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Refinements of bounds for Neuman means with applications

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Abstract

In this article, we present the sharp bounds for the Neuman means derived from the Schwab-Borchardt, geometric, arithmetic and quadratic means in terms of the arithmetic and geometric combinations of harmonic, arithmetic and contra-harmonic means.

Keywords: Neuman mean, Schwab-Borchardt mean, harmonic mean, geometric mean, quadratic mean, contra-harmonic mean, arithmetic mean.
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1. Introduction

Let $a, b > 0$ with $a \neq b$. Then the Schwab-Borchardt mean $SB(a,b)$ [2, 3] of a and b is respectively defined by

$$SB(a,b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\arccos(a/b)}, & a < b, \\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases}$$

where $\cosh^{-1}(x) = \log(x + \sqrt{x^2-1})$ is the inverse hyperbolic cosine function.

It is well known that the Schwab-Borchardt mean $SB(a,b)$ is strictly increasing in both a and b, nonsymmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special

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cases of the Schwab-Borchardt mean. For example, $SB[G(a,b),A(a,b)] = (a-b)/[2\arcsin((a-b)/(a+b))] = P(a,b)$ is the first Seiffert mean, $SB[A(a,b),Q(a,b)] = (a-b)/[2\arctan((a-b)/(a+b))] = T(a,b)$ is the second Seiffert mean, $SB[Q(a,b),A(a,b)] = (a-b)/[2\sinh^{-1}((a-b)/(a+b))] = M(a,b)$ is the Neuman-Sándor mean, $SB[A(a,b),G(a,b)] = (a-b)/[2\tanh^{-1}((a-b)/(a+b))] = L(a,b)$ is the logarithmic mean, where $G(a,b) = \sqrt{ab}$, $A(a,b) = (a+b)/2$ and $Q(a,b) = \sqrt{(a^2+b^2)/2}$ are respectively the geometric, arithmetic and quadratic means, $\sinh^{-1}(x) = \log(x+\sqrt{x^2+1})$ is the inverse hyperbolic sine function and $\tanh^{-1}(x) = \log[(1+x)/(1-x)]/2$ is the inverse hyperbolic tangent function.

Let $X(a,b)$ and $Y(a,b)$ be the symmetric bivariate means of a and b. Then the Neuman mean $N_{XY}(a,b)$ [1] is defined by

$$N_{XY}(a,b) = \frac{1}{2}\left(X(a,b) + \frac{Y^2(a,b)}{SB(X(a,b),Y(a,b))}\right). \tag{1.1}$$

Let $a > b > 0$, $v = (a-b)/(a+b) \in (0,1)$. Then the following explicit formulas and inequalities can be found in the literature [3].

$$N_{AG}(a,b) = \frac{A(a,b)}{2}\left[1 + (1-v^2)\frac{\tanh^{-1}(v)}{v}\right],$$
$$N_{GA}(a,b) = \frac{A(a,b)}{2}\left[\sqrt{1-v^2} + \frac{\arcsin(v)}{v}\right],$$
$$N_{AQ}(a,b) = \frac{A(a,b)}{2}\left[1 + (1+v^2)\frac{\arctan(v)}{v}\right],$$
$$N_{QA}(a,b) = \frac{A(a,b)}{2}\left[\sqrt{1+v^2} + \frac{\sinh^{-1}(v)}{v}\right],$$

$$\begin{aligned} H(a,b) < G(a,b) < L(a,b) < N_{AG}(a,b) < P(a,b) < N_{GA}(a,b) < A(a,b) \\ < M(a,b) < N_{QA}(a,b) < T(a,b) < N_{AQ}(a,b) < Q(a,b) < C(a,b), \end{aligned}$$

where $H(a,b) = 2ab/(a+b)$ and $C(a,b) = (a^2+b^2)/2(a+b)$ are respectively the harmonic and contra-harmonic means.

Recently, the Neuman means $N_{AG}(a,b)$, $N_{GA}(a,b)$, $N_{AQ}(a,b)$ and $N_{QA}(a,b)$ have attracted the attention of many researchers.

