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Feasible iterative algorithms and strong convergence theorems for bi-level fixed point problems

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Abstract

The purpose of this paper is to introduce and study *the bi-level split fixed point problems* in the setting of infinite-dimensional Hilbert spaces. For solving this kind problems, some new simultaneous iterative algorithms are proposed. Under suitable conditions, some strong convergence theorems for the sequences generated by the proposed algorithm are proved. As applications, we shall utilize the results presented in the paper to study *bi-level split equilibrium problem, bi-level split optimization problems* and *the bi-level split variational inequality problems*. The results presented in the paper are new which also extend and improve many recent results.

Keywords: Bi-level split fixed point problem, bi-level split equilibrium problem, bi-level split optimization problem, bi-level split variational inequality problem, split feasibility problem.
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1. Introduction

Let C and Q be a nonempty closed and convex subset of real Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* (SFP) is formulated as:

$$\text{to find } x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{1.1}$$

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where $A : H_1 \to H_2$ is a bounded linear operator. In 1994, Censor and Elfving [5] first introduced the (SFP) in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [3]. It has been found that the (SFP) can also be used in various disciplines such as image restoration, computer tomograph and radiation therapy treatment planning [4, 6, 7]. The (SFP) in an infinite dimensional real Hilbert space can be found in [3, 6, 11, 15, 27, 28, 29].

Recently, Moudafi [23, 24, 25] introduced the following *split equality feasibility problem* (SEFP):

$$\text{to find} \quad x \in C, \quad y \in Q \quad \text{such that} \quad Ax = By, \tag{1.2}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. Obviously, if $B = I$ (identity mapping on H_2) and $H_3 = H_2$, then (1.2) reduces to (1.1). The kind of split equality feasibility problems (1.2) allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, such as decomposition methods for PDEs, applications in game theory and intensity-modulated radiation therapy.

In order to solve split equality feasibility problem (1.2), Moudafi [24] introduced the following simultaneous iterative method:

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \beta B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.3}$$

and under suitable conditions he proved the weak convergence of the sequence $\{(x_n, y_n)\}$ to a solution of (1.2) in Hilbert spaces.

Very recently, Eslamian et al. [19], Chen et al. [17], Chuang [18] and Chang et al. [10, 13, 14] introduced and studied some kinds of *general split feasibility problem*, *general split equality problem*, *general split equality variational inclusion problems* and *general split equality optimization problems* in real Hilbert spaces. Under suitable conditions some strong convergence theorems are proved.

Motivated by the above works and related literatures, the purpose of this paper is to introduce and study the following *bi-level split fixed point problems* in the setting of infinite-dimensional real Hilbert space:

Let H_1, H_2 and H_3 be three real Hilbert spaces. In the sequel we always denote by $F(K)$ the fixed point set of a mapping K. Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ and $U : H_3 \to H_3$ be three nonlinear operators with nonempty fixed point sets $F(T)$, $F(S)$ and $F(U)$, respectively. Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. The "so-called" *bi-lever split fixed point problem* (BLSFPP) is to find:

$$p \in F(T), q \in F(S) \text{ such that } Ap = Bq \text{ and } u := Ap = Bq \in F(U). \tag{1.4}$$

It is easy to know that the (BLSFPP) can be regarded as a new development of the split fixed point theory and it contains several important problems, for example, *bi-lever split equilibrium problem, split equality equilibrium problem*, *bi-lever split optimization problems*, *bi-lever split variational inequalities problems* and etc. as its special cases.

Example 1.1 (Bi-lever split equilibrium problems). Let D be a nonempty closed and convex subset of a real Hilbert space H. A bifunction $g : D \times D \to (-\infty, +\infty)$ is said to be a *equilibrium function*, if it satisfies the following conditions:

(A1) $g(x, x) = 0$ for all $x \in D$;

(A2) g is monotone, i.e., $g(x, y) + g(y, x) \le 0$ for all $x, y \in D$;

(A3) $\limsup_{t\downarrow 0} g(tz + (1 - t)x, y) \le g(x, y)$ for all $x, y, z \in D$;

(A4) for each $x \in D$, $y \mapsto g(x, y)$ is convex and lower semi-continuous.

The "so-called" *equilibrium problem with respective to the equilibrium function* g is:

$$\text{to find } x^* \in D \text{ such that } g(x^*, y) \ge 0 \ \forall y \in D. \tag{1.5}$$

Its solution set is denoted by $EP(g)$. For given $\lambda > 0$ and $x \in H$, the *resolvent of the equilibrium function*

g is the operator $R_{\lambda,g} : H \to D$ defined by

$$R_{\lambda,g}(x) := \{z \in D : g(z,y) + \frac{1}{\lambda}\langle y - z,\ z - x\rangle \geq 0\ \forall y \in D\},\ x \in H. \tag{1.6}$$

Proposition 1.2 ([2])**.** *The resolvent operator $R_{\lambda,g}$ of the equilibrium function g has the following properties:*
(1) $R_{\lambda,g}$ is single-valued;
(2) $F(R_{\lambda,g}) = EP(g)$ and $EP(g)$ is a nonempty closed and convex subset of D;
(3) $R_{\lambda,g}$ is a firmly nonexpansive mapping.

Let $H_1,\ H_2$ and H_3 be three real Hilbert spaces. Let C be a nonempty closed convex subset of H_1, Q be a nonempty closed convex subset of H_2 and K be a nonempty closed convex subset of H_3. Let $h : C \times C \to \mathbb{R}$, $g : Q \times Q \to \mathbb{R}$ and $j : K \times K \to \mathbb{R}$ be three equilibrium functions. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with adjoint operator A^* and B^*, respectively. For given $\lambda > 0$, let $R_{\lambda,h}$, $R_{\lambda,g}$ and $R_{\lambda,j}$ be the resolvent of h, g and j (defined by (1.6)), respectively.

The "so-called" *bi-level split equilibrium problem with respective to h, g, j* (BLSEP(h, g, j)) is to find $x^* \in C,\ y^* \in Q$ such that

$$\begin{cases} (i)\ h(x^*, x) \geq 0\ \forall x \in C \text{ and } g(y^*, y) \geq 0\ \ \forall y \in Q; \\ (ii)\ Ax^* = By^* := u; \\ (iii)\ j(u, z) \geq 0\ \forall\ z \in K. \end{cases} \tag{1.7}$$

Especially, the "so-called" *split equality equilibrium problem with respective to h, g ($SEEP(h, g)$)* is to find $x^* \in C,\ y^* \in Q$ such that

$$\begin{cases} (i)\ h(x^*, x) \geq 0\ \forall x \in C \text{ and } g(y^*, y) \geq 0\ \forall\ y \in Q; \\ (ii)\ Ap = Bq. \end{cases} \tag{1.8}$$

This problem was first introduced and studied by Moudafi [12].

