}$. It follows that $z = tz_1 + (1-t)z_2 \in C_{(m,i)}$, where $t \in (0, 1)$. Notice that

$$\phi(z_1, u_{(m,i)}) \leq \phi(z_1, x_m) + \xi_{m,i}$$

and

$$\phi(z_2, u_{(m,i)}) \leq \phi(z_2, x_m) + \xi_{m,i}.$$

The above inequalities are equivalent to

$$2\langle z_1, Jx_m - Ju_{(m,i)}\rangle \leq \|x_m\|^2 - \|u_{(m,i)}\|^2 + \xi_{m,i}$$

and

$$2\langle z_2, Jx_m - Ju_{(m,i)}\rangle \leq \|x_m\|^2 - \|u_{(m,i)}\|^2 + \xi_{m,i}.$$

Multiplying t and $(1-t)$ on the both sides of the inequalities above, respectively yields that

$$2\langle z, Jx_m - Ju_{(m,i)}\rangle \leq \|x_m\|^2 - \|u_{(m,i)}\|^2 + \xi_{m,i}.$$

That is,

$$\phi(z, u_{(m,i)}) \leq \phi(z, x_m) + \xi_{m,i},$$

where $z \in C_{(m,i)}$. This finds that $C_{(m+1,i)}$ is closed and convex. We conclude that $C_{(n,i)}$ is closed and convex. This in turn implies that $C_n = \cap_{i\in\Lambda} C_{(n,i)}$ is closed, and convex. This implies that $\Pi_{C_n} x_1$ is well defined. Note that

$$\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i) \subset C_1 = C$$

is clear. Suppose that $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i) \subset C_{(m,i)}$ for some positive integer m. For any

$$w \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i) \subset C_{(m,i)},$$

we see from the definition of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense that

$$\begin{aligned}\phi(w,u_{(m,i)}) &= \phi(w,S_{r_{(m,i)}}T_i^m x_m)\\ &\le \phi(w,T_i^m x_m))\\ &\le \phi(w,x_m)+\xi_{(m,i)}.\end{aligned}$$

This implies $w \in C_{(m+1,i)}$. Hence,

$$\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)\subset\cap_{i\in\Lambda}C_{(n,i)}.$$

Since $x_n = \Pi_{C_n}x_1$, we find from Lemma 1.6 that $\langle x_n - z, Jx_1 - Jx_n\rangle \ge 0$, for any $z \in C_n$. Since $\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)\subset C_n$, we find that

$$\langle x_n - w, Jx_1 - Jx_n\rangle \ge 0, \quad \forall w \in \cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i). \tag{2.1}$$

Note that $x_n = \Pi_{C_{n+1}}x_1$. Using Lemma 1.5, one sees that

$$\begin{aligned}&\phi(\Pi_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)}x_1,x_1)\\ &\ge \phi(\Pi_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)}x_1,x_1)-\phi(\Pi_{\cap_{i\in\Lambda}Fix(T_i)\bigcap\cap_{i\in\Lambda}Sol(G_i)}x_1,\Pi_{C_{n+1}}x_1)\\ &\ge \phi(\Pi_{C_{n+1}}x_1,x_1)\\ &= \phi(x_n,x_1).\end{aligned}$$

This implies that $\{\phi(x_n,x_1)\}$ is bounded. Hence, $\{x_n\}$ is also bounded. Since the space is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$. Since C_n is closed and convex, we find that $\bar{x}\in C_n$. This implies from $x_n = \Pi_{C_n}x_1$ that $\phi(x_n,x_1)\le\phi(\bar{x},x_1)$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$\begin{aligned}\phi(\bar{x},x_1) &\ge \limsup_{n\to\infty}\phi(x_n,x_1)\\ &\ge \liminf_{n\to\infty}\phi(x_n,x_1)\\ &= \liminf_{n\to\infty}(\|x_n\|^2-2\langle x_n,Jx_1\rangle+\|x_1\|^2)\\ &\ge \|\bar{x}\|^2-2\langle\bar{x},Jx_1\rangle+\|x_1\|^2\\ &= \phi(\bar{x},x_1),\end{aligned}$$