Neuman [1] proved that the double inequalities

$$\alpha_1 A(a,b) + (1-\alpha_1)G(a,b) < N_{GA}(a,b) < \beta_1 A(a,b) + (1-\beta_1)G(a,b),$$
$$\alpha_2 Q(a,b) + (1-\alpha_2)A(a,b) < N_{AQ}(a,b) < \beta_2 Q(a,b) + (1-\beta_2)A(a,b),$$
$$\alpha_3 A(a,b) + (1-\alpha_3)G(a,b) < N_{AG}(a,b) < \beta_3 A(a,b) + (1-\beta_3)G(a,b),$$
$$\alpha_4 Q(a,b) + (1-\alpha_4)A(a,b) < N_{QA}(a,b) < \beta_4 Q(a,b) + (1-\beta_4)A(a,b)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/3$, $\beta_1 \geq \pi/4$, $\alpha_2 \leq 2/3$, $\beta_2 \geq (\pi-2)/[4(\sqrt{2}-1)] = 0.689\ldots$, $\alpha_3 \leq 1/3$, $\beta_3 \geq 1/2$, $\alpha_4 \leq 1/3$ and $\beta_4 \geq [\log(1+\sqrt{2}) + \sqrt{2} - 2]/[2(\sqrt{2}-1)] = 0.356\ldots$.

In [7], Zhang et al. presented the best possible parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1/2]$ and $\alpha_3, \alpha_4, \beta_3, \beta_4 \in [1/2,1]$ such that the double inequalities

$$G(\alpha_1 a + (1-\alpha_1)b, \alpha_1 b + (1-\alpha_1)a) < N_{AG}(a,b) < G(\beta_1 a + (1-\beta_1)b, \beta_1 b + (1-\beta_1)a),$$
$$G(\alpha_2 a + (1-\alpha_2)b, \alpha_2 b + (1-\alpha_2)a) < N_{GA}(a,b) < G(\beta_2 a + (1-\beta_2)b, \beta_2 b + (1-\beta_2)a),$$
$$Q(\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a) < N_{QA}(a,b) < Q(\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a),$$
$$Q(\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a) < N_{AQ}(a,b) < Q(\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a)$$

hold for all $a,b > 0$ with $a \neq b$.

Qian et. al. [4] proved that the double inequalities

$$\alpha_1 A(a,b) + (1-\alpha_1)L(a,b) < N_{AG}(a,b) < \beta_1 A(a,b) + (1-\beta_1)L(a,b),$$
$$\alpha_2 A(a,b) + (1-\alpha_2)P(a,b) < N_{GA}(a,b) < \beta_2 A(a,b) + (1-\beta_2)P(a,b),$$
$$\alpha_3 Q(a,b) + (1-\alpha_3)M(a,b) < N_{QA}(a,b) < \beta_3 Q(a,b) + (1-\beta_3)M(a,b),$$
$$\alpha_4 Q(a,b) + (1-\alpha_4)T(a,b) < N_{AQ}(a,b) < \beta_4 Q(a,b) + (1-\beta_4)T(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leq 0$, $\beta_1\geq 1/2$, $\alpha_2\leq 0$, $\beta_2\geq(\pi^2-8)/(4\pi-8)$, $\alpha_3\leq 0$, $\beta_3\geq[\sqrt{2}\log^2(1+\sqrt{2})+2\log(1+\sqrt{2})-2\sqrt{2}]/[4\log(1+\sqrt{2})-2\sqrt{2}]$, $\alpha_4\leq 0$ and $\beta_4\geq(\pi^2+2\pi-16)/(4\sqrt{2}\pi-16)$.

In [5, 6], the authors presented the double inequalities

$$H^{1/3}(a,b)A^{2/3}(a,b) < N_{AG}(a,b) < \frac{1}{3}H(a,b) + \frac{2}{3}A(a,b), \tag{1.2}$$

$$C^{1/3}(a,b)A^{2/3}(a,b) < N_{AQ}(a,b) < \frac{1}{3}C(a,b) + \frac{2}{3}A(a,b), \tag{1.3}$$

$$H^{1/6}(a,b)A^{5/6}(a,b) < N_{GA}(a,b) < \frac{1}{6}H(a,b) + \frac{5}{6}A(a,b), \tag{1.4}$$

$$C^{1/6}(a,b)A^{5/6}(a,b) < N_{QA}(a,b) < \frac{1}{6}C(a,b) + \frac{5}{6}A(a,b) \tag{1.5}$$

for all $a,b>0$ with $a\neq b$.