By Proposition 1.2, the (BLSEP(h, g, j)) (1.7) is equivalent to find

$$\begin{aligned} x^* \in EP(h, C),\ \ y^* \in EP(g, Q) \text{ such that } Ax^* = By^* \text{ and } Ap = Bq := u \in EP(j, K) \\ \Leftrightarrow x^* \in F(R_{\lambda h}),\ \ y^* \in F(R_{\lambda g}) \text{ such that } Ax^* = By^* \\ \text{and } Ax^* = By^* := u^* \in F(R_{\lambda j}) \text{ for each } \lambda > 0. \end{aligned} \tag{1.9}$$

Especially, the (SEEP(h, g)) (1.8) is equivalent to find

$$x^* \in EP(h, C),\ \ y^* \in EP(g, Q) \text{ such that } Ax^* = By^*. \tag{1.10}$$

Example 1.3 (Bi-level split convex optimization problems). Let $H_1,\ H_2$ and H_3 be three real Hilbert spaces, $C \subset H_1,\ Q \subset H_2$ and $K \subset H_3$ be three nonempty closed and convex subsets and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Let $h : C \to \mathbb{R}$, $g : Q \to \mathbb{R}$ and $j : K \to \mathbb{R}$ be three proper convex and lower semi-continuous functions. Then the "so-called" *bi-level split convex optimization problem* (BLSCOP) is to find $p \in C,\ q \in Q,\ Ap = Bq := u$ such that

$$h(p) = \min_{x \in C} h(x),\ \ g(q) = \min_{z \in Q} g(z),\ \ j(u) = \min_{s \in K} j(s). \tag{1.11}$$

Denote by $U = \partial h$, $V = \partial g$ and $M = \partial j$, then the mappings U, V and M all are maximal monotone. For any given $\gamma > 0$, let $J_\gamma^U := (I + \gamma U)^{-1}$, $J_\gamma^V := (I + \gamma V)^{-1}$ and $J_\gamma^M := (I + \gamma M)^{-1}$ be the resolvent of U, V and M, respectively. Therefore we have

$$U^{-1}(0) = F(J_\gamma^U),\ V^{-1}(0) = F(J_\gamma^V),\ M^{-1}(0) = F(J_\gamma^M), \tag{1.12}$$

where $U^{-1}(0),\ V^{-1}(0)$ and $M^{-1}(0)$ are the null point set of the mapping $U,\ V,$ and M, respectively. Then the (BLSCOP) (1.11) is equivalent to find $p \in C$, $q \in Q$ and $Ap = Bq := u$ such that

$$\begin{aligned} & p \in U^{-1}(0), \;\; q \in V^{-1}(0) \text{ and } u \in M^{-1}(0) \\ & \Leftrightarrow p \in F(J_\gamma^U), \;\; q \in F(J_\gamma^V) \text{ and } u \in F(J_\gamma^M). \end{aligned} \tag{1.13}$$

Example 1.4 (Bi-level split variational inequality problems). Let $H_1,\ H_2,\ H_3,\ C,\ Q,\ K,\ A$ and B be the same as in (II). Let $T : C \to H_1,\ S : Q \to H_2$ and $U : K \to H_3$ be three nonlinear operators. The "so-called" *Bi-level split variational inequality problem* (BLSVIP) is to find $p \in C$, $q \in Q$ and $Ap = Bq := u$ such that

$$\langle Tp, x-p\rangle \geq 0\ \forall x \in C, \;\; \langle Sq, -q\rangle \geq 0 \;\; \forall y \in Q \text{ and } \langle Us, v-s\rangle \geq 0\ \forall v \in K. \tag{1.14}$$

It is well known that $p \in C$ is a solution of the variational inequality $\langle Tp, x-p\rangle \geq 0\ \forall x \in C$ if and only for any $\gamma > 0$, $p = P_C(I - \gamma T)p$, i.e., $p \in F(P_C(I - \gamma T))$. This implies that (BLSVIP) (1.14) is equivalent to find $p \in C$, $q \in Q$ and $Ap = Bq := u$ such that

$$p \in F(P_C(I - \gamma T)), \;\; q \in F(P_C(I - \gamma S)) \text{ and } u \in F(P_C(I - \gamma U)). \tag{1.15}$$

For solving (BLSFPP)(1.4), in Section 3, we propose a new simultaneous type iterative algorithm. Under suitable conditions some strong convergence theorems for the sequences generated by the algorithm are proved in the setting of infinite-dimensional Hilbert spaces. As special cases, we shall utilize our results to study *the Bi-lever split equilibrium problems*, *Bi-level split convex optimization problems* and *Bi-level split variational inequality problems*. The results presented in the paper extend and improve the corresponding results announced by Censor et al. [4, 5, 6, 7, 8], Moudafi et al. [23, 24, 25], Eslamian and Latif [19], Chen et al. [17], Chuang [18], Chang and wang [13], Chang and Agarwal [10] and Chang et al. [14], Naraghirad [26], He and Du [21], Ansari and Rehan [1].

2. Preliminaries

First we recall some definitions, notations and conclusions. Throughout this section, we assume that H is a real Hilbert space, C is a nonempty closed and convex subset of H. Denote by $x_n \to x^*$ and $x_n \rightharpoonup x^*$, the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point x^*, respectively.

For a closed convex subset C of H and for each $x \in H$, *the (metric) projection* $P_C : H \to C$ is defined as the unique element $P_C x \in C$ such that

$$||x - P_C x|| = inf_{y \in C} ||x - y||.$$

For given $x \in H$, $y = P_C(x)$ if and only if

$$\langle y - z,\ x - y\rangle \geq 0 \;\; \forall x \in H,\ z \in C. \tag{2.1}$$

Recall that a mapping $T : H \to H$ is said to be nonexpansive, if $||Tx - Ty|| \leq ||x - y||\ \forall x, y \in H$.

A mapping $T : H \to H$ is said to be *quasi nonexpansive*, if $F(T) \neq \emptyset$ and

$$||Tx - p|| \leq ||x - p|| \text{ for each } x \in H \text{ and } p \in F(T).$$

It is easy to see that if T is a quasi nonexpansive mapping, then $F(T)$ is a nonexpansive closed and convex subset. T is said to be a *firmly nonexpansive mapping*, if

$$\begin{aligned} & ||Tx - Ty||^2 \leq \langle x - y,\ Tx - Ty\rangle \;\; \forall x, y \in C; \\ & \Leftrightarrow ||Tx - Ty||^2 \leq ||x - y||^2 - ||(I - T)x - (I - T)y||^2 \;\; \forall x, y \in C. \end{aligned} \tag{2.2}$$

A typical example of firmly nonexpansive mappings is the metric projection P_C from H onto C.