which implies that $\lim_{n\to\infty}\phi(x_n,x_1)=\phi(\bar{x},x_1)$. Hence, we have $\lim_{n\to\infty}\|x_n\|=\|\bar{x}\|$. Since E has the KKP one has $x_n\to\bar{x}\in C_n$ as $n\to\infty$. Using Lemma 1.5, one has

$$\phi(x_{n+1},x_1)\ge\phi(x_{n+1},x_n)+\phi(x_n,x_1).$$

This implies that $\lim_{n\to\infty}\phi(x_{n+1},x_n)=0$. Since $x_{n+1}=\Pi_{C_{n+1}}x_1\in C_{n+1}$, we find that

$$\phi(x_{n+1},x_n)+\xi_{(n,i)}\ge\phi(x_{n+1},u_{(n,i)}),$$

which further implies $\lim_{n\to\infty}\phi(x_{n+1},u_{(n,i)})=0$. Hence, we have $\lim_{n\to\infty}(\|x_{n+1}\|-\|u_{(n,i)}\|)=0$. Therefore, one has $\lim_{n\to\infty}\|u_{(n,i)}\|=\|\bar{x}\|$. On the other hand, we have

$$\lim_{n\to\infty}\|Ju_{(n,i)}\|=\lim_{n\to\infty}\|u_{(n,i)}\|=\|\bar{x}\|=\|J\bar{x}\|.$$

This implies that $\{Ju_{(n,i)}\}$ is bounded. Since both E and E^* are reflexive, we may assume that $Ju_{(n,i)}\rightharpoonup u^{(*,i)}\in E^*$. Since E is reflexive, we see $J(E)=E^*$. This shows that there exists an element $u^i\in E$ such that $Ju^i=u^{(*,i)}$. It follows that

$$\phi(x_{n+1},u_{(n,i)})=\|x_{n+1}\|^2-2\langle x_{n+1},Ju_{(n,i)}\rangle+\|Ju_{(n,i)}\|^2.$$

Therefore, one has

$$\begin{aligned}0 &\geq \|\bar{x}\|^2 + \|u^{(*,i)}\|^2 - 2\langle \bar{x}, u^{(*,i)}\rangle \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Ju^i\rangle + \|u^i\|^2 \\ &= \phi(\bar{x}, u^i) \\ &\geq 0.\end{aligned}$$

That is, $\bar{x} = u^i$, which in turn implies that $u^{(*,i)} = J\bar{x}$. It follows that $Ju_{(n,i)} \rightharpoonup J\bar{x} \in E^*$. Since E^* has the KKP, we obtain that $\lim_{n\to\infty} Ju_{(n,i)} = J\bar{x}$. On the other hand, one has $\lim_{n\to\infty} \phi(x_{n+1}, T_i^n x_n) = 0$. It follows that $\lim_{n\to\infty} \|T_i^n x_n\| = \|\bar{x}\|$. On the other hand, we have

$$\lim_{n\to\infty} \|JT_i^n x_n\| = \lim_{n\to\infty} \|T_i^n x_n\| = \|\bar{x}\| = \|J\bar{x}\|.$$

This implies that $\{JT_i^n x_n\}$ is bounded. Since both E and E^* are reflexive, we may assume that $JT_i^n x_n \rightharpoonup y^{(*,i)} \in E^*$. Since E is reflexive, we see $J(E) = E^*$. This shows that there exists an element $y^i \in E$ such that $Jy^i = y^{(*,i)}$. It follows that

$$\phi(x_{n+1}, T_i^n x_n) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, JT_i^n x_n\rangle + \|JT_i^n x_n\|^2.$$

Therefore, one has

$$\begin{aligned}0 &\geq \|\bar{x}\|^2 - 2\langle \bar{x}, y^{(*,i)}\rangle + \|y^{(*,i)}\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy^i\rangle + \|y^i\|^2 \\ &= \phi(\bar{x}, y^i) \\ &\geq 0.\end{aligned}$$