Motivated by inequalities (1.2)-(1.5), it is natural to ask what are the best possible parameters λ_1, μ_1, λ_2, μ_2, λ_3, μ_3, λ_4 and μ_4 such that the double inequalities

$$\lambda_1[H(a,b)/3+2A(a,b)/3] + (1-\lambda_1)\,H^{1/3}(a,b)A^{2/3}(a,b) < N_{AG}(a,b)$$
$$< \mu_1[H(a,b)/3+2A(a,b)/3] + (1-\mu_1)\,H^{1/3}(a,b)A^{2/3}(a,b),$$

$$\lambda_2[C(a,b)/3+2A(a,b)/3] + (1-\lambda_2)\,C^{1/3}(a,b)A^{2/3}(a,b) < N_{AQ}(a,b)$$
$$< \mu_2[C(a,b)/3+2A(a,b)/3] + (1-\mu_2)\,C^{1/3}(a,b)A^{2/3}(a,b),$$

$$\lambda_3[H(a,b)/6+5A(a,b)/6] + (1-\lambda_3)\,H^{1/6}(a,b)A^{5/6}(a,b) < N_{GA}(a,b)$$
$$< \mu_3[H(a,b)/6+5A(a,b)/6] + (1-\mu_3)\,H^{1/6}(a,b)A^{5/6}(a,b),$$

$$\lambda_4[C(a,b)/6+5A(a,b)/6] + (1-\lambda_4)\,C^{1/6}(a,b)A^{5/6}(a,b) < N_{QA}(a,b)$$
$$< \mu_4[C(a,b)/6+5A(a,b)/6] + (1-\mu_4)\,C^{1/6}(a,b)A^{5/6}(a,b)$$

hold for all $a,b>0$ with $a\neq b$. The main purpose of this paper is to answer this question and present sharp bounds for the logarithmic mean $L(a,b)$, first Seiffert mean $P(a,b)$, second Seiffert mean $T(a,b)$ and Neuman-Sándor mean $M(a,b)$.

2. Lemmas

In order to prove our main results we need two lemmas, which we present in this section.

Lemma 2.1. *Let* $p\in(0,1)$ *and*

$$f(x) = px^4 + 2px^3 + (4p-1)x^2 + 2(2p-1)x + 4p - 3. \tag{2.1}$$

Then the following statements are true:

(1) if $p = 2/5$, then $f(x) < 0$ for all $x \in (0, 1)$ and $f(x) > 0$ for all $x \in (1, \sqrt[3]{2})$;
(2) if $p = 3/4$, then $f(x) > 0$ for all $x \in (0, 1)$;
(3) if $p = \left(3\pi + 6 - 12\sqrt[3]{2}\right) / \left(16 - 12\sqrt[3]{2}\right) = 0.3470\cdots$, then there exists $\xi_1 \in (1, \sqrt[3]{2})$ such that $f(x) < 0$ for $x \in (1, \xi_1)$ and $f(x) > 0$ for $x \in (\xi_1, \sqrt[3]{2})$.

Proof. For part (1), if $p = 2/5$, then (2.1) becomes

$$f(x) = \frac{1}{5}(x-1)(2x^3 + 6x^2 + 9x + 7). \tag{2.2}$$

Therefore, part (1) follows easily from (2.2).

For part (2), if $p = 3/4$, then (2.1) becomes

$$f(x) = \frac{1}{4}x(3x^3 + 6x^2 + 8x + 4). \tag{2.3}$$

Therefore, part (2) follows easily from (2.3).

For part (3), if $p = \left(3\pi + 6 - 12\sqrt[3]{2}\right) / \left(16 - 12\sqrt[3]{2}\right)$, then simple computations lead to

$$11p - 2 = 1.8174\cdots > 0, \tag{2.4}$$

$$2p - 1 = -0.3059\cdots < 0, \tag{2.5}$$

$$f(1) = 3(5p - 2) = -0.7943\cdots < 0, \tag{2.6}$$

$$f(\sqrt[3]{2}) = 0.4961\cdots > 0 \tag{2.7}$$

and

$$f'(x) = 4px^3 + 6px^2 + 2(4p-1)x + 2(2p-1). \tag{2.8}$$

It follows from (2.4), (2.5) and (2.8) that

$$f'(x) > 4px + 6px + 2(4p-1)x + 2(2p-1)x = 2(11p-2)x > 0 \tag{2.9}$$

for $x \in (1, \sqrt[3]{2})$.