Lemma 2.1 ([12])**.** *Let H be a real Hilbert space and $\{x_n\}$ be a sequence in H. Then, for any given sequence $\{\lambda_n\}$ of positive numbers with $\sum_{i=1}^{\infty}\lambda_n = 1$ such that for any positive integers i, j with $i < j$ the following holds*

$$||\sum_{i=1}^{\infty}\lambda_n x_n||^2 \leq \sum_{i=1}^{\infty}\lambda_n||x_n||^2 - \lambda_i\lambda_j||x_i - x_j||^2.$$

Lemma 2.2 ([9])**.** *Let H be a real Hilbert space. For any $x, y \in H$, the following inequality holds*

$$||x + y||^2 \leq ||x||^2 + 2\langle y,\ x + y\rangle.$$

Lemma 2.3 ([20])**.** *If $T : C \to C$ is a nonexpansive mapping with $F(T) \neq \emptyset$, then $I - T$ is demi-closed at zero, i.e., for any sequence $\{x_n\} \subset C$ with $x_n \rightharpoonup x$ and $||x_n - Tx_n|| \to 0$, then $x = Tx$.*

Lemma 2.4 ([21])**.** *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of positive real numbers satisfying $a_{n+1} \leq (1 - b_n)a_n + c_n$ for all $n \geq 1$. If the following conditions are satisfied:*

(1) $b_n \in (0, 1)$ *and* $\sum_{n=1}^{\infty} b_n = \infty$,

(2) $\sum_{n=1}^{\infty} c_n < \infty$, *or* $\limsup_{n\to\infty}\frac{c_n}{b_n} \leq 0$,

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 ([22])**.** *Let $\{t_n\}$ be a sequence of real numbers. If there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \geq 1$, then there exists a nondecreasing sequence $\{\tau(n)\}$ with $\tau(n) \to \infty$ such that for all (sufficiently large) positive integer number n, the following holds:*

$$t_{\tau(n)} \leq t_{\tau(n)+1} \quad \text{and} \quad t_n \leq t_{\tau(n)+1}.$$

In fact,

$$\tau(n) = max\{k \leq n : t_k \leq t_{k+1}\}.$$

3. Bi-level split fixed point problems and some strong convergence theorems

Throughout this section we assume that H_1, H_2 and H_3 are three real Hilbert spaces, $T : H_1 \to H_1$, $S : H_2 \to H_2$ and $U : H_3 \to H_3$ are three quasi nonexpensive mappings (therefore their fixed point sets $F(T)$, $F(S)$ and $F(U)$ all are nonempty closed and convex) and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators with adjoint operators A^* and B^*, respectively.

Denote by

$$\begin{aligned}
&\mathbb{H}_1 := H_1 \times H_2,\ \ \mathbb{H}_2 := H_3 \times H_3,\\
&\mathbb{C} := \{(p, q) \in F(T) \times F(S) \text{ and } A(p) = B(q)\} \subset \mathbb{H}_1,\\
&\mathbb{Q} := F(U) \times F(U) \subset \mathbb{H}_2,\\
&P_{\mathbb{C}}(x, y) = (P_{F(T)}x,\ P_{F(S)}y)\ \forall(x, y) \in \mathbb{H}_1,\\
&P_{\mathbb{Q}}(z, w) = (P_{F(U)}(z), P_{F(U)}(w))\ \forall(z, w) \in \mathbb{H}_2.
\end{aligned}$$

Define a linear bounded operator $\mathbb{A} : \mathbb{H}_1 \to \mathbb{H}_2$ and its adjoint operator $\mathbb{A}^* : \mathbb{H}_2 \to \mathbb{H}_1$ by

$$\mathbb{A}(x, y) = (Ax, By)\ \forall(x, y) \in \mathbb{H}_1,\ \ \mathbb{A}^*(z, w) = (A^*(z), B^*(w))\ \forall(z, w) \in \mathbb{H}_2.$$

Now we consider the bi-lever split fixed point problem (1.4). We have the following Lemma.

Lemma 3.1. *If the solution set of the bi-lever split fixed point problem* (1.4)*:*

$$\Gamma := \{(p, q) \in F(T) \times F(S) \text{ such that } A(p) = B(q) \text{ and } u =: A(p) = B(q) \in F(U)\} \tag{3.1}$$

is nonempty, then $w = (p, q) \in \mathbb{H}_1$ is a solution of (BLSFPP) (1.4) *if and only if w is a solution of the following equation*

$$w = P_{\mathbb{C}}(I - \gamma\mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w)\ for\ any\ \gamma > 0. \tag{3.2}$$

Proof. Indeed, if $w := (p, q) \in \mathbb{H}_1$ is a solution of (BLSFPP) (1.4), then $p \in F(T)$, $q \in F(S)$ and $Ap = Bq$ such that $u := Ap = Bq \in F(U)$. Hence we have

$$\begin{aligned}\mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A}(p, q) &= \mathbb{A}^*(\mathbb{A}(p, q) - P_{\mathbb{Q}}\mathbb{A}(p, q)) \\ &= \mathbb{A}^*((Ap, Bq) - P_{\mathbb{Q}}((Ap, Bq))) \\ &= \mathbb{A}^*((Ap, Bq) - (P_{F(U)}(Ap), P_{F(U)}(Bq))) \\ &= \mathbb{A}^*((Ap, Bq) - (Ap, Bq)) = 0.\end{aligned}$$

Therefore we have

$$\begin{aligned}&P_{\mathbb{C}}(I - \gamma\mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(p, q) = P_{\mathbb{C}}(p, q) \\ &= (P_{F(T)}p,\ P_{F(S)}q) = (p, q).\end{aligned}$$

Conversely, if $w = (p, q) \in \mathbb{H}_1$ is a solution of equation (3.2), then $w \in \mathbb{C}$. This implies that $p \in F(T)$, $q \in F(S)$, $Ap = Bq$ and

$$\begin{cases} p = P_{F(T)}[I - \gamma A^*(I - P_{F(U)})A](p), \\ q = P_{F(S)}[I - \gamma B^*(I - P_{F(U)})B](q). \end{cases} \tag{3.3}$$

It follows from (2.1) and (3.3) that

$$\langle p - (p - \gamma A^*((I - P_{F(U)})A(p))),\ x - p\rangle \geq 0\ \forall x \in F(T).$$

Simplifying, we have

$$\langle Ap - P_{F(U)}A(p),\ Ax - Ap\rangle \geq 0\ \forall x \in F(T). \tag{3.4}$$