That is, $\bar{x} = y^i$, which in turn implies that $y^{(*,i)} = J\bar{x}$. It follows that $JT_i^n x_n \rightharpoonup J\bar{x} \in E^*$. Since E^* has the KKP, one has $\lim_{n\to\infty} JT_i^n x_n = J\bar{x}$. Since J^{-1} is demi-continuous, we have $T_i^n x_n \rightharpoonup \bar{x}$, for every $i \in \Lambda$. Since

$$|\|T_i^n x_n\| - \|\bar{x}\|| \leq \|J(T_i^n x_n) - J\bar{x}\|,$$

one has $\|T_i^n x_n\| \to \|\bar{x}\|$, as $n \to \infty$, for every $i \in \Lambda$. Since E has the Kadec-Klee property, one obtains $\lim_{n\to\infty} \|T_i^n x_n - \bar{x}\| = 0$. On the other hand, we have

$$\|T_i^{n+1} x_n - T_i^n x_n\| + \|T_i^n x_n - \bar{x}\| \geq \|T_i^{n+1} x_n - \bar{x}\| \geq 0.$$

In view of the uniformly asymptotic regularity of T_i, one has

$$\lim_{n\to\infty} \|T_i^{n+1} x_n - \bar{x}\| = 0,$$

that is, $T_i T_i^n x_n - \bar{x} \to 0$ as $n \to \infty$. Since every T_i is closed, we find that $T_i\bar{x} = \bar{x}$ for every $i \in \Lambda$.

On the other hand, we have

$$\langle y - u_{(n,i)}, Ju_{(n,i)} - JT_i^n x_n\rangle + r_{(n,i)} G_i(u_{(n,i)}, y) \geq 0, \quad \forall y \in C_n,$$

we see that

$$\|y - u_{(n,i)}\| \|Ju_{(n,i)} - JT_i^n x_n\| \geq r_{(n,i)} G_i(y, u_{(n,i)}), \quad \forall y \in C_n.$$

In view of (R-3), one has $G_i(y, \bar{x}) \leq 0$. For $0 < t_i < 1$, define

$$y_{(t,i)} = t_i y + (1 - t_i)\bar{x}.$$

It follows that $y_{(t,i)} \in C$, which yields that $G_i(y_{(t,i)}, \bar{x}) \leq 0$. It follows from the (R-3) and (R-4) that

$$\begin{aligned}0 &= G_i(y_{(t,i)}, y_{(t,i)}) \\ &\leq t_i G_i(y_{(t,i)}, y) + (1 - t_i) G_i(y_{(t,i)}, \bar{x}) \\ &\leq t_i G_i(y_{(t,i)}, y).\end{aligned}$$

That is, $G_i(y_{(t,i)}, y) \geq 0$. Letting $t_i \downarrow 0$, we obtain from (R-1) that $G_i(\bar{x}, y) \geq 0$, $\forall y \in C$. This implies that $\bar{x} \in Sol(G_i)$ for every $i \in \Lambda$. This shows that $\bar{x} \in \cap_{i\in\Lambda} Sol(G_i)$. Finally, we prove $\bar{x} = \Pi_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)} x_1$. Letting $n \to \infty$ in 2.1, we see that

$$\langle \bar{x} - w, Jx_1 - J\bar{x} \rangle \geq 0, \quad \forall w \in \cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i).$$

Using Lemma 1.6, we find that $\bar{x} = \Pi_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)} x_1$. This completes the proof. □

For the class of quasi-ϕ-nonexpansive mapping, we have the following result.

Corollary 2.2. *Let E be a reflexive strictly convex and smooth Banach space such that both E and E^* have the KKP and let C be a convex and closed subset of E. Let Λ be an index set and let G_i be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let $T_i : C \to C$ be a closed quasi-ϕ-nonexpansive mapping for every $i \in \Lambda$. Assume that $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_{(1,i)} = C, C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \\ x_1 = \Pi_{C_1} x_0, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \phi(z, x_n) \geq \phi(z, u_{(n,i)})\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{u_{(n,i)}\}$ is a sequence in C_n such that $r_{(n,i)} G_i(u_{(n,i)}, y) \geq \langle u_{(n,i)} - y, Ju_{(n,i)} - JT_i x_n \rangle$, $y \in C_n$, and $\{r_{(n,i)}\}$ is a real sequence in $[a_i, \infty)$, where $\{a_i\}$ is a positive real number sequence, for every $i \in \Lambda$. Then sequence $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)} x_1$.

