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A viscosity of Cesàro mean approximation method for split generalized equilibrium, variational inequality and fixed point problems

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Abstract

In this paper, we introduce and study an iterative viscosity approximation method by modified Cesàro mean approximation for finding a common solution of split generalized equilibrium, variational inequality and fixed point problems. Under suitable conditions, we prove a strong convergence theorem for the sequences generated by the proposed iterative scheme. The results presented in this paper generalize, extend and improve the corresponding results of Shimizu and Takahashi [K. Shimoji, W. Takahashi, Taiwanese J. Math., **5** (2001), 387–404].

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1. Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $\{x_n\}$ be a sequence in H_1, then $x_n \to x$ (respectively,

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$x_n \rightharpoonup x$) will denote strong (respectively, weak) convergence of the sequence $\{x_n\}$. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$.

The *fixed point problem* (FPP) for the mapping T is to find $x \in C$ such that

$$Tx = x. \tag{1.1}$$

We denote $Fix(T) := \{x \in C : Tx = x\}$, the set of solutions of FPP.

Assumed throughout the paper that T is a nonexpansive mapping such that $Fix(T) \neq \emptyset$. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$.

Given a nonlinear mapping $A : C \to H_1$. Then the *variational inequality problem* (VIP) is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \leq 0, \quad \forall y \in C. \tag{1.2}$$

The solution of VIP (1.2) is denoted by $VI(C, A)$. It is well known that if A is strongly monotone and Lipschitz continuous mapping on C then VIP (1.2) has a unique solution. There are several different approaches towards solving this problem in finite dimensional and infinite dimensional spaces see [6, 7, 8, 14, 16, 20, 31, 35, 40] and the research in this direction is intensively continued.

Variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral and equilibrium problems, which arise in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see, e.g., [1, 13, 18] and the references therein.

For finding a common element of $Fix(T) \cap VI(C, A)$, Takahashi and Toyoda [34] introduced the following iterative scheme:

$$\begin{cases} x_0 \ \text{ chosen arbitrary}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)TP_C(x_n - \lambda_n Ax_n), \forall n \geq 0, \end{cases} \tag{1.3}$$

where A is an ρ-inverse-strongly monotone, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\rho)$. They showed that if $Fix(T) \cap VI(C, A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.3) converges weakly to $z_0 \in Fix(T) \cap VI(C, A)$.

On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$, Korpelevich [18] introduced the following so-called *Korpelevich's extragradient method* and which generates a sequence $\{x_n\}$ via the recursion;

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \quad n \geq 0, \end{cases} \tag{1.4}$$

where P_C is the metric projection from $\mathbb{R}^n$ onto $C, A : C \to H_1$ is a monotone operator and λ is a constant. Korpelevich [18] prove that the sequence $\{x_n\}$ converges strongly to a solution of $VI(C, A)$.

In this paper, we will present article, our main purpose is to study the split problem. First, we recall some background in the literature.

Problem 1: the split feasibility problem (SFP)

Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ be a bounded linear operator. The *split feasibility problem* (SFP) is formulated as finding a point

$$x^* \in C \ \text{ such that } \ Ax^* \in Q, \tag{1.5}$$

which was first introduced by Censor and Elfving [9] in medical image reconstruction.

A special case of the SFP is the *convexly constrained linear inverse problem* (CLIP) in a finite dimensional real Hilbert space [12]:

$$\text{find } x^* \in C \text{ such that } Ax^* = b, \tag{1.6}$$

where C is a nonempty closed convex subset of a real Hilbert space H_1 and b is a given element of a real Hilbert space H_2, which has extensively been investigated by using the Landweber iterative method [19]:

$$x_{n+1} = x_n + \gamma A^T(b - Ax_n),\ n \in \mathbb{N}.$$

Assume that the SFP (1.5) is consistent (i.e., (1.5) has a solution), it is not hard to see that $x^* \in C$ solves (1.5) if and only if it solves the following *fixed point equation*;

$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*,\ x^* \in C, \tag{1.7}$$

where P_C and P_Q are the (Orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A. Moreover, for sufficiently small $\gamma > 0$, the operator $P_C(I - \gamma A^*(I - P_Q)A)$ which defines the fixed point equation in (1.7) is nonexpansive.

An iterative method for solving the SFP, called the CQ algorithm, has the following iterative step:

$$x_{k+1} = P_C(x_k + \gamma A^T(P_Q - I)Ax_k). \tag{1.8}$$

The operator

$$T = P_C(I - \gamma A^T(I - P_Q)A), \tag{1.9}$$

is averaged whenever $\gamma \in (0, \frac{2}{L})$ with L being the largest eigenvalue of the matrix A^TA (T stands for matrix transposition), and so the CQ algorithm converges to a fixed point of T, whenever such fixed points exist.

When the SFP has a solution, the CQ algorithm converges to a solution; when it does not, the CQ algorithm converges to a minimizer, over C, of the proximity function $g(x) = \|P_QAx - Ax\|$, whenever such minimizer exists. The function $g(x)$ is convex and according to [2], its gradient is

$$\nabla g(x) = A^T(I - P_Q)Ax. \tag{1.10}$$

Problem 2: the split equilibrium problem (SEP)

.

In 2011, Moudafi [25] introduced the following split equilibrium problem (SEP):

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the *split equilibrium problem* (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{1.11}$$

and such that

$$y^* = Ax^* \in Q \ \text{ solves } \ F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{1.12}$$

When looked separately, (1.11) is the classical equilibrium problem (EP) and we denoted its solution set by $EP(F_1)$. The SEP (1.11) and (1.12) constitutes a pair of equilibrium problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A, of the solution x^* of the EP (1.11) in H_1 is the solution of another EP (1.12) by $EP(F_2)$.

The solution set SEP (1.11) and (1.12) is denoted by $\Theta = \{x^* \in EP(F_1) : Ax^* \in EP(F_2)\}$.

Problem 3: the split generalized equilibrium problem (SGEP)

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In 2013, Kazmi and Rivi [17] consider the *split generalized equilibrium problem* (SGEP):

Let $F_1, h_1 : C \times C \to \mathbb{R}$ and $F_2, h_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the *split generalized equilibrium problem* (SGEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) + h_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{1.13}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } F_2(y^*, y) + h_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{1.14}$$

They denoted the solution set of generalized equilibrium problem (GEP) (1.13) and GEP (1.14) by $GEP(F_1, h_1)$ and $GEP(F_2, h_2)$, respectively. The solution set of SGEP (1.13)-(1.14) is denoted by $\Gamma = \{x^* \in GEP(F_1, h_1) : Ax^* \in GEP(F_2, h_2)\}$.

If $h_1 = 0$ and $h_2 = 0$, then SGEP (1.13)-(1.14) reduces to SEP (1.11)-(1.12). If $h_2 = 0$ and $F_2 = 0$, then SGEP (1.13)-(1.14) reduces to the equilibrium problem considered by Cianciaruso et al. [10].

In 1975, Baillon [3] proved the first non-linear ergodic theorem.

Theorem 1.1 (Baillons ergodic theorem). *Suppose that C is a nonempty closed convex subset of Hilbert space H_1 and $T : C \to C$ is nonexpansive mapping such that $Fix(T) \neq \emptyset$ then $\forall x \in C$, the* ***Cesàro mean***

$$T_n x = \frac{1}{n+1} \sum_{i=0}^{n} T^i x, \tag{1.15}$$

weakly converges to a fixed point of T.

In 1997, Shimizu and Takahashi [29] studied the convergence of an iteration process sequence $\{x_n\}$ for a family of nonexpansive mappings in the framework of a real Hilbert space. They restate the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^{n} T^j x_n, \quad \text{for } n = 0, 1, 2, \ldots, \tag{1.16}$$

where x_0 and x are all elements of C and α_n is an appropriate point in $[0, 1]$. They proved that x_n converges strongly to an element of fixed point of T which is the nearest to x.

In 2000, for T a nonexpansive self-mapping with $Fix(T) \neq \emptyset$ and f a fixed contractive self-mapping, Moudafi [23] introduced the following viscosity approximations method for T:

$$x_{n+1} = \alpha_n f(x) + (1 - \alpha_n) T x_n, \tag{1.17}$$

and prove that $\{x_n\}$ converges to a fixed point p of T in a Hilbert space.