Therefore, part (3) follows easily from (2.6) and (2.7) together with (2.9). □

Lemma 2.2. *Let $q \in (0, 1)$ and*

$$g(x) = 3qx^9 + 6qx^8 + 9qx^7 + 6(2q-1)x^6 + 3(5q-4)x^5 + 2(q-1)\left(5x^4 + 4x^3 + 3x^2 + 2x + 1\right). \tag{2.10}$$

Then the following statements are true:

(1) if $q = 16/25$, then $g(x) < 0$ for all $x \in (0, 1)$ and $g(x) > 0$ for all $x \in (1, \sqrt[6]{2})$;
(2) if $q = 3\pi/10$, then exists $\xi_2 \in (0, 1)$ such that $g(x) < 0$ for all $x \in (0, \xi_2)$ and $g(x) > 0$ for all $x \in (\xi_2, 1)$;
(3) if $q = \left[3\sqrt{2} + 3\log(1+\sqrt{2}) - 6\sqrt[6]{2}\right]/\left(7 - 6\sqrt[6]{2}\right) = 0.5730\cdots$, then there exists $\xi_3 \in (1, \sqrt[6]{2})$ such that $g(x) < 0$ for $x \in (1, \xi_3)$ and $g(x) > 0$ for $x \in (\xi_3, \sqrt[6]{2})$.

Proof. For part (1), if $q = 16/25$, then (2.10) becomes

$$g(x) = \frac{6}{25}(x-1)\left(8x^8 + 24x^7 + 48x^6 + 55x^5 + 45x^4 + 30x^3 + 18x^2 + 9x + 3\right). \tag{2.11}$$

Therefore, part (1) follows easily from (2.11).

For part (2), if $q = 3\pi/10$, then simple computations lead to

$$2q - 1 = \frac{3\pi - 5}{5} > 0, \tag{2.12}$$

$$5q-4=\frac{3\pi-8}{2}>0, \tag{2.13}$$

$$g(0)=2(q-1)=\frac{3\pi-10}{5}<0, \tag{2.14}$$

$$g(1)=3(25q-16)=\frac{3(15\pi-32)}{2}>0, \tag{2.15}$$

$$g'(x)=27qx^8+48qx^7+63qx^6+36(2q-1)x^5+15(5q-4)x^4+4(q-1)(10x^3+6x^2+3x+1). \tag{2.16}$$

Let $g_1(x)=g'(x)$, $g_2(x)=g_1'(x)/6$, $g_3(x)=g_2'(x)$, $g_4(x)=g_3'(x)/4$, $g_5(x)=g_4'(x)/15$. Then simple computations lead to

$$g_1(0)=4(q-1)<0, \tag{2.17}$$

$$g_1(1)=365q-176=\frac{219\pi-352}{2}>0, \tag{2.18}$$

$$g_2(x)=36qx^7+56qx^6+63qx^5+30(2q-1)x^4+10(5q-4)x^3+2(q-1)(10x^2+4x+1),$$

$$g_2(0)=2(q-1)<0, \tag{2.19}$$

$$g_2(1)=5(59q-20)=\frac{177\pi-200}{2}>0, \tag{2.20}$$

$$g_3(x)=252qx^6+336qx^5+315qx^4+120(2q-1)x^3+30(5q-4)x^2+8(q-1)(5x+1),$$

$$g_3(0)=8(q-1)<0, \tag{2.21}$$

$$g_3(1)=9(149q-32)=\frac{4023\pi-2880}{10}>0, \tag{2.22}$$

$$g_4(x)=378qx^5+420qx^4+315qx^3+90(2q-1)x^2+15(5q-4)x+10(q-1),$$

$$g_4(0)=10(q-1)<0, \tag{2.23}$$

$$g_4(1)=2(689q-80)=\frac{2067\pi-800}{5}>0, \tag{2.24}$$

$$g_5(x)=126qx^4+112qx^3+63qx^2+12(2q-1)x+(5q-4). \tag{2.25}$$

From (2.12) and (2.13) together with (2.25) we clearly see that g_4 is strictly increasing on $(0,1)$. Then (2.23) and (2.24) lead to the conclusion that there exists $x_0\in(0,1)$ such that g_3 is strictly decreasing on $(0,x_0)$ and strictly increasing on $(x_0,1)$.

It follows from (2.21) and (2.22) together with the piecewise monotonicity of g_3 that there exists $x_1\in(0,1)$ such that g_2 is strictly decreasing on $(0,x_1)$ and strictly increasing on $(x_1,1)$.