Similarly, from (2.1) and (3.3) we also have

$$\langle Bq - P_{F(U)}B(q),\ By - Bq\rangle \geq 0\ \forall y \in F(S). \tag{3.5}$$

By the assumption that the solution set Γ of (BLSFPP) (1.4) is nonempty. Taking $(p^*, q^*) \in \Gamma$, hence $Ap^* = Bq^*$ and $(Ap^*, Bq^*) \in F(U) \times F(U)$. In (3.4) taking $x = p^*$, we have

$$\langle Ap - P_{F(U)}A(p),\ P_{F(U)}A(p) - P_{F(U)}A(p) + Ap^* - Ap\rangle \geq 0. \tag{3.6}$$

Since $Ap^* \in F(U)$, it follows from (3.6) and (2.1) that

$$||Ap - P_{F(U)}A(p)||^2 \leq \langle Ap - P_{F(U)}A(p),\ Ap^* - P_{F(U)}A(p)\rangle \leq 0.$$

This implies that $Ap \in F(U)$. Similarly, we can also prove that $Bq \in F(U)$.

This completes the proof of Lemma 3.1. □

Lemma 3.2. *If $\gamma \in (0, \frac{2}{L})$, where $L = ||\mathbb{A}||^2$, then $(I - \gamma\mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A}) : H_1 \times H_2 \to H_1 \times H_2$ is a nonexpansive mapping.*

Proof. In fact, since $P_{\mathbb{Q}}$ is a firmly nonexpansive mapping, $I - P_{\mathbb{Q}}$ is also a firmly nonexpansive mapping. Again since $\gamma \in (0, \frac{2}{L})$, for any $w, u \in H_1 \times H_2$, we have

$$\begin{aligned}&||(I - \gamma\mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(u) - (I - \gamma\mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w)||^2 \\ &\quad = ||(u - w) - \gamma\mathbb{A}^*\{(I - P_{\mathbb{Q}})\mathbb{A}(u) - (I - P_{\mathbb{Q}})\mathbb{A}(w)\}||^2 \\ &\quad = ||u - w||^2 + \gamma^2||\mathbb{A}^*\{(I - P_{\mathbb{Q}})\mathbb{A}(u) - (I - P_{\mathbb{Q}})\mathbb{A}(w)\}||^2 \\ &\quad - 2\gamma\langle u - w,\ \mathbb{A}^*\{(I - P_{\mathbb{Q}})\mathbb{A}(u) - (I - P_{\mathbb{Q}})\mathbb{A}(w)\}\rangle \\ &\quad \leq ||u - w||^2 + \gamma^2 L||(I - P_{\mathbb{Q}})\mathbb{A}(u) - (I - P_{\mathbb{Q}})\mathbb{A}(w)||^2 \\ &\quad - 2\gamma\langle\mathbb{A}(u) - \mathbb{A}(w),\ (I - P_{\mathbb{Q}})\mathbb{A}(u) - (I - P_{\mathbb{Q}})\mathbb{A}(w)\rangle \\ &\quad \leq ||u - w||^2 - \gamma(2 - L\gamma)||(I - P_{\mathbb{Q}})\mathbb{A}(u) - (I - P_{\mathbb{Q}})\mathbb{A}(w)||^2 \\ &\quad \leq ||u - w||^2.\end{aligned}$$

This completes the proof. □

We are now in a position to give the following main result.

Theorem 3.3. *Let H_1, H_2, H_3, T, S, U, A, B, A^*, B^*, $\mathbb{H}_1$, $\mathbb{H}_2$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{A}$ and $\mathbb{A}^*$ be the same as above. For any given $w_0 \in H_1 \times H_2$, let the simultaneous iterative sequence $\{w_n\} \subset H_1 \times H_2$ be generated by*

$$w_{n+1} = \alpha_n w_n + \beta_n f(w_n) + \lambda_n(P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)),\ n \geq 0, \tag{3.7}$$

where $f(x,y) := (f_1(x), f_2(y)) : H_1 \times H_2 \to H_1 \times H_2$ and $f_i : H_i \to H_i$, $i = 1,2$ is a contractive mapping with a contractive constant $k \in (0,1)$. If the solution set Γ of (BLSFPP) (1.4) defined by (3.1) is nonempty and the following conditions are satisfied

(i) $\alpha_n + \beta_n + \lambda_n = 1$ *for each* $n \geq 0$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\liminf_{n\to\infty} \alpha_n \lambda_n > 0$;

(iv) $\{\gamma_n\} \subset (0, \frac{2}{L})$, *where* $L = ||\mathbb{A}||^2$,

then the sequence $\{w_n\}$ converges strongly to $w^* = P_\Gamma f(w^*)$ which is a solution of (BLSFPP) (1.4).

Proof. (I) First we prove that the sequence $\{w_n\}$ is bounded.

In fact, for any given $w \in \Gamma$, by Lemma 3.1 and Lemma 3.2, we know that

$$w = P_{\mathbb{C}}(I - \gamma \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w)$$

and $(I - \gamma \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A}) : H_1 \times H_2 \to H_1 \times H_2$ is a nonexpensive mapping. Hence from condition (iv) we have

$$\begin{aligned}
||w_{n+1} - w|| &= ||(\alpha_n w_n + \beta_n f(w_n) + \lambda_n(P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)) - w|| \\
&\leq \alpha_n ||w_n - w|| + \beta_n ||f(w_n) - w|| + \lambda_n ||P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n) - w|| \\
&\leq \alpha_n ||w_n - w|| + \beta_n ||f(w_n) - w|| + \lambda_n ||w_n - w|| \\
&= (1 - \beta_n)||w_n - w|| + \beta_n ||f(w_n) - w|| \\
&\leq (1 - \beta_n)||w_n - w|| + \beta_n ||f(w_n) - f(w)|| + \beta_n ||f(w) - w|| \\
&\leq (1 - \beta_n)||w_n - w|| + k\beta_n ||w_n - w|| + \beta_n ||f(w) - w|| \\
&= (1 - (1-k)\beta_n)||w_n - w|| + (1-k)\beta_n \frac{1}{1-k}||f(w) - w|| \\
&\leq max\{||w_n - w||,\ \frac{1}{1-k}||f(w) - w||\}.
\end{aligned}$$

By induction, we can prove that

$$||w_n - w|| \leq max\{||w_0 - w||,\ \frac{1}{1-k}||f(w) - w||\}\ \forall n \geq 0. \tag{3.8}$$

This shows that $\{w_n\}$ is bounded, so is $\{f(w_n)\}$.