On the other hand, iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [11, 36, 37] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.18}$$

where C is the fixed point set of a nonexpansive mapping T on H_1 and b is a given point in H_1. Assume A is strongly positive; that is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x \rangle \leq \bar{\gamma} \|x\|^2, \quad \forall x \in H_1. \tag{1.19}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H_1:

$$\min_{x \in Fix(T)} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{1.20}$$

where A is strongly positive linear bounded operator and h is a potential function for γf i.e., $(h'(x) = \gamma f(x)$ for $x \in H_1)$.

In [37] (see also [39]), it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily

$$x_{n+1} = (I - \alpha_n A) T x_n + \alpha_n b, \quad n \geq 0, \tag{1.21}$$

converges strongly to the unique solution of the minimization problem (1.18).

Using the viscosity approximation method, Xu [38], develops Moudafi [23] in both Hilbert and Banach spaces.

Theorem 1.2 ([38]). *Let H_1 be a Hilbert space, C a closed convex subset of $H_1, T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$, and $f : C \to C$ a contraction. Let $\{x_n\}$ be generated by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n f(x_n), n \geq 0, \end{cases} \tag{1.22}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies:

(H1) $\alpha_n \to 0$;

(H2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(H3) either $\sum_{n=\infty}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\frac{\alpha_{n+1}}{\alpha_n}) = 1$.

Then under the hypotheses $(H1)-(H3)$, $x_n \to \tilde{x}$, where $\tilde{x}$ is the unique solution of the variational inequality

$$\langle (I-f)\tilde{x}, \tilde{x} - x\rangle \leq 0, x \in Fix(T).$$

Marino and Xu [22], combine the iterative method (1.21) with the viscosity approximation method (1.22).

Theorem 1.3 ([22]). *Let H_1 be a real Hilbert space, A be a bounded operator on H_1, T be a nonexpansive mapping on H_1 and $f : H_1 \to H_1$ be a contraction mapping. Assume that the set of fixed point of H_1 is nonempty. Let $\{x_n\}$ be generated by*

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.23}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(N1) $\alpha_n \to 0$;

(N2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(N3) either $\sum_{n=\infty}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\frac{\alpha_{n+1}}{\alpha_n}) = 1$.

Then $\{x_n\}$ converges strongly to $\tilde{x}$ of T which solves the variational inequality:

$$\langle (A - \gamma f)\tilde{x}, \tilde{x} - z\rangle \leq 0, z \in Fix(T).$$

Equivalently, $P_{Fix(T)}(I - A + \gamma f)\tilde{x} = \tilde{x}$.

Inspired and motivated by Korpelevich [18], Kazmi and Rivi [17], Shimizu and Takahashi [29], and Marino and Xu [22], we introduce the general Cesàro mean iterative method for a nonexpansive mapping in a real Hilbert space as follows:

$$\begin{cases} u_n = T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)\frac{1}{n+1}\sum_{i=0}^{n} S^i y_n, \ \forall n \geq 0, \end{cases} \tag{1.24}$$

under our conditions, we suggest and analyze an iterative method for approximating a common solution of FPP (1.1), $VI(C,B)$ (1.2) and SGEP (1.13)-(1.14). Furthermore, we prove that the sequences generated by the iterative scheme converge strongly to a common solution of FPP (1.1), $VI(C,B)$ (1.2) and SGEP (1.13)-(1.14).

2. Preliminaries

Let H_1 be a real Hilbert space. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \tag{2.1}$$

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.2}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2.3}$$

for all $x, y \in H_1$ and $y \in [0,1]$. It is also known that H_1 satisfies the *Opial's condition* [26], i.e., for any sequence $\{x_n\} \subset H_1$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.4}$$

holds for every $y \in H_1$ with $x \neq y$. Hilbert space H_1 satisfies the *Kadee-Klee property* [15] that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

We recall some concepts and results which are needed in sequel. A mapping P_C is said to be *metric projection* of H_1 onto C if for every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.5}$$

It is well known that P_C is a nonexpansive mapping and is characterized by the following property:

$$\|P_C x - P_C y\|^2 \leq \langle x-y, P_C x - P_C y\rangle, \quad \forall x, y \in H_1. \tag{2.6}$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.7}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H_1, y \in C, \tag{2.8}$$

and

$$\|(x-y) - (P_C x - P_C y)\|^2 \geq \|x-y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H_1. \tag{2.9}$$

It is known that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $(x,y) \in H_1 \times H_1$, the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \leq \frac{1}{2}\|(T(x) - x) - (T(y) - y)\|^2, \tag{2.10}$$

and therefore, we get, for all $(x,y) \in H_1 \times Fix(T)$,

$$\langle x - T(x), y - T(x)\rangle \leq \frac{1}{2}\|T(x) - x\|^2, \tag{2.11}$$

(see, e.g., Theorem 3 in [32] and Theorem 1 in [30]).

Let B be a monotone mapping of C into H_1. In the context of the variational inequality problem the characterization of projection (2.7) implies the following:

$$u \in VI(C,B) \Leftrightarrow u = P_C(u - \lambda Bu), \lambda > 0.$$

Lemma 2.1 ([21]). *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

(i) $F(x,x) \geq 0, \forall x \in C$;

(ii) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0, \forall x \in C$;

(iii) F is upper hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y); \tag{2.12}$$

(iv) For each $x \in C$ fixed, the function $y \mapsto F(x, y)$ is convex and lower semicontinuous;

let $h : C \times C \to \mathbb{R}$ such that

(i) $h(x, y) \geq 0, \forall x \in C$;

(ii) For each $y \in C$ fixed, the function $x \to h(x, y)$ is upper semicontinuous;

(iii) For each $x \in C$ fixed, the function $y \to h(x, y)$ is convex and lower semicontinuous;

and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that

$$F(y, x) + h(y, x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \quad \forall y \in C \setminus K. \tag{2.13}$$

The proof of the following lemma is similar to the proof of Lemma 2.13 in [21] and hence omitted.

Lemma 2.2. *Assume that $F_1, h_1 : C \times C \to \mathbb{R}$ satisfy* Lemma 2.1. *Let $r > 0$ and $x \in H_1$. Then, there exists $z \in C$ such that*

$$F_1(z, y) + h_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.14}$$

Lemma 2.3 ([9])**.** *Assume that the bifunctions $F_1, h_1 : C \times C \to \mathbb{R}$ satisfy* Lemma 2.1 *and h_1 is monotone. For $r > 0$ and for all $x \in H_1$, define a mapping $T_r^{(F_1,h_1)} : H_1 \to C$ as follows:*

$$T_r^{(F_1,h_1)}(x) = \left\{ z \in C : F_1(z, y) + h_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}. \tag{2.15}$$

Then, the following hold:

(1) $T_r^{(F_1,h_1)}$ is single-valued.

(2) $T_r^{(F_1,h_1)}$ is firmly nonexpansive, i.e.,

$$\|T_r^{(F_1,h_1)}x - T_r^{(F_1,h_1)}y\|^2 \leq \langle T_r^{(F_1,h_1)}x - T_r^{(F_1,h_1)}y, x - y\rangle, \quad \forall x, y \in H_1. \tag{2.16}$$

(3) $Fix(T_r^{(F_1,h_1)}) = GEP(F_1, h_1)$.

(4) $GEP(F_1, h_1)$ is compact and convex.

Further, assume that $F_2, h_2 : Q \times Q \to \mathbb{R}$ satisfy Lemma 2.1. For $s > 0$ and for all $w \in H_2$, define a mapping $T_s^{(F_2,h_2)} : H_2 \to Q$ as follows:

$$T_s^{(F_2,h_2)}(w) = \left\{ d \in Q : F_2(d, e) + h_2(d, e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0, \quad \forall e \in Q \right\}. \tag{2.17}$$

Then, we easily observe that $T_s^{(F_2,h_2)}$ is single-valued and firmly nonexpansive, $GEP(F_2, h_2, Q)$ is compact and convex, and $Fix(T_s^{(F_2,h_2)}) = GEP(F_2, h_2, Q)$, where $GEP(F_2, h_2, Q)$ is the solution set of the following generalized equilibrium problem:

Find $y^* \in Q$ such that $F_2(y^*, y) + h_2(y^*, y) \geq 0, \forall y \in Q$.

We observe that $GEP(F_2, h_2) \subset GEP(F_2, h_2, Q)$. Further, it is easy to prove that Γ is a closed and convex set.