Inequalities (2.19) and (2.20) together with the piecewise monotonicity of g_2 imply that there exists $x_2\in(0,1)$ such that g_1 is strictly decreasing on $(0,x_2)$ and strictly increasing on $(x_2,1)$.

From (2.17) and (2.18) together with the piecewise monotonicity of g_1 we clearly see that exists $x_3\in(0,1)$ such that g is strictly decreasing on $(0,x_3)$ and strictly increasing on $(x_3,1)$.

Therefore, part (2) follows easily from (2.14) and (2.15) together with the piecewise monotonicity of g.

For part (3), if $q=\left[3\sqrt{2}+3\log(1+\sqrt{2})-6\sqrt[6]{2}\right]/\left(7-6\sqrt[6]{2}\right)=0.5730\cdots$, then numerical computations lead to

$$5q-4=-1.1347\cdots<0, \tag{2.26}$$

$$365q-176=33.1638\cdots>0, \tag{2.27}$$

$$g(1)=3(25q-16)=-5.0211\cdots<0, \tag{2.28}$$

$$g(\sqrt[6]{2})=3.1861\cdots>0. \tag{2.29}$$

From (2.16), (2.26) and (2.27) we have

$$\begin{aligned}g'(x)>&27qx^5+48qx^5+63qx^5+36(2q-1)x^5+15(5q-4)x^5+4(q-1)(10x^5+6x^5+3x^5+x^5)\\=&(365q-176)x^5>0\end{aligned} \tag{2.30}$$

for $x\in(1,\sqrt[6]{2})$.

Therefore, part (3) follows from (2.28)-(2.30). □

3. Main Results

Theorem 3.1. *The double inequality*

$$\lambda_1[H(a,b)/3+2A(a,b)/3]+(1-\lambda_1)\,H^{1/3}(a,b)A^{2/3}(a,b)<N_{AG}(a,b)$$
$$<\mu_1[H(a,b)/3+2A(a,b)/3]+(1-\mu_1)\,H^{1/3}(a,b)A^{2/3}(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\lambda_1\leq 2/5$ and $\mu_1\geq 3/4$.

Proof. Since $H(a,b)$, $N_{AG}(a,b)$ and $A(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a>b>0$. Let $v=(a-b)/(a+b)\in(0,1)$, $x=\sqrt[3]{1-v^2}\in(0,1)$ and $p\in(0,1)$. Then simple computations lead to

$$H^{1/3}(a,b)A^{2/3}(a,b)=A(a,b)\sqrt[3]{1-v^2},$$
$$\frac{1}{3}H(a,b)+\frac{2}{3}A(a,b)=A(a,b)\left(1-\frac{1}{3}v^2\right),$$

$$\frac{N_{AG}(a,b)-H^{1/3}(a,b)A^{2/3}(a,b)}{H(a,b)/3+2A(a,b)/3-H^{1/3}(a,b)A^{2/3}(a,b)}=\frac{\frac{1}{2}\left[1+(1-v^2)\frac{\tanh^{-1}(v)}{v}\right]-\sqrt[3]{1-v^2}}{(1-\frac{1}{3}v^2)-\sqrt[3]{1-v^2}},\tag{3.1}$$

$$\begin{aligned}&N_{AG}(a,b)-\left[p\Big(\frac{1}{3}H(a,b)+\frac{2}{3}A(a,b)\Big)+(1-p)H^{1/3}(a,b)A^{2/3}(a,b)\right]\\&=A(a,b)\left[\frac{1}{2}\left(1+(1-v^2)\frac{\tanh^{-1}(v)}{v}\right)-\frac{1}{3}p(3-v^2)-(1-p)\sqrt[3]{1-v^2}\right]\\&=\frac{A(a,b)(1-v^2)}{2v}F(x),\end{aligned}\tag{3.2}$$

where

$$F(x)=\tanh^{-1}\left(\sqrt{1-x^3}\right)-\frac{\sqrt{1-x^3}\left(2px^3+6(1-p)x+4p-3\right)}{3x^3},$$
$$F(1)=0,\tag{3.3}$$
$$F'(x)=\frac{(x-1)^2}{x^4\sqrt{1-x^3}}f(x),\tag{3.4}$$

where $f(x)$ is defined as in Lemma 2.1.

We divide the proof into two cases.