Using Theorem 2.1, we also have the following result on equilibrium problem (1.1).

Corollary 2.3. *Let E be a reflexive strictly convex and smooth Banach space such that both E and E^* have the KKP and let C be a convex and closed subset of E. Let Λ be an index set and let G_i be a bifunction with (R-1), (R-2), (R-3) and (R-4). Assume that $\cap_{i\in\Lambda} Sol(G_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_{(1,i)} = C, C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \\ x_1 = \Pi_{C_1} x_0, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \phi(z, x_n) + \xi_{n,i} \geq \phi(z, u_{(n,i)})\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{u_{(n,i)}\}$ is a sequence in C_n such that $r_{(n,i)} G_i(u_{(n,i)}, y) \geq \langle u_{(n,i)} - y, Ju_{(n,i)} - Jx_n \rangle$, $y \in C_n$, and $\{r_{(n,i)}\}$ is a real sequence in $[a_i, \infty)$, where $\{a_i\}$ is a positive real number sequence, for every $i \in \Lambda$. Then sequence $\{x_n\}$ converges strongly to $\Pi_{\cap_{i\in\Lambda} Sol(G_i)} x_1$.

In the framework of Hilbert spaces, $\sqrt{\phi(x, y)} = \|x - y\|$, $\forall x, y \in E$. The generalized projection is reduced to the metric projection and asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense is reduced to asymptotically quasi-nonexpansive mappingss in the intermediate sense. Using Theorem 2.1, we have the following result.

Corollary 2.4. *Let E be a Hilbert space and let C be a convex and closed subset of E. Let Λ be an index set and let G_i be a bifunction with (R-1), (R-2), (R-3) and (R-4). Let $T_i : C \to C$ be an asymptotically quasi-nonexpansive mapping in the intermediate sense for every $i \in \Lambda$. Assume that T_i is closed and uniformly asymptotically regular on C for every $i \in \Lambda$ and $\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \\ x_1 = \Pi_{C_1} x_0, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \|x_n - z\|^2 + \xi_{n,i} \geq \|z - u_{(n,i)}\|^2\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\xi_{(n,i)} = \max\{0, \sup_{p\in Fix(T_i), x\in C} \left(\|p - T_i^n x\|^2 - \|p - x\|^2\right)\}$, $\{u_{(n,i)}\}$ *is a sequence in* C_n *such that* $r_{(n,i)} G_i(u_{(n,i)}, y) \geq \langle u_{(n,i)} - y, u_{(n,i)} - T_i^n x_n\rangle$, $y \in C_n$, *and* $\{r_{(n,i)}\}$ *is a real sequence in* $[a_i, \infty)$, *where* $\{a_i\}$ *is a positive real number sequence, for every* $i \in \Lambda$. *Then sequence* $\{x_n\}$ *converges strongly to* $P_{\cap_{i\in\Lambda} Fix(T_i) \bigcap \cap_{i\in\Lambda} Sol(G_i)} x_1$.

If T_i is the identity operator for each $i \in \Lambda$, we have the following result.

Corollary 2.5. *Let* E *be a Hilbert space and let* C *be a convex and closed subset of* E. *Let* Λ *be an index set and let* G_i *be a bifunction with (R-1), (R-2), (R-3) and (R-4). Assume that* $\cap_{i\in\Lambda} Sol(G_i)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_{(1,i)} = C, C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \\ x_1 = \Pi_{C_1} x_0, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \|x_n - z\| \geq \|z - u_{(n,i)}\|\}, \\ C_{n+1} = \cap_{i\in\Lambda} C_{(n+1,i)}, x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{u_{(n,i)}\}$ *is a sequence in* C_n *such that* $r_{(n,i)} G_i(u_{(n,i)}, y) \geq \langle u_{(n,i)} - y, u_{(n,i)} - x_n\rangle$, $y \in C_n$, *and* $\{r_{(n,i)}\}$ *is a real sequence in* $[a_i, \infty)$, *where* $\{a_i\}$ *is a positive real number sequence, for every* $i \in \Lambda$. *Then sequence* $\{x_n\}$ *converges strongly to* $P_{\cap_{i\in\Lambda} Sol(G_i)} x_1$.

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