Remark 2.4. Lemmas 2.2 and 2.3 are slight generalizations of Lemma 3.5 in [10] where the equilibrium condition $F_1(\hat{x}, x) = h_1(\hat{x}, x) = 0$ has been relaxed to $F_1(\hat{x}, x) \geq 0$ and $h_1(\hat{x}, x) \geq 0$ for all $x \in C$. Further, the monotonicity of h_1 in Lemma 2.2 is not required.

Lemma 2.5 ([10])**.** *Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying* Lemma 2.1 *hold and let $T_r^{F_1}$ be defined as in* Lemma 2.3 *for $r > 0$. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then*

$$\|T_{r_2}^{F_1} y - T_{r_1}^{F_1} x\| \leq \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \|T_{r_2}^{F_1} y - y\|.$$

Lemma 2.6 ([22])**.** *Assume A is a strongly positive linear bounded operator on Hilbert space H_1 with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then, $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.7 ([33])**.** *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.8 ([27])**.** *Let X be an inner product space. Then, for any $x, y, z \in X$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

Lemma 2.9 ([4])**.** *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space E and $T : C \to C$ a nonexpansive mapping. For each $x \in C$ and the Cesàro means $T_n x = \frac{1}{n+1}\sum_{i=0}^n T^i x$, then $\limsup_{n\to\infty} \|T_n x - T(T_n x)\| = 0$.*

Lemma 2.10 ([38])**.** *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n,\ n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^\infty |\delta_n| < \infty$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11 ([26])**.** *Each Hilbert space H_1 satisfies the Opial condition that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$, holds for every $y \in H$ with $y \neq x$.*

3. Main Result

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces and $C \subset H_1$ and $Q \subset H_2$ be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $F_1, h_1 : C \times C \to \mathbb{R}$ and $F_2, h_2 : Q \times Q \to \mathbb{R}$ satisfy* Lemma 2.1*; h_1, h_2 are monotone and F_2 is upper semicontinuous. Let B be β-inverse-strongly monotone mapping from C into H_1. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$ and let D be a strongly positive linear bounded operator on H_1 with coefficient $\bar{\gamma} > 0$ and $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\{S^i\}_{i=1}^n$ be a sequence of nonexpansive mappings from C into itself such that*

$$\Omega := \cap_{i=1}^n Fix(S^i) \cap VI(C, B) \cap \Gamma \neq \emptyset.$$

Let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$ and

$$\begin{cases} u_n = T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n D)\frac{1}{n+1}\sum_{i=0}^n S^i y_n,\ \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$, $\{\lambda_n\} \in [a, b] \subset (0, 2\beta)$ and $\{r_n\} \subset (0, \infty)$ and $\xi \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(C3) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;

(C4) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = P_\Omega(I - D + \gamma f)(q)$, which is the unique solution of the variational inequality problem

$$\langle (D - \gamma f)q, x - q\rangle \ge 0, \ \forall x \in \Omega,$$

or, equivalently, q is the unique solution to the minimization problem

$$\min_{x\in\Omega} \frac{1}{2}\langle Dx, x\rangle - h(x),$$

where h is a potential function for γf such that $h'(x) = \gamma f(x)$ for $x \in H_1$.

Proof. From the condition (C1), we may assume without loss of generality that $\alpha_n \le (1 - \beta_n)\|D\|^{-1}$ for all $n \in \mathbb{N}$. By Lemma 2.6, we know that if $0 \le \rho \le \|D\|^{-1}$, then $\|I - \rho D\| \le 1 - \rho\bar{\gamma}$. We will assume that $\|I - D\| \le 1 - \bar{\gamma}$. Since D is a strongly positive linear bounded operator on H, we have

$$\|D\| = \sup\{|\langle Dx, x\rangle| : x \in H_1, \|x\| = 1\}.$$

Observe that

$$\begin{aligned}\Big\langle \big((1-\beta_n)I - \alpha_n D\big)x, x\Big\rangle &= 1 - \beta_n - \alpha_n\langle Dx, x\rangle \\ &\ge 1 - \beta_n - \alpha_n\|D\| \\ &\ge 0,\end{aligned}$$

this show that $(1 - \beta_n)I - \alpha_n D$ is positive. It follows that

$$\begin{aligned}\|(1-\beta_n)I - \alpha_n D\| &= \sup\left\{\left|\Big\langle \big((1-\beta_n)I - \alpha_n D\big)x, x\Big\rangle\right| : x \in H_1, \|x\| = 1\right\} \\ &= \sup\Big\{1 - \beta_n - \alpha_n\langle Dx, x\rangle : x \in H_1, \|x\| = 1\Big\} \\ &\le 1 - \beta_n - \alpha_n\bar{\gamma}.\end{aligned}$$

Since $\lambda_n \in (0, 2\beta)$ and B is β-inverse-strongly monotone mapping. For any $x, y \in C$, we have

$$\begin{aligned}\|(I - \lambda_n B)x - (I - \lambda_n B)y\|^2 &= \|(x - y) - \lambda_n(Bx - By)\|^2 \\ &= \|x - y\|^2 - 2\lambda_n\langle x - y, Bx - By\rangle + \lambda_n^2\|Bx - By\|^2 \\ &\le \|x - y\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bx - By\|^2 \\ &\le \|x - y\|^2. \end{aligned} \tag{3.2}$$

It follows that $\|(I - \lambda_n B)x - (I - \lambda_n B)y\| \le \|x - y\|$, hence $I - \lambda_n B$ is nonexpansive.

Step 1. We will show that $\{x_n\}$ is bounded.

Since $x^* \in \Omega$, i.e., $x^* \in \Gamma$, and we have $x^* = T_{r_n}^{(F_1,h_1)}x^*$ and $Ax^* = T_{r_n}^{(F_2,h_2)}Ax^*$.

We estimate

$$\|u_n - x^*\|^2 = \|T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n) - x^*\|^2$$

$$
\begin{aligned}
&= \|T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n) - T_{r_n}^{(F_1,h_1)}x^*\|^2 \\
&\le \|x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n - x^*\|^2 \\
&\le \|x_n - x^*\|^2 + \xi^2\|A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2 + 2\xi\langle x_n - x^*, A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\rangle. \qquad (3.3)
\end{aligned}
$$

Thus, we have

$$
\|u_n-x^*\|^2 \le \|x_n-x^*\|^2+\xi^2\langle(T_{r_n}^{(F_2,h_2)}-I)Ax_n, AA^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle+2\xi\langle x_n-x^*, A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle. \quad (3.4)
$$

Now, we have

$$
\begin{aligned}
\xi^2\langle(T_{r_n}^{(F_2,h_2)} - I)Ax_n, AA^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\rangle &\le L\xi^2\langle(T_{r_n}^{(F_2,h_2)} - I)Ax_n, (T_{r_n}^{(F_2,h_2)} - I)Ax_n\rangle \\
&= L\xi^2\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2. \qquad (3.5)
\end{aligned}
$$

Denoting $\Lambda := 2\xi\langle x_n - x^*, A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\rangle$ and using (2.11), we have

$$
\begin{aligned}
\Lambda &= 2\xi\langle x_n - x^*, A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\rangle \\
&= 2\xi\langle A(x_n - x^*), (T_{r_n}^{(F_2,h_2)} - I)Ax_n\rangle \\
&= 2\xi\langle A(x_n - x^*) + (T_{r_n}^{(F_2,h_2)} - I)Ax_n - (T_{r_n}^{(F_2,h_2)} - I)Ax_n, (T_{r_n}^{(F_2,h_2)} - I)Ax_n\rangle \\
&= 2\xi\left\{\langle T_{r_n}^{(F_2,h_2)}Ax_n - Ax^*, (T_{r_n}^{(F_2,h_2)} - I)Ax_n\rangle - \|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2\right\} \\
&\le 2\xi\left\{\frac{1}{2}\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2 - \|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2\right\} \\
&\le -\xi\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2. \qquad (3.6)
\end{aligned}
$$

Using (3.4), (3.5) and (3.6), we obtain

$$
\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 + \xi(L\xi - 1)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2. \qquad (3.7)
$$