(II) Now we prove that the following inequality holds.

$$\alpha_n \lambda_n ||w_n - P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)||^2 \leq ||w_n - w||^2 - ||w_{n+1} - w||^2 + \beta_n ||f(w_n) - w||^2. \tag{3.9}$$

Indeed, it follows from (3.7) and Lemma 2.1 that for each $i \geq 1$

$$\begin{aligned}
||w_{n+1} - w||^2 &= ||\alpha_n(w_n - w) + \beta_n(f(w_n) - w) \\
&\quad + \lambda_n(P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)) - w)||^2 \\
&\leq \alpha_n ||w_n - w||^2 + \beta_n ||f(w_n) - w||^2 \\
&\quad + \lambda_n ||P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n) - w||^2 \\
&\quad - \alpha_n \lambda_n ||w_n - P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)||^2 \\
&\leq \alpha_n ||w_n - w||^2 + \beta_n ||f(w_n) - w||^2 + \lambda_n ||w_n - w||^2 \\
&\quad - \alpha_n \lambda_n ||w_n - P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)||^2 \\
&= (1 - \beta_n)||w_n - w||^2 + \beta_n ||f(w_n) - w||^2 \\
&\quad - \alpha_n \lambda_n ||w_n - P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)||^2.
\end{aligned}$$

This implies that

$$\alpha_n \lambda_n ||w_n - P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)||^2 \le ||w_n - w||^2 - ||w_{n+1} - w||^2 + \beta_n ||f(w_n) - w||^2. \tag{3.10}$$

The conclusion is proved.

It is easy to see that the solution set Γ of (BLSFPP) (1.4) is a closed and convex subset in $H_1 \times H_2$. By the assumption that Γ is nonempty, so it is a nonempty closed and convex subset in $H_1 \times H_2$. Hence the metric projection P_Γ is well defined. Again since $P_\Gamma f : H_1 \times H_2 \to \Gamma$ is a contractive mapping, there exists a unique $w^* \in \Gamma$ such that

$$w^* = P_\Gamma f(w^*). \tag{3.11}$$

(III) Now we prove that $\{w_n\}$ converges strongly to w^*. For the purpose, we consider two cases.

Case I: Suppose that the sequence $\{||w_n - w^*||\}$ is monotone. Since $\{||w_n - w^*||\}$ is bounded, $\{||w_n - w^*||\}$ is convergent. Since $w^* \in \Gamma$, taking $w = w^*$ in (3.9), it follows from conditions (ii) and (iii) that

$$lim_{n\to\infty} ||w_n - P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)||^2 = 0. \tag{3.12}$$

On the other hand, by Lemma 2.2 and (3.7), we have

$$\begin{aligned}
||w_{n+1} - w^*||^2 &= ||\alpha_n w_n + \beta_n f(w_n) + \lambda_n (P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)) - w^*||^2 \\
&= ||\alpha_n (w_n - w^*) + \beta_n (f(w_n) - w^*) \\
&\quad + \lambda_n (P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)) - w^*)||^2 \\
&\le ||\alpha_n (w_n - w^*) + \lambda_n (P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A}(w_n) - w^*)||^2 \\
&\quad + 2\beta_n \langle f(w_n) - w^*,\ w_{n+1} - w^* \rangle \ (by\ Lemma\ 2.2) \\
&\le \{\alpha_n ||w_n - w^*|| + \lambda_n ||w_n - w^*||\}^2 \\
&\quad + 2\beta_n \langle f(w_n) - f(w^*),\ w_{n+1} - w^* \rangle + 2\beta_n \langle f(w^*) - w^*,\ w_{n+1} - w^* \rangle \\
&= (1 - \beta_n)^2 ||w_n - w^*||^2 + 2\beta_n k ||w_n - w^*|| ||w_{n+1} - w^*|| \\
&\quad + 2\beta_n \langle f(w^*) - w^*,\ w_{n+1} - w^* \rangle \\
&\le (1 - \beta_n)^2 ||w_n - w^*||^2 + \beta_n k \{||w_n - w^*||^2 + ||w_{n+1} - w^*||^2\} \\
&\quad + 2\beta_n \langle f(w^*) - w^*,\ w_{n+1} - w^* \rangle.
\end{aligned}$$

Simplifying it we have

$$\begin{aligned}
||w_{n+1} - w^*||^2 &\le \frac{(1 - \beta_n)^2 + \beta_n k}{1 - \beta_n k} ||w_n - w^*||^2 + \frac{2\beta_n}{1 - \beta_n k} \langle f(w^*) - w^*,\ w_{n+1} - w^* \rangle \\
&= \frac{1 - 2\beta_n + \beta_n k}{1 - \beta_n k} ||w_n - w^*||^2 + \frac{\beta_n^2}{1 - \beta_n k} ||w_n - w^*||^2 \\
&\quad + \frac{2\beta_n}{1 - \beta_n k} \langle f(w^*) - w^*,\ w_{n+1} - w^* \rangle \\
&= (1 - \frac{2(1 - k)\beta_n}{1 - \beta_n k}) ||w_n - w^*||^2 \\
&\quad + \frac{2(1 - k)\beta_n}{1 - \beta_n k} \{\frac{\beta_n M}{2(1 - k)} + \frac{1}{1 - k} \langle f(w^*) - w^*,\ w_{n+1} - w^* \rangle\} \\
&= (1 - \eta_n) ||w_n - w^*||^2 + \eta_n \delta_n,
\end{aligned} \tag{3.13}$$

where

$$\eta_n = \frac{2(1 - k)\beta_n}{1 - \beta_n k}, \quad \delta_n = \frac{\beta_n M}{2(1 - k)} + \frac{1}{1 - k} \langle f(w^*) - w^*,\ w_{n+1} - w^* \rangle \text{ and } M = \sup_{n \ge 0} ||w_n - w^*||^2.$$

By condition (ii), $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}\beta_n = \infty$, so is $\sum_{n=1}^{\infty}\eta_n = \infty$.