Since $\xi \in (0, \frac{1}{L})$, we obtain

$$
\|u_n - x^*\|^2 \le \|x_n - x^*\|^2. \qquad (3.8)
$$

By the fact that P_C and $I - \lambda_n B$ are nonexpansive and $x^* = P_C(x^* - \lambda_n Bx^*)$, then we get

$$
\begin{aligned}
\|y_n - x^*\| &= \|P_C(u_n - \lambda_n Bu_n) - x^*\| \\
&\le \|P_C(u_n - \lambda_n Bu_n) - P_C(x^* - \lambda_n Bx^*)\| \\
&\le \|(I - \lambda_n B)u_n - (I - \lambda_n B)x^*\| \\
&\le \|u_n - x^*\| \\
&\le \|x_n - x^*\|. \qquad (3.9)
\end{aligned}
$$

Let $S_n = \frac{1}{n+1}\sum_{i=0}^{n} S^i$, it follows that

$$
\begin{aligned}
\|S_n x - S_n y\| &= \Big\|\frac{1}{n+1}\sum_{i=0}^{n} S^i x - \frac{1}{n+1}\sum_{i=0}^{n} S^i y\Big\| \\
&\le \frac{1}{n+1}\sum_{i=0}^{n}\|S^i x - S^i y\| \\
&\le \frac{1}{n+1}\sum_{i=0}^{n}\|x - y\| \\
&= \frac{n+1}{n+1}\|x - y\| = \|x - y\|,
\end{aligned}
$$

which implies that S_n is nonexpansive. Since $x^* \in \Omega$, we have

$$S_n x^* = \frac{1}{n+1}\sum_{i=0}^{n} S^i x^* = \frac{1}{n+1}\sum_{i=0}^{n} x^* = x^*, \forall x, y \in C.$$

By (3.9),we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(\gamma f(x_n) - Dx^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n D) \\
&\quad \times (S_n y_n - x^*)\| \\
&\le \alpha_n\|\gamma f(x_n) - Dx^*\| + \beta_n\|x_n - x^*\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - x^*\| \\
&\le \alpha_n\|\gamma f(x_n) - Dx^*\| + \beta_n\|x_n - x^*\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - x^*\| \\
&\le \alpha_n\gamma\|f(x_n) - f(x^*)\| + \alpha_n\|\gamma f(x^*) - Dx^*\| + (1 - \alpha_n\bar{\gamma})\|x_n - x^*\| \\
&\le \alpha_n\gamma\alpha\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Dx^*\| + (1 - \alpha_n\bar{\gamma})\|x_n - x^*\| \\
&= (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x_n - x^*\| + \alpha_n(\bar{\gamma} - \gamma\alpha)\frac{\|\gamma f(x^*) - Dx^*\|}{(\bar{\gamma} - \gamma\alpha)} \\
&\le \max\Big\{\|x_n - x^*\|, \frac{\|\gamma f(x^*) - Dx^*\|}{(\bar{\gamma} - \gamma\alpha)}\Big\}.
\end{aligned}$$

It follows from induction that

$$\|x_{n+1} - x^*\| \le \max\Big\{\|x_0 - x^*\|, \frac{\|\gamma f(x^*) - Dx^*\|}{(\bar{\gamma} - \gamma\alpha)}\Big\}.$$

Hence, $\{x_n\}$ is bounded, so are $\{u_n\}, \{y_n\}$ and $\{S_n y_n\}$.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Since $T_{r_{n+1}}^{(F_1,h_1)}$ and $T_{r_{n+1}}^{(F_2,h_2)}$ both are firmly nonexpansive, for $\xi \in (0, \frac{1}{L})$, the mapping $T_{r_{n+1}}^{(F_1,h_1)}(I + \xi A^*(T_{r_{n+1}}^{(F_2,h_2)} - I)A)$ is nonexpansive, see [5, 24]. Further, since $u_n = T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n)$ and $u_{n+1} = T_{r_{n+1}}^{(F_1,h_1)}(x_{n+1} + \xi A^*(T_{r_{n+1}}^{(F_2,h_2)} - I)Ax_{n+1})$, it follows from Lemma 2.5 that

$$\begin{aligned}
\|u_{n+1} - u_n\| &\le \|T_{r_{n+1}}^{(F_1,h_1)}(x_{n+1} + \xi A^*(T_{r_{n+1}}^{(F_2,h_2)} - I)Ax_{n+1}) - T_{r_{n+1}}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_{n+1}}^{(F_2,h_2)} - I)Ax_n)\| \\
&\quad + \|T_{r_{n+1}}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_{n+1}}^{(F_2,h_2)} - I)Ax_n) - T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n)\| \\
&\le \|x_{n+1} - x_n\| + \|(x_n + \xi A^*(T_{r_{n+1}}^{(F_2,h_2)} - I)Ax_n) - (x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n)\| \\
&\quad + \Big|1 - \frac{r_n}{r_{n+1}}\Big|\|T_{r_{n+1}}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n) - (x_n + \xi A^*(T_{r_{n+1}}^{(F_2,h_2)} - I)Ax_n)\| \\
&\le \|x_{n+1} - x_n\| + \xi\|A\|\|T_{r_{n+1}}^{(F_2,h_2)}Ax_n - T_{r_n}^{(F_2,h_2)}Ax_n\| + \varsigma_n \\
&\le \|x_{n+1} - x_n\| + \xi\|A\|\Big|1 - \frac{r_n}{r_{n+1}}\Big|\|T_{r_{n+1}}^{(F_2,h_2)}Ax_n - Ax_n\| + \varsigma_n \\
&= \|x_{n+1} - x_n\| + \xi\|A\|\sigma_n + \varsigma_n,
\end{aligned} \tag{3.10}$$

where

$$\sigma_n := \Big|1 - \frac{r_n}{r_{n+1}}\Big|\|T_{r_n}^{(F_2,h_2)}Ax_n - Ax_n\|$$

and

$$\varsigma_n := \Big|1 - \frac{r_n}{r_{n+1}}\Big|\|T_{r_{n+1}}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n) - (x_n + \xi A^*(T_{r_{n+1}}^{(F_2,h_2)} - I)Ax_n)\|.$$

On the other hand, it follows that

$$\|y_{n+1} - y_n\| = \|P_C(u_{n+1} - \lambda_{n+1}Du_{n+1}) - P_C(u_n - \lambda_n Du_n)\|$$

$$\begin{aligned}
&\leq \|(u_{n+1} - \lambda_{n+1} D u_{n+1}) - (u_n - \lambda_n D u_n)\| \\
&= \|(u_{n+1} - u_n) - \lambda_{n+1}(D u_{n+1} - D u_n) + (\lambda_{n+1} - \lambda_n) D u_n\| \\
&\leq \|(u_{n+1} - u_n) - \lambda_{n+1}(D u_{n+1} - D u_n)\| + |\lambda_{n+1} - \lambda_n| \|D u_n\| \\
&\leq \|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n| \|D u_n\|. \qquad (3.11)
\end{aligned}$$

So from (3.10) and (3.11), we get

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \xi \|A\| \sigma_n + \varsigma_n + |\lambda_{n+1} - \lambda_n| \|D u_n\|. \qquad (3.12)$$

We compute that

$$\begin{aligned}
\|S_{n+1} y_{n+1} - S_n y_n\| &\leq \|S_{n+1} y_{n+1} - S_{n+1} y_n\| + \|S_{n+1} y_n - S_n y_n\| \\
&\leq \|y_{n+1} - y_n\| + \left\| \frac{1}{n+2} \sum_{i=0}^{n+1} S^i y_n - \frac{1}{n+1} \sum_{i=0}^{n} S^i y_n \right\| \\
&= \|y_{n+1} - y_n\| + \left\| \frac{1}{n+2} \sum_{i=0}^{n} S^i y_n + \frac{1}{n+2} S^{n+1} y_n - \frac{1}{n+1} \sum_{i=0}^{n} S^i y_n \right\| \\
&= \|y_{n+1} - y_n\| + \left\| - \frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} S^i y_n + \frac{1}{n+2} S^{n+1} y_n \right\| \\
&\leq \|y_{n+1} - y_n\| + \frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} \|S^i y_n\| + \frac{1}{n+2} \|S^{n+1} y_n\| \\
&\leq \|y_{n+1} - y_n\| + \frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} (\|S^i y_n - S^i x^*\| + \|x^*\|) \\
&\quad + \frac{1}{n+2} (\|S^{n+1} y_n - S^{n+1} x^*\| + \|x^*\|) \\
&\leq \|y_{n+1} - y_n\| + \frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} (\|y_n - x^*\| + \|x^*\|) \\
&\quad + \frac{1}{n+2} (\|y_n - x^*\| + \|x^*\|) \\
&\leq \|y_{n+1} - y_n\| + \frac{n+1}{(n+1)(n+2)} (\|y_n - x^*\| + \|x^*\|) \\
&\quad + \frac{1}{n+2} \|y_n - x^*\| + \frac{1}{n+2} \|x^*\| \\
&= \|y_{n+1} - y_n\| + \frac{2}{n+2} \|y_n - x^*\| + \frac{2}{n+2} \|x^*\| \\
&\leq \|x_{n+1} - x_n\| + \xi \|A\| \sigma_n + \varsigma_n + |\lambda_{n+1} - \lambda_n| \|D u_n\| \\
&\quad + \frac{2}{n+2} \|y_n - x^*\| + \frac{2}{n+2} \|x^*\|.
\end{aligned}$$

Let $x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n$, it follows that

$$\begin{aligned}
z_n &= \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
&= \frac{\alpha_n \gamma f(x_n) + ((1 - \beta_n) I - \alpha_n D) S_n y_n}{1 - \beta_n},
\end{aligned}$$

and hence

$$\|z_{n+1} - z_n\| = \left\| \frac{\alpha_{n+1} \gamma f(x_{n+1}) + ((1 - \beta_{n+1}) I - \alpha_{n+1} D) S_{n+1} y_{n+1}}{1 - \beta_{n+1}} \right.$$

$$
\begin{aligned}
&- \frac{\alpha_n\gamma f(x_n) + ((1-\beta_n)I - \alpha_n D)S_n y_n}{1-\beta_n}\Big\| \\
=&\Big\|\frac{\alpha_{n+1}\gamma f_(x_{n+1})}{1-\beta_{n+1}} + \frac{(1-\beta_{n+1})S_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_{n+1}DS_{n+1}y_{n+1}}{1-\beta_{n+1}} \\
&- \frac{\alpha_n\gamma f(x_n)}{1-\beta_n} - \frac{(1-\beta_n)S_n y_n}{1-\beta_n} + \frac{\alpha_n DS_n y_n}{1-\beta_n}\Big\| \\
=&\Big\|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - DS_{n+1}y_{n+1}) \\
&+ \frac{\alpha_n}{1-\beta_n}(DS_n y_n - \gamma f(x_n)) + S_{n+1}y_{n+1} - S_n y_n\Big\| \\
\le&\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - DS_{n+1}y_{n+1}\| \\
&+ \frac{\alpha_n}{1-\beta_n}\|DS_n y_n - \gamma f(x_n)\| + \|S_{n+1}y_{n+1} - S_n y_n\| \\
\le&\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - DS_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|DS_n y_n - \gamma f(x_n)\| \\
&+ \|x_{n+1} - x_n\| + \xi\|A\|\sigma_n + \varsigma_n + |\lambda_{n+1} - \lambda_n|\|Du_n\| \\
&+ \frac{2}{n+2}\|y_n - x^*\| + \frac{2}{n+2}\|x^*\|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - DS_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|DS_n y_n - \gamma f(x_n)\| \\
&+ \xi\|A\|\sigma_n + \varsigma_n + |\lambda_{n+1} - \lambda_n|\|Du_n\| + \frac{2}{n+2}\|y_n - x^*\| + \frac{2}{n+2}\|x^*\|.
\end{aligned}
$$

It follows from $n \to \infty$ and the conditions (C1)-(C4), that

$$
\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.
$$

From Lemma 2.7, we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$ and also

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{3.13}
$$

Step 3. We will show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$.

For $x^* \in \Omega, x^* = T_{r_n}^{(F_1,h_1)}x^*$ and $T_{r_n}^{(F_1,h_1)}$ is firmly nonexpansive, we obtain

$$
\begin{aligned}
\|u_n - x^*\|^2 =& \|T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n) - x^*\|^2 \\
=& \|T_{r_n}^{(F_1,h_1)}(x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n) - T_{r_n}^{(F_1,h_1)}x^*\|^2 \\
\le& \langle u_n - x^*, x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n - x^*\rangle \\
=& \frac{1}{2}\Big\{\|u_n - x^*\|^2 + \|x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n - x^*\|^2 \\
&- \|(u_n - x^*) - [x_n + \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n - x^*]\|^2\Big\} \\
=& \frac{1}{2}\Big\{\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|u_n - x_n - \xi A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2\Big\} \\
=& \frac{1}{2}\Big\{\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - [\|u_n - x_n\|^2 + \xi^2\|A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2 \\
&- 2\xi\langle u_n - x_n, A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\rangle]\Big\}.
\end{aligned}
$$

Hence, we obtain

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2\xi\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|. \tag{3.14}$$

Using (3.7), (3.9) and Lemma 2.8, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 =& \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)S_n y_n - x^*\|^2 \\
=& \|\alpha_n(\gamma f(x_n) - Dx^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n D) \times (S_n y_n - x^*)\|^2 \\
\leq& \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - x^*\|^2 \\
\leq& \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - x^*\|^2 \\
\leq& \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&+ (1 - \beta_n - \alpha_n\bar{\gamma})(\|x_n - x^*\|^2 + \xi(L\xi - 1)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2) \\
=& \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - x^*\|^2 \\
&- (1 - \beta_n - \alpha_n\bar{\gamma})\xi(1 - L\xi)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(1 - \beta_n - \alpha_n\bar{\gamma})\xi(1 - L\xi)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\|\gamma f(x_n) - Dx^*\|^2 - \alpha_n\bar{\gamma}\|x_n - x^*\|^2 \\
&\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - Dx^*\|^2 - \alpha_n\bar{\gamma}\|x_n - x^*\|^2.
\end{aligned}$$

Since $\alpha_n \to 0, (1 - \beta_n - \alpha_n\bar{\gamma})\xi(1 - L\xi) > 0$ and $lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$, we obtain

$$\lim_{n\to\infty} \|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\| = 0. \tag{3.15}$$

Using (3.9), (3.14) and Lemma 2.8, we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)S_n y_n - x^*\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Dx^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n D) \times (S_n y_n - x^*)\|^2 \\
&\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - x^*\|^2 \\
&\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - x^*\|^2 \\
&\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + (1 - \beta_n - \alpha_n\bar{\gamma})(\|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2\xi\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|) \\
&= \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - x^*\|^2 \\
&\quad - (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - x_n\|^2 + 2\xi(1 - \beta_n - \alpha_n\bar{\gamma})\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|).
\end{aligned}$$

Then, we have

$$\begin{aligned}
&(1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - u_n\|^2 \\
&\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2\xi(1 - \beta_n - \alpha_n\bar{\gamma})\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\| \\
&\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 - \alpha_n\bar{\gamma}\|x_n - x^*\|^2 \\
&\quad + 2\xi(1 - \beta_n - \alpha_n\bar{\gamma})\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|) \\
&\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) - \alpha_n\bar{\gamma}\|x_n - x^*\|^2 \\
&\quad + 2\xi(1 - \beta_n - \alpha_n\bar{\gamma})\|A(u_n - x_n)\|\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|).
\end{aligned}$$

By condition (C1), (3.13) and (3.15), then we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.16}$$

Step 4. We will show that $\lim_{n\to\infty} \|S_n y_n - x_n\| = 0$.
Indeed, observe that

$$\begin{aligned}\|x_n - S_n y_n\| \leq& \|x_n - x_{n+1}\| + \|x_{n+1} - S_n y_n\|\\ =& \|x_n - x_{n+1}\| + \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)S_n y_n - S_n y_n\|\\ =& \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - \alpha_n D S_n y_n + \alpha_n D S_n y_n + \beta_n x_n - \beta_n S_n y_n + \beta_n S_n y_n\\ &+ ((1-\beta_n)I - \alpha_n D)S_n y_n - S_n y_n\|\\ \leq& \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - D S_n y_n\| + \beta_n\|x_n - S_n y_n\|\end{aligned}$$

and then

$$\|x_n - S_n y_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - D S_n y_n\|.$$

Since from condition (C1), (C2) and (3.13), we get

$$\lim_{n\to\infty} \|x_n - S_n y_n\| = 0. \tag{3.17}$$

Step 5. We will show that

(i) $\lim_{n\to\infty} \|y_n - u_n\| = 0$;

(ii) $\lim_{n\to\infty} \|S_n y_n - y_n\| = 0$.