Next we prove that

$$\limsup_{n\to\infty}\delta_n \le 0. \tag{3.14}$$

In fact, since $\{w_n\}$ is bounded in $H_1 \times H_2$, there exists a subsequence $\{w_{n_k}\} \subset \{w_n\}$ with $w_{n_k} \rightharpoonup v^*$ (some point in $H_1 \times H_2$) and $\lambda_{n_k,i} \to \lambda_i \in (0,\ \frac{2}{L})$ such that

$$\lim_{n\to\infty}\langle f(w^*) - w^*,\ w_{n_k} - w^*\rangle = \limsup_{n\to\infty}\langle f(w^*) - w^*,\ w_n - w^*\rangle.$$

Since $P_{\mathbb{C}}(I - \gamma\mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})$ is a nonexpansive mapping and from (3.12)

$$||w_{n_k} - P_{\mathbb{C}}(I - \gamma_n\mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_{n_k})|| \to 0.$$

It follows from Lemma 2.3 that

$$v^* = P_{\mathbb{C}}(I - \gamma_n\mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(v^*). \tag{3.15}$$

By Lemma 3.1, this implies that $v^* \in \Gamma$. Since $w^* = P_\Omega f(w^*)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(w^*) - w^*,\ w_n - w^*\rangle &= \lim_{n\to\infty}\langle f(w^*) - w^*,\ w_{n_k} - w^*\rangle \\ &= \langle f(w^*) - w^*,\ v^* - w^*\rangle \le 0.\end{aligned}$$

This shows that (3.14) is true. Taking $a_n = ||w_n - w^*||^2$, $b_n = \eta_n$ and $c_n = \delta_n\eta_n$ in Lemma 2.4 , all conditions in Lemma 2.4 are satisfied. Hence $w_n \to w^*$.

Case II: If the sequence $\{||w_n - w^*||\}$ is not monotone, by Lemma 2.5, there exists a sequence of positive integers: $\{\tau(n)\}$, $n \ge n_0$ (where n_0 large enough) such that

$$\tau(n) = max\{k \le n : ||w_k - w^*|| \le ||w_{k+1} - w^*||\}. \tag{3.16}$$

Clearly $\{\tau(n)\}$ is a nondecreasing, $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \ge n_0$

$$||w_{\tau(n)} - w^*|| \le ||w_{\tau(n)+1} - w^*|| \quad \text{and} \quad ||w_n - w^*|| \le ||w_{\tau(n)+1} - w^*||. \tag{3.17}$$

Therefore $\{||w_{\tau(n)} - w^*||\}$ is a nondecreasing sequence. According to the Case (I), $\lim_{n\to\infty}||w_{\tau(n)} - w^*|| = 0$ and $\lim_{n\to\infty}||w_{\tau(n)+1} - w^*|| = 0$. Hence we have

$$0 \le ||w_n - w^*|| \le max\{||w_n - w^*||,\ ||w_{\tau(n)} - w^*||\} \le ||w_{\tau(n)+1} - w^*|| \to 0,\ as\ n \to \infty.$$

This implies that $w_n \to w^*$ and $w^* = P_\Omega f(w^*)$ is a solution of (GSEVIP) (1.5).

This completes the proof of Theorem 3.3. □

Remark 3.4. Theorem 3.3 extend and improve the main results in Moudafi et al. [23, 24, 25], Eslamian and Latif [19], Chen et al. [17], Chuang [18], Chang et al. [10, 13, 14, 16], Naraghirad [26] and He and Du [21].

4. Strong convergence theorems for some other bi-level split problems

In this section we shall utilize the results presented in Theorem 3.3 to study some other *bi-lever split problems* in Hilbert spaces

4.1. Bi-lever split equilibrium problems

Let H_1, H_2 and H_3 be three real Hilbert spaces, $C \subset H_1$, $Q \subset H_2$ and $K \subset H_3$ be three nonempty closed and convex subsets. Let $h : C \times C \to \mathbb{R}$, $g : Q \times Q \to \mathbb{R}$ and $j : K \times K \to \mathbb{R}$ be three equilibrium functions. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with adjoint operator A^*

and B^*, respectively. For given $\lambda > 0$, let $R_{\lambda,h}$, $R_{\lambda,g}$ and $R_{\lambda,j}$ be the resolvents of h, g and j (defined by (1.6)), respectively.

As pointed out in Example 1.1 (see Section 1) that the bi-level split equilibrium problem with respective to h, g, j, i.e., to find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{cases} (i)\ h(x^*, x) \geq 0\ \forall x \in C \text{ and } g(y^*, y) \geq 0\ \forall y \in Q; \\ (ii)\ Ax^* = By^* := u; \\ (iii)\ j(u, z) \geq 0\ \forall\ z \in K, \end{cases} \tag{4.1}$$

is equivalent to the following *bi-level split fixed point problem*:

$$\begin{aligned} &\text{to find } x^* \in F(R_{\lambda h}),\ \ y^* \in F(R_{\lambda g}) \text{ such that } Ax^* = By^* \text{ and} \\ &Ax^* = By^* := u^* \in F(R_{\lambda j}) \text{ for each } \lambda > 0, \end{aligned} \tag{4.2}$$

where $R_{\lambda h}$, $R_{\lambda g}$ and $R_{\lambda j}$ are the resolvent operators of h, g and j defined by (1.6), respectively. Take $T = R_{\lambda h}$, $S = R_{\lambda g}$ and $U = R_{\lambda j}$ in Theorem 3.3. Then all T, S and U are firmly nonexpansive mappings with nonempty closed and convex fixed point sets. Therefore all conditions in Theorem 3.3 are satisfied. Hence the following result can be obtained from Theorem 3.3 immediately.

Theorem 4.1. *Let H_1, H_2, H_3, C, Q, K, h, g, j, A, B, A^*, B^*, T, S and U be the same as above. Denote by $\mathbb{H}_1 := C \times Q$ and $\mathbb{H}_2 := K \times K$. Let $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{A}$ and $\mathbb{A}^*$ be the same as in* Theorem 3.3. *For given $w_0 \in C \times Q$, let $\{w_n\}$ be the sequence generated by*

$$w_{n+1} = \alpha_n w_n + \beta_n f(w_n) + \lambda_n (P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)),\ n \geq 0, \tag{4.3}$$

where $f(x, y) := (f_1(x), f_2(y))$ and $f_1 : C \to C$, $f_2 : Q \to Q$ both are contractive mapping with a contractive constant $k \in (0, 1)$. If the solution set Γ_1 of (BLSEP) (4.1) *is nonempty and the following conditions are satisfied*

(i) $\alpha_n + \beta_n + \lambda_n = 1$ *for each $n \geq 0$;*

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$*;*

(iii) $\liminf_{n\to\infty} \alpha_n \lambda_n > 0$*;*

(iv) $\{\gamma_n\} \subset (0, \frac{2}{L})$*, where* $L = ||\mathbb{A}||^2$*,*

then the sequence $\{w_n\}$ converges strongly to $w^* = P_{\Gamma_1} f(w^*)$ which is a solution of (BLSFPP) (4.1).

Remark 4.2. Theorem 4.1 is a generalization and improvement of Theorem 4.1 in He and Du [21].