From (3.2), (3.8) and Lemma 2.8, we obtain

$$\begin{aligned}&\|x_{n+1} - x^*\|^2\\ &\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + ((1-\beta_n)I - \alpha_n D)\|S_n y_n - x^*\|^2\\ &\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\|y_n - x^*\|^2\\ &\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\|P_C(u_n - \lambda_n B u_n) - P_C(x^* - \lambda_n B x^*)\|^2\\ &\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\|(u_n - \lambda_n B u_n) - (x^* - \lambda_n B x^*)\|^2\\ &\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\{\|u_n - x^*\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bu_n - Bx^*\|^2\}\\ &\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + (1-\alpha_n\bar{\gamma})\|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\lambda_n(\lambda_n - 2\beta)\|Bu_n - Bx^*\|^2\\ &\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})\lambda_n(\lambda_n - 2\beta)\|Bu_n - Bx^*\|^2\\ &\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \|x_n - x^*\|^2 + (1-\beta_n-\alpha_n\bar{\gamma})a(b - 2\beta)\|Bu_n - Bx^*\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}0 &\leq (1-\beta_n-\alpha_n\bar{\gamma})a(2\beta - b)\|Bu_n - Bx^*\|^2\\ &\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\\ &\leq \alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|).\end{aligned}$$

Since $\alpha_n \to 0, \|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, so we get

$$\lim_{n\to\infty} \|Bu_n - Bx^*\| = 0. \tag{3.18}$$

Next, we will show that $\lim_{n\to\infty} \|u_n - y_n\| = 0$.

Further, we observe that

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \|P_C(u_n - \lambda_n Bu_n) - P_C(x^* - \lambda_n Bx^*)\|^2 \\
&\leq \langle (u_n - \lambda_n Bu_n) - (x^* - \lambda_n Bx^*), y_n - x^* \rangle \\
&\leq \frac{1}{2}\{\|(u_n - \lambda_n Bu_n) - (x^* - \lambda_n Bx^*)\|^2 + \|y_n - x^*\|^2 - \|(u_n - \lambda_n Bu_n) - (x^* - \lambda_n Bx^*) - (y_n - x^*)\|^2\} \\
&\leq \frac{1}{2}\{\|u_n - x^*\|^2 + \|y_n - x^*\|^2 - \|(u_n - y_n) - \lambda_n(Bu_n - Bx^*)\|^2\} \\
&\leq \frac{1}{2}\{\|u_n - x^*\|^2 + \|y_n - x^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Bu_n - Bx^*\rangle - \lambda_n^2\|Bu_n - Bx^*\|^2\},
\end{aligned}$$

so, we obtain

$$\|y_n - x^*\|^2 \leq \|u_n - x^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Bu_n - Bx^*\rangle - \lambda_n^2\|Bu_n - Bx^*\|^2, \tag{3.19}$$

and hence from (3.9) and (3.19), we get

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
\leq&\alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - x^*\|^2 \\
\leq&\alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 \\
&+ (1 - \beta_n - \alpha_n\bar{\gamma})\{\|u_n - x^*\|^2 - \|u_n - y_n\|^2 + 2\lambda_n\langle u_n - y_n, Bu_n - Bx^*\rangle - \lambda_n^2\|Bu_n - Bx^*\|^2\} \\
=&\alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - x^*\|^2 \\
&- (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - y_n\|^2 + 2\lambda_n(1 - \beta_n - \alpha_n\bar{\gamma})\langle u_n - y_n, Bu_n - Bx^*\rangle \\
&- (1 - \beta_n - \alpha_n\bar{\gamma})\lambda_n^2\|Bu_n - Bx^*\|^2 \\
\leq&\alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\bar{\gamma})\|x_n - x^*\|^2 \\
&- (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - y_n\|^2 + 2\lambda_n(1 - \beta_n - \alpha_n\bar{\gamma})\langle u_n - y_n, Bu_n - Bx^*\rangle \\
&- (1 - \beta_n - \alpha_n\bar{\gamma})\lambda_n^2\|Bu_n - Bx^*\|^2 \\
\leq&\alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \|x_n - x^*\|^2 - \alpha_n\bar{\gamma}\|x_n - x^*\|^2 - (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - y_n\|^2 \\
&+ 2\lambda_n(1 - \beta_n - \alpha_n\bar{\gamma})\langle u_n - y_n, Bu_n - Bx^*\rangle - (1 - \beta_n - \alpha_n\bar{\gamma})\lambda_n^2\|Bu_n - Bx^*\|^2 \\
\leq&\alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \|x_n - x^*\|^2 - \alpha_n\bar{\gamma}\|x_n - x^*\|^2 - (1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - y_n\|^2 \\
&+ 2\lambda_n(1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - y_n\|\|Bu_n - Bx^*\| - (1 - \beta_n - \alpha_n\bar{\gamma})\lambda_n^2\|Bu_n - Bx^*\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - y_n\|^2 \\
&\quad\leq\alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 - \alpha_n\bar{\gamma}\|x_n - x^*\|^2 \\
&\qquad+ 2\lambda_n(1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - y_n\|\|Bu_n - Bx^*\| - (1 - \beta_n - \alpha_n\bar{\gamma})\lambda_n^2\|Bu_n - Bx^*\|^2 \\
&\quad\leq\alpha_n\|\gamma f(x_n) - Dx^*\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) - \alpha_n\bar{\gamma}\|x_n - x^*\|^2 \\
&\qquad+ 2\lambda_n(1 - \beta_n - \alpha_n\bar{\gamma})\|u_n - y_n\|\|Bu_n - Bx^*\| - (1 - \beta_n - \alpha_n\bar{\gamma})\lambda_n^2\|Bu_n - Bx^*\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\|Bu_n - Bx^*\| = 0, \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and the conditions (C1)-(C3), we have

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.20}$$

Consequently, from (3.16), (3.17) and (3.20), we observe that

$$\|S_n y_n - y_n\| \leq \|S_n y_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\| \to 0 \ \text{ as } \ n \to \infty. \tag{3.21}$$

By Lemma 2.9, we have $\limsup_{n\to\infty}\|S_ny_n - S(S_ny_n)\| = 0$.

Step 6. We claim that $\limsup_{n\to\infty}\langle (D-\gamma f)q, q-x_n\rangle \leq 0$, where q is the unique solution of the variational inequality $\langle (D-\gamma f)q, x_n - q\rangle \geq 0$.

To show this inequality, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$, such that

$$\lim_{i\to\infty}\langle (D-\gamma f)q, q-y_{n_i}\rangle = \limsup_{n\to\infty}\langle (D-\gamma f)q, q-y_n\rangle.$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_k}}\}$ of $\{y_{n_i}\}$ which converge weakly to $z\in C$. Without loss of generality, we can assume that $y_{n_i}\rightharpoonup z$. From $\|S_ny_n - S(S_ny_n)\|\to 0$, as $n\to\infty$, we obtain $S(S_{n_i}y_{n_i})\rightharpoonup z$.

Step 7. We will show that $z\in\Omega$. **Step 7.1** First, we show that $z\in Fix(S_n) = \frac{1}{n+1}\sum_{i=0}^{n} Fix(S^i)$. Assume that $z\notin \frac{1}{n+1}\sum_{i=0}^{n} Fix(S^i)$. Since $y_{n_i}\rightharpoonup z$ and $Tz\neq z$. From Lemma 2.11, we have

$$\begin{aligned}\liminf_{i\to\infty}\|y_{n_i}-z\| &< \liminf_{i\to\infty}\|y_{n_i}-Sz\|\\ &\leq \liminf_{i\to\infty}(\|y_{n_i}-Sy_{n_i}\| + \|Sy_{n_i}-Sz\|)\\ &\leq \liminf_{i\to\infty}\|y_{n_i}-z\|,\end{aligned}$$

which is a contradiction. Thus, we obtain $z\in Fix(S_n) = \frac{1}{n+1}\sum_{i=0}^{n} Fix(S^i)$.

Step 7.2 We will show that $z\in\Gamma$.

First, we will show $z\in GEP(F_1,h_1)$.

Since $u_n = T_{r_n}^{(F_1,h_1)}x_n$, we have

$$F_1(u_n,w) + h_1(u_n,w) + \frac{1}{r_n}\langle w-u_n, u_n-x_n\rangle \geq 0, \quad \forall w\in C.$$

It follows from the monotonicity of F_1 that

$$h_1(u_n,w) + \frac{1}{r_n}\langle w-u_n, u_n-x_n\rangle \geq F_1(w,u_n),$$

and hence replacing n by n_i, we get

$$h_1(u_{n_i},w) + \left\langle w-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\right\rangle \geq F_1(w,u_{n_i}).$$

Since $\|u_n - x_n\|\to 0$, and $x_n\rightharpoonup z$, we get $u_{n_i}\rightharpoonup z$ and $\frac{u_{n_i}-x_{n_i}}{r_{n_i}}\to 0$. It follows by Lemma 2.1 (iv) that $0\geq F_1(w,z), \forall z\in C$. For any t with $0<t\leq 1$ and $w\in C$, let $w_t = tw + (1-t)z$. Since $w\in C, z\in C$, we have $w_t\in C$, and hence, $F_1(w_t,z)\leq 0$. So, from Lemma 2.1 (i) and (iv), we have

$$\begin{aligned}0 &= F_1(w_t,w_t) + h_1(w_t,w_t)\\ &\leq t[F_1(w_t,w) + h_1(w_t,w)] + (1-t)[F_1(w_t,z) + h_1(w_t,z)]\\ &\leq t[F_1(w_t,w) + h_1(w_t,w)] + (1-t)[F_1(z,w_t) + h_1(z,w_t)]\\ &\leq [F_1(w_t,w) + h_1(w_t,w)].\end{aligned}$$

Therefore, $0\leq F_1(w_t,w) + h_1(w_t,w)$. From Lemma 2.1 (iii), we have $0\leq F_1(z,w) + h_1(z,w)$. This implies that $z\in GEP(F_1,h_1)$.

Next, we show that $Az\in GEP(F_2,h_2)$. Since $\|u_n - x_n\|\to 0, u_n\rightharpoonup z$ as $n\to\infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_i\}$ such that $x_{n_i}\rightharpoonup z$, and since A is bounded linear operator, so $Ax_{n_i}\rightharpoonup Az$.

Now, setting $k_{n_i} = Ax_{n_i} - T_{r_{n_i}}^{(F_2,h_2)}Ax_{n_i}$. It follows from (3.15) that $\lim_{i\to\infty} k_{n_i} = 0$ and $Ax_{n_i} - k_{n_i} = T_{r_{n_i}}^{(F_2,h_2)}Ax_{n_i}$.

Therefore, from Lemma 2.3, we have

$$F_2(Ax_{n_i}-k_{n_i},\tilde{z})+h_2(Ax_{n_i}-k_{n_i},\tilde{z})+\frac{1}{r_{n_i}}\langle\tilde{z}-(Ax_{n_i}-k_{n_i}),(Ax_{n_i}-k_{n_i})-Ax_{n_i}\rangle\geq 0,\quad\forall\tilde{z}\in Q.$$

Since F_2 and h_2 are upper semicontinuous taking $\limsup$ to above inequality as $i\to\infty$ and using condition (iv), we obtain

$$F_2(Az,\tilde{z})+h_2(Ax,\tilde{z})\geq 0,\quad\forall\tilde{z}\in Q,$$

which means that $Az\in GEP(F_2,h_2)$ and hence $z\in\Gamma$.

Step 7.3 We will show that $z\in VI(C,B)$.

Let $M:H\to 2^H$ be a set-valued mapping defined by

$$Mv=\begin{cases}Bv+N_Cv, & v\in C;\\ \emptyset, & v\notin C,\end{cases}$$

where $N_Cv:=\{z\in H_1:\langle v-u,z\rangle\geq 0,\forall u\in C\}$ is the normal cone to C at $v\in C$. Then M is maximal monotone and $0\in Mv$ if and only if $v\in VI(C,B)$; (see [28]) for more details. Let $(v,u)\in G(M)$. Then we have

$$u\in Mv=Bv+N_Cv,$$

and hence

$$u-Bv\in N_Cv.$$

Since $y_n\in C,\forall n$, so we have

$$\langle v-y_n,u-Bv\rangle\geq 0. \tag{3.22}$$

On the other hand, from $y_n=P_C(u_n-\lambda_nBu_n)$, we have

$$\langle v-y_n,y_n-(u_n-\lambda_nBu_n)\rangle\geq 0,$$

that is

$$\left\langle v-y_n,\frac{y_n-u_n}{\lambda_n}+Bu_n\right\rangle\geq 0.$$

Therefore, we have

$$\begin{aligned}\langle v-y_{n_i},u\rangle&\geq\langle v-y_{n_i},Bv\rangle\\&\geq\langle v-y_{n_i},Bv\rangle-\left\langle v-y_{n_i},\frac{y_{n_i}-u_{n_i}}{\lambda_{n_i}}+Bu_{n_i}\right\rangle\\&=\left\langle v-y_{n_i},Bv-\frac{y_{n_i}-u_{n_i}}{\lambda_{n_i}}-Bu_{n_i}\right\rangle\\&=\langle v-y_{n_i},Bv-By_{n_i}\rangle+\langle v-y_{n_i},By_{n_i}-Bu_{n_i}\rangle-\left\langle v-y_{n_i},\frac{y_{n_i}-u_{n_i}}{\lambda_{n_i}}\right\rangle\\&\geq\langle v-y_{n_i},By_{n_i}-Bu_{n_i}\rangle-\left\langle v-y_{n_i},\frac{y_{n_i}-u_{n_i}}{\lambda_{n_i}}\right\rangle.\end{aligned}\tag{3.23}$$

Note that $y_{n_i}\rightharpoonup z$, $\|y_{n_i}-u_{n_i}\|\to 0$ as $i\to\infty$ and B is β-inverse-strongly monotone, hence from (3.23), we obtain $\langle v-z,u\rangle\geq 0$ as $i\to\infty$. Since M is maximal monotone, we have $z\in M^{-1}0$, and hence $z\in VI(C,B)$. Therefore $z\in\Omega$.

Since $q=P_\Omega(I-D+\gamma f)(q)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma f-D)q,x_n-q\rangle&=\limsup_{n\to\infty}\langle(\gamma f-D)q,S_ny_n-q\rangle\\&=\lim_{i\to\infty}\langle(\gamma f-D)q,S_{n_i}y_{n_i}-q\rangle\\&=\langle(\gamma f-D)q,z-q\rangle\leq 0.\end{aligned}\tag{3.24}$$