4.2. Bi-level split convex optimization problems

Let H_1, H_2, H_3, C, Q, K, A and B be the same as in Section 4.1. Let $h : C \to \mathbb{R}$, $g : Q \to \mathbb{R}$ and $j : K \to \mathbb{R}$ be three proper convex and lower semi-continuous functions. Denote by $U = \partial h$, $V = \partial g$ and $M = \partial j$. It is well-known that U, V and M all are maximal monotone mappings. For any given $\gamma > 0$, denote by $J_\gamma^U := (I + \gamma U)^{-1}$, $J_\gamma^V := (I + \gamma V)^{-1}$ and $J_\gamma^M := (I + \gamma M)^{-1}$ the resolvent of U, V and M, respectively.

As pointed out in Example 1.4 (see Section 1) that the Bi-level split convex optimization problems, i.e., to find $p \in C$, $q \in Q$ and $Ap = Bq := u$ such that

$$h(p) = \min_{x \in C} h(x),\ \ g(q) = \min_{z \in Q} g(z) \text{ and } j(u) = \min_{s \in K} j(s) \tag{4.4}$$

is equivalent to the following bi-level split fixed point problem: to find $p \in C$, $q \in Q$ and $Ap = Bq := u$ such that

$$p \in F(J_\gamma^U),\ \ q \in F(J_\gamma^V) \text{ and } u \in F(J_\gamma^M). \tag{4.5}$$

Letting $T = J_\gamma^U$, $S = J_\gamma^V$ and $U = J_\gamma^M$, then T, S and U all are quasi nonexpansive mappings with a nonempty closed and convex fixed point set. Therefore all conditions in Theorem 4.1 are satisfied. From Theorem 4.1 we have the following

Theorem 4.3. *Let H_1, H_2, H_3, C, Q, K, h, g, j, A, B, A^*, B^*, T, S and U be the same as above. Denote by $\mathbb{H}_1 := C \times Q$ and $\mathbb{H}_2 := K \times K$. Let $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{A}$, $\mathbb{A}^*$ and f be the same as in* Theorem 4.1. *For given $w_0 \in C \times Q$, let $\{w_n\}$ be the sequence generated by*

$$w_{n+1} = \alpha_n w_n + \beta_n f(w_n) + \lambda_n (P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)), \ n \geq 0. \tag{4.6}$$

If the solution set Γ_2 of (BLSCOP) (4.4) *is nonempty and the conditions (i)-(iv) in* Theorem 4.1 *are satisfied, then the sequence $\{w_n\}$ converges strongly to $w^* = P_{\Gamma_2} f(w^*)$ which is a solution of (BLSCOP)* (4.4).

4.3. Bi-level split variational inequality problems

Let H_1, H_2, H_3, C, Q, K, A and B be the same as in (II). Let $T^* : C \to H_1$, $S^* : Q \to H_2$ and $U^* : K \to H_3$ be three nonlinear operators. As point out in 1.4 (see Section 1), the bi-level split variational inequality problem (BLSVIP), i.e., to find $p \in C$, $q \in Q$ and $Ap = Bq := u$ such that

$$\langle T^* p, x - p \rangle \geq 0 \ \forall x \in C, \ \ \langle S^* q, -q \rangle \geq 0 \ \ \forall y \in Q \text{ and } \langle U^* s, v - s \rangle \geq 0 \ \forall v \in K \tag{4.7}$$

is equivalent to the following bi-level split fixed point problem: i.e., to find $p \in C$, $q \in Q$ and $Ap = Bq := u$ such that

$$p \in F(P_C(I - \gamma T^*)), \ \ q \in F(P_C(I - \gamma S^*)) \text{ and } u \in F(P_C(I - \gamma U^*)). \tag{4.8}$$

Denote by $T := P_C(I - \gamma T^*)$, $S := P_C(I - \gamma S^*)$ and $U := P_C(I - \gamma U^*)$, then T, S and U all are quasi nonexpansive mappings. Therefore all conditions in Theorem 4.3 are satisfied. hence the following theorem can be obtained from Theorem 4.3 immediately.

Theorem 4.4. *Let H_1, H_2, H_3, C, Q, K, A, B, A^*, B^*, T^*, S^* and U^* be the same as above. Denote by $\mathbb{H}_1 := C \times Q$ and $\mathbb{H}_2 := K \times K$. Let $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{A}$, $\mathbb{A}^*$ and f be the same as in* Theorem 4.1. *For given $w_0 \in C \times Q$, let $\{w_n\}$ be the sequence generated by*

$$w_{n+1} = \alpha_n w_n + \beta_n f(w_n) + \lambda_n (P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)), \ n \geq 0. \tag{4.9}$$

If the solution set Γ_3 of bi-level split variational inequality problem (4.7) *is nonempty and the conditions (i)-(iv) in* Theorem 4.1 *are satisfied, then the sequence $\{w_n\}$ converges strongly to $w^* = P_{\Gamma_3} f(w^*)$ which is a solution of (BLSVIP)* (4.7).

5. A concrete application

In this section, we shall give a concrete application of Theorem 3.3 and Theorem 4.1.

Let $H_1 = \mathbb{R}^2$, $H_2 = \mathbb{R}^3$ and $H_3 = \mathbb{R}^4$ with standard norm and inner product. For each $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $z = (z_1, z_2, z_3, z_4) \in \mathbb{R}^4$ define operators A and A^* by

$$A(\alpha) = (\alpha_1, \ \alpha_2, \ \alpha_1 + \alpha_2, \ \alpha_1 - \alpha_2) \tag{5.1}$$

and

$$A^*(z) = (z_1 + z_3 + z_4, \ z_2 + z_3 - z_4).$$

Then A is a bounded linear operator from $\mathbb{R}^2$ into $\mathbb{R}^4$ and $A^* : \mathbb{R}^4 \to \mathbb{R}^2$ is the adjoint operator of A. The norm of A is

$$\begin{aligned} ||A|| &= \sup_{||\alpha||_{\mathbb{R}^2} \leq 1} ||A\alpha||_{\mathbb{R}^4} \\ &= \sup_{||\alpha||_{\mathbb{R}^2} \leq 1} ||(\alpha_1, \ \alpha_2, \ \alpha_1 + \alpha_2, \ \alpha_1 - \alpha_2)||_{\mathbb{R}^4} \\ &= \sqrt{\alpha_1^2 + \alpha_2^2 + (\alpha_1 + \alpha_2)^2 + (\alpha_1 - \alpha_2)^2} \\ &= \sqrt{3}. \end{aligned}$$

Hence also $||A^*|| = \sqrt{3}$. For each $\beta = (\beta_1,\ \beta_2,\ \beta_3) \in \mathbb{R}^3$ and $z = (z_1,\ z_2,\ z_3,\ z_4) \in \mathbb{R}^4$, define operators B and B^* by

$$B(\beta) = (\beta_1,\ \beta_2,\ \beta_3,\ \beta_1 - \beta_2) \tag{5.2}$$

and

$$B^*(z) = (z_1 + z_4,\ z_2 - z_4,\ z_3).$$

Then B is a bounded linear operator from $\mathbb{R}^3$ into $\mathbb{R}^4$. We can also prove that $||B|| = \sqrt{3}$ and $B^* : \mathbb{R}^4 \to \mathbb{R}^3$ is the adjoint operator of B with $||B^*|| = \sqrt{3}$. Put

$$\begin{cases} C := \{\alpha = (\alpha_1, \alpha_2) \in [-1,\ 2] \times [2,\ 4]\} \subset \mathbb{R}^2, \\ Q := \{\beta = (\beta_1,\ \beta_2,\ \beta_3) \in [-1,\ 1] \times [3,\ 4] \times [3,\ 5]\} \subset \mathbb{R}^3, \\ K := \{z = (z_1,\ z_2,\ z_3,\ z_4) \in [0,\ 1] \times [3,\ 6] \times [3,\ 5] \times [-5,\ -3]\} \subset \mathbb{R}^4. \end{cases} \tag{5.3}$$

For each $\alpha = (\alpha_1,\ \alpha_2) \in C$, $\beta = (\beta_1,\ \beta_2,\ \beta_3) \in Q$ and $z = (z_1,\ z_2,\ z_3,\ z_4) \in K$, define functions:

$$h^*(\alpha) = \alpha_1^2 + \alpha_2^2,\ \ g^*(\beta) = \beta_1^2 + \beta_2^2 + \beta_3^2,\ \ j^*(z) = z_1^2 + z_2^2 + z_3^2 + z_4^2.$$

Let

$$\begin{cases} h(\alpha, x) = h^*(x) - h^*(\alpha)\ for\ each\ \ \alpha,\ x \in C; \\ g(\beta, y) = g^*(y) - g^*(\beta)\ for\ each\ \ \beta,\ y \in Q; \\ j(\eta, z) = j^*(z) - j^*(\eta)\ for\ each\ \ \eta,\ z \in K. \end{cases} \tag{5.4}$$

It is easy to know that $h : C \times C \to \mathbb{R},\ g : Q \times Q \to \mathbb{R}\ and\ j : K \times K \to \mathbb{R}$ all are the equilibrium functions satisfying conditions (A1)-(A4). Let $EP(h)$ (resp. $EP(g)$ and $EP(j)$) be the set of solutions of equilibrium problem with respective to h (resp. g and j). It is not hard to verify that

$$\begin{cases} EP(h) = \{x^* = (0, 3)\}, \\ EP(g) = \{y^* = (0, 3, 3)\}, \\ Ax^* \quad = By^* = \{(0, 3, 3, -3)\} := u^*, \\ EP(j) = \{u^* = (0, 3, 3, -3)\}. \end{cases} \tag{5.5}$$

This implies that $(x^*, y^*) = ((0, 3), (0, 3, 3)) \in C \times Q$ is the unique solution of the following bi-level split equilibrium problem with respective to h, g and j

$$\begin{cases} (i)\ h(x^*, x) \geq 0\ \ \forall x \in C \text{ and } g(y^*, y) \geq 0\ \forall y \in Q; \\ (ii)\ Ax^* = By^* := u^*; \\ (iii)\ j(u^*, z) \geq 0\ \forall\ z \in K. \end{cases} \tag{5.6}$$

Denote by Ω the set of solutions of the bi-level split equilibrium problem (5.6). Hence we have

$$\Omega = \{(p, q) \in EP(h) \times EP(g) : Ap = Bq \in EP(j)\} = \{((0, 3), (0, 3, 3))\}.$$

For given $\lambda > 0$, let $R_{\lambda,h}$, $R_{\lambda,g}$ and $R_{\lambda,j}$ be the resolvent of h, g and j (defined by (1.6)), respectively. Let $T = R_{\lambda,h}$, $S = R_{\lambda,g}$ and $U = R_{\lambda,j}$, therefore we have $F(T) = EP(h)$, $F(S) = EP(g)$ and $F(U) = EP(j)$. Denote by

$$\begin{cases} \mathbb{H}_1 := \mathbb{R}^2 \times \mathbb{R}^3,\ \ \mathbb{H}_2 := \mathbb{R}^4 \times \mathbb{R}^4, \\ \mathbb{C} := \{(p, q) \in F(T) \times F(S) \text{ and } A(p) = B(q)\} \subset \mathbb{H}_1, \\ \mathbb{Q} := F(U) \times F(U) \subset \mathbb{H}_2, \\ P_{\mathbb{C}}(x, y) = (P_{F(T)}x,\ P_{F(S)}y)\ \forall (x, y) \in \mathbb{H}_1, \\ P_{\mathbb{Q}}(z, w) = (P_{F(U)}(z), P_{F(U)}(w))\ \forall (z, w) \in \mathbb{H}_2. \end{cases} \tag{5.7}$$

From Theorem 3.3 we can obtain the following

Theorem 5.1 Let $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$, T, S, U, A, B, A^*, B^* and $\mathbb{H}_1$, $\mathbb{H}_2$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{A}$, $\mathbb{A}^*$, $P_{\mathbb{C}}$, $P_{\mathbb{Q}}$ be the same as above. For any given $w_0 \in \mathbb{R}^2 \times \mathbb{R}^3$, let the simultaneous iterative sequence $\{w_n\} \subset \mathbb{R}^2 \times \mathbb{R}^3$ be generated by

$$w_{n+1} = \alpha_n w_n + \beta_n f(w_n) + \lambda_n (P_{\mathbb{C}}(I - \gamma_n \mathbb{A}^*(I - P_{\mathbb{Q}})\mathbb{A})(w_n)), \ n \geq 0, \tag{5.8}$$

where $f(x, y) := (f_1(x), f_2(y)) : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2 \times \mathbb{R}^3$, $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $f_2 : \mathbb{R}^3 \to \mathbb{R}^3$ are contractive mappings with a contractive constant $k \in (0, 1)$. If the following conditions are satisfied

(i) $\alpha_n + \beta_n + \lambda_n = 1$ for each $n \geq 0$;

(ii) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\liminf_{n\to\infty} \alpha_n \lambda_n > 0$;

(iv) $\{\gamma_n\} \subset (0, \frac{2}{3})$, where $3 = ||\mathbb{A}||^2$,

then the sequence $\{w_n\}$ converges strongly to $(x^*, y^*) = ((0, 3), (0, 3, 3))$ which is a solution of the bi-level split equilibrium problem (5.6).

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