Step 8. Finally, we show that $\{x_n\}$ converge strongly to q, we obtain that

$$\begin{aligned}
\|x_{n+1}-q\|^2 =&\|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n D)S_n y_n - q\|^2\\
=&\|\alpha_n(\gamma f(x_n)-Dq)+\beta_n(x_n-q)+((1-\beta_n)I-\alpha_n D)(S_n y_n-q)\|^2\\
=&\alpha_n^2\|\gamma f(x_n)-Dq\|^2+\|\beta_n(x_n-q)+((1-\beta_n)I-\alpha_n D)(S_n y_n-q)\|^2\\
&+2\langle\beta_n(x_n-q)+((1-\beta_n)I-\alpha_n D)(S_n y_n-q),\alpha_n(\gamma f(x_n)-Dq)\rangle\\
\le&\alpha_n^2\|\gamma f(x_n)-Dq\|^2+\{\beta_n\|x_n-q\|+(1-\beta_n-\alpha_n\bar\gamma)\|y_n-q\|\}^2\\
&+2\beta_n\alpha_n\langle x_n-q,\gamma f(x_n)-Dq\rangle+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle S_n y_n-q,\gamma f(x_n)-Dq\rangle\\
=&\alpha_n^2\|\gamma f(x_n)-Dq\|^2+\{\beta_n\|x_n-q\|+(1-\beta_n-\alpha_n\bar\gamma)\|x_n-q\|\}^2\\
&+2\beta_n\alpha_n\langle x_n-q,\gamma f(x_n)-\gamma f(q)+\gamma f(q)-D(q)\rangle\\
&+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle S_n y_n-q,\gamma f(x_n)-\gamma f(q)+\gamma f(q)-Dq\rangle\\
\le&\alpha_n^2\|\gamma f(x_n)-Dq\|^2+\{\beta_n\|x_n-q\|+(1-\beta_n-\alpha_n\bar\gamma)\|x_n-q\|\}^2\\
&+2\beta_n\alpha_n\langle x_n-q,\gamma f(x_n)-\gamma f(q)\rangle+2\alpha_n\beta_n\langle x_n-q,\gamma f(q)-Dq\rangle\\
&+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle S_n y_n-q,\gamma f(x_n)-\gamma f(q)\rangle\\
&+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle S_n y_n-q,\gamma f(q)-Dq\rangle\\
\le&\alpha_n^2\|\gamma f(x_n)-Dq\|^2+(1-\alpha_n\bar\gamma)^2\|x_n-q\|^2+2\alpha_n\beta_n\gamma\|x_n-q\|\|f(x_n)-f(q)\|\\
&+2\alpha_n\beta_n\langle x_n-q,\gamma f(q)-Dq\rangle+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\gamma\|S_n y_n-q\|\|f(x_n)-f(q)\|\\
&+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle S_n y_n-q,\gamma f(q)-Dq\rangle\\
\le&\alpha_n^2\|\gamma f(x_n)-Dq\|^2+(1-\alpha_n\bar\gamma)^2\|x_n-q\|^2+2\alpha_n\beta_n\gamma\alpha\|x_n-q\|^2\\
&+2\alpha_n\beta_n\langle x_n-q,\gamma f(q)-Dq\rangle+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\gamma\alpha\|x_n-q\|^2\\
&+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle S_n y_n-q,\gamma f(q)-Dq\rangle\\
=&\alpha_n^2\|\gamma f(x_n)-Dq\|^2+(1-2\alpha_n\bar\gamma+\alpha_n^2\bar\gamma^2)\|x_n-q\|^2+2\alpha_n\beta_n\gamma\alpha\|x_n-q\|^2\\
&+2\alpha_n\beta_n\langle x_n-q,\gamma f(q)-Dq\rangle+(2\alpha_n\gamma\alpha-2\alpha_n\beta_n\gamma\alpha-2\alpha_n^2\bar\gamma\gamma\alpha)\|x_n-q\|^2\\
&+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle S_n y_n-q,\gamma f(q)-Dq\rangle\\
=&\alpha_n^2\|\gamma f(x_n)-Dq\|^2+(1-2\alpha_n\bar\gamma+\alpha_n^2\bar\gamma^2+2\alpha_n\gamma\alpha-2\alpha_n^2\bar\gamma\gamma\alpha)\|x_n-q\|^2\\
&+2\alpha_n\beta_n\langle x_n-q,\gamma f(q)-Dq\rangle+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle S_n y_n-q,\gamma f(q)-Dq\rangle\\
\le&(1-\alpha_n(2\bar\gamma-\alpha_n\bar\gamma^2-2\gamma\alpha+2\alpha_n\bar\gamma\gamma\alpha))\|x_n-q\|^2+\alpha_n^2\|\gamma f(x_n)-Dq\|^2\\
&+2\alpha_n\beta_n\langle x_n-q,\gamma f(q)-Dq\rangle+2\alpha_n(1-\beta_n-\alpha_n\bar\gamma)\langle S_n y_n-q,\gamma f(q)-Dq\rangle\\
\le&(1-\alpha_n(2\bar\gamma^2-\alpha_n\bar\gamma-2\gamma\alpha+2\alpha_n\bar\gamma\gamma\alpha))\|x_n-q\|^2+\alpha_n\delta_n, \qquad (3.25)
\end{aligned}$$

where $\delta_n:=\alpha_n\|\gamma f(x_n)-Dq\|^2+2\beta_n\langle x_n-q,\gamma f(q)-Dq\rangle+2(1-\beta_n-\alpha_n\bar\gamma)\langle S_n y_n-q,\gamma f(q)-Dq\rangle$.

By (3.24), the conditions (C1) and (C2), we get $\limsup_{n\to\infty}\delta_n\le 0$. Applying Lemma 2.10 to (3.25) we conclude that $x_n\to q$. This completes the proof. □

4. Consequently results

Corollary 4.1. *Let H_1 and H_2 be two real Hilbert spaces and $C\subset H_1$ and $Q\subset H_2$ be nonempty closed convex subsets of H_1 and H_2, respectively. Let $A:H_1\to H_2$ be a bounded linear operator. Let $F_1:C\times C\to\mathbb{R}$ and $F_2:Q\times Q\to\mathbb{R}$ satisfy* Lemma 2.1 *and F_2 is upper semicontinuous. Let B be β-inverse-strongly monotone mapping from C into H_1. Let f be a contraction of C into itself with coefficient $\alpha\in(0,1)$ and let D be a strongly positive linear bounded operator on H_1 with coefficient $\bar\gamma>0$ and $0<\gamma<\frac{\bar\gamma}{\alpha}$. Let $\{S^i\}_{i=1}^n$ be a sequence of nonexpansive mappings from C into itself such that*

$$\Omega:=\cap_{i=1}^n Fix(S^i)\cap VI(C,B)\cap\Theta\neq\emptyset.$$

Let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$ and

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \xi A^*(T_{r_n}^{F_2} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n D)\frac{1}{n+1}\sum_{i=0}^{n} S^i y_n, \ \forall n \geq 0, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \in [a,b] \subset (0, 2\beta)$ and $\{r_n\} \subset (0,\infty)$ and $\xi \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A and A^* is the adjoint of A satisfy the following conditions (C1)-(C4). Then $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = P_\Omega(I - D + \gamma f)(q)$, which is the unique solution of the variational inequality problem

$$\langle (D - \gamma f)q, x - q\rangle \geq 0, \ \forall x \in \Omega,$$

or, equivalently, q is the unique solution to the minimization problem

$$\min_{x\in\Omega} \frac{1}{2}\langle Dx, x\rangle - h(x),$$

where h is a potential function for γf such that $h'(x) = \gamma f(x)$ for $x \in H_1$.

Proof. Taking $h_1 = h_2 = 0$ in Theorem 3.1, then the conclusion of Corollary 4.1 is obtained. □

Corollary 4.2. *Let H be real Hilbert spaces and $C \subset H$. Let $F : C \times C \to \mathbb{R}$ satisfying* Lemma 2.1*. Let B be β-inverse-strongly monotone mapping from C into H. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$ Let $S : C \to C$ be nonexpansive mapping such that*

$$\Omega := Fix(S) \cap VI(C,B) \cap EP(F) \neq \emptyset.$$

Let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$ and

$$\begin{cases} u_n = T_{r_n}^{F} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + (1-\beta_n - \alpha_n) S y_n, \ \forall n \geq 0, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \in [a,b] \subset (0,2\beta)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions (C1)-(C4). Then $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = P_\Omega f(q)$.

Proof. Taking $S^i = S$, for $i = 0,1,2,\ldots,n, F_1 = F_2 = F, H_1 = H_2 = H, h_1 = h_2 = 0, A = 0$ and $D = I$ in Theorem 3.1, then the conclusion of Corollary 4.2 is obtained. □

Corollary 4.3. *Let H be real Hilbert space and $C \subset H$. Let $F : C \times C \to \mathbb{R}$ satisfy* Lemma 2.1*. Let B be β-inverse-strongly monotone mapping from C into H. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Let $S : C \to C$ be nonexpansive mapping such that*

$$\Omega := Fix(S) \cap VI(C,B) \cap EP(F) \neq \emptyset.$$

Let $\{x_n\}, \{y_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$ and

$$\begin{cases} u_n = T_{r_n}^{F} x_n, \\ y_n = P_C(u_n - \lambda_n B u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + (1-\beta_n - \alpha_n) S y_n, \ \forall n \geq 0, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, $\{\lambda_n\} \in [a,b] \subset (0,2\beta)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions (C1)-(C4). Then $\{x_n\}$ converges strongly to $q \in \Omega$, where $q = P_\Omega(q)$.

Proof. Taking $\gamma = 1$ and $f(x_n) = v$ in Corollary 4.2, then the conclusion of Corollary 4.3 is obtained. □

Corollary 4.4. *Let H be real Hilbert space and $C \subset H$. Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$. Let $S : C \to C$ be nonexpansive mapping such that $Fix(S) \neq \emptyset$. Let $\{x_n\}$ be sequences generated by $x_0 \in C$, and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n) S x_n, \ \forall n \geq 0, \tag{4.4}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$, satisfy the following conditions (C1)-(C2). Then $\{x_n\}$ converges strongly to $q \in Fix(S)$, where $q = P_{Fix(S)} f(q)$.

Proof. Taking $S^i = S$, for $i = 0, 1, 2, \ldots, n, H_1 = H_2 = H, F_1 = F_2 = h_1 = h_2 = 0, A = 0, y_n = u_n = x_n, D = P_C = I$ and $B = 0$ in Theorem 3.1, then the conclusion of Corollary 4.4 is obtained. □

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