Case 1.1 ($p=2/5$). Then from Lemma 2.1 (1) and (3.4) we know that F is strictly decreasing on $(0,1)$. Therefore,

$$N_{AG}(a,b)>\frac{2}{5}\left[\frac{1}{3}H(a,b)+\frac{2}{3}A(a,b)\right]+\frac{3}{5}H^{1/3}(a,b)A^{2/3}(a,b)\tag{3.5}$$

for all $a,b>0$ with $a\neq b$ follows from (3.2) and (3.3) together with the monotonicity of F.

Case 1.2 ($p=3/4$). Then from Lemma 2.1 (2) and (3.4) we clearly see that F is strictly increasing on $(0,1)$. Therefore,

$$N_{AG}(a,b)<\frac{3}{4}\left[\frac{1}{3}H(a,b)+\frac{2}{3}A(a,b)\right]+\frac{1}{4}H^{1/3}(a,b)A^{2/3}(a,b)\tag{3.6}$$

for all $a, b > 0$ with $a \neq b$ follows from (3.2) and (3.3) together with the monotonicity of F.

Note that

$$\lim_{v\to 0^+} \frac{\frac{1}{2}\left[1+(1-v^2)\frac{\tanh^{-1}(v)}{v}\right]-\sqrt[3]{1-v^2}}{(1-\frac{1}{3}v^2)-\sqrt[3]{1-v^2}} = \frac{2}{5}, \tag{3.7}$$

$$\lim_{v\to 1^-} \frac{\frac{1}{2}\left[1+(1-v^2)\frac{\tanh^{-1}(v)}{v}\right]-\sqrt[3]{1-v^2}}{(1-\frac{1}{3}v^2)-\sqrt[3]{1-v^2}} = \frac{3}{4}. \tag{3.8}$$

Therefore, Theorem 3.1 follows from (3.5) and (3.6) together with the following statements.

- If $\lambda_1 > 2/5$, then (3.1) and (3.7) imply that there exists small enough $\delta_1 > 0$ such that $N_{AG}(a,b) < \lambda_1 (H(a,b)/3 + 2A(a,b)/3) + (1-\lambda_1) H^{1/3}(a,b)A^{2/3}(a,b)$ for all $a > b > 0$ with $(a-b)/(a+b) \in (0, \delta_1)$.

•• If $\mu_1 < 3/4$, then (3.1) and (3.8) imply that there exists small enough $0 < \delta_2 < 1$ such that $N_{AG}(a,b) > \mu_1 (H(a,b)/3 + 2A(a,b)/3) + (1-\mu_1) H^{1/3}(a,b)A^{2/3}(a,b)$ for all $a > b > 0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$.

□

Theorem 3.2. *The double inequality*

$$\lambda_2[C(a,b)/3 + 2A(a,b)/3] + (1-\lambda_2)\, C^{1/3}(a,b)A^{2/3}(a,b) < N_{AQ}(a,b)$$

$$< \mu_2[C(a,b)/3 + 2A(a,b)/3] + (1-\mu_2)\, C^{1/3}(a,b)A^{2/3}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_2 \leq (3\pi + 6 - 12\sqrt[3]{2})/(16 - 12\sqrt[3]{2}) = 0.3470\cdots$ and $\mu_2 \geq 2/5$.

Proof. Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$, $x = \sqrt[3]{1+v^2} \in (1, \sqrt[3]{2})$ and $p \in (0,1)$. Then simple computations lead to

$$C^{1/3}(a,b)A^{2/3}(a,b) = A(a,b)\sqrt[3]{1+v^2},$$
$$\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b) = A(a,b)\left(1+\frac{1}{3}v^2\right),$$

$$\frac{N_{AQ}(a,b) - C^{1/3}(a,b)A^{2/3}(a,b)}{C(a,b)/3 + 2A(a,b)/3 - C^{1/3}(a,b)A^{2/3}(a,b)} = \frac{\frac{1}{2}\left[1+(1+v^2)\frac{\arctan(v)}{v}\right]-\sqrt[3]{1+v^2}}{(1+\frac{1}{3}v^2)-\sqrt[3]{1+v^2}}, \tag{3.9}$$

$$\lim_{v\to 0^+} \frac{\frac{1}{2}\left[1+(1+v^2)\frac{\arctan(v)}{v}\right]-\sqrt[3]{1+v^2}}{(1+\frac{1}{3}v^2)-\sqrt[3]{1+v^2}} = \frac{2}{5}, \tag{3.10}$$

$$\lim_{v\to 1^-} \frac{\frac{1}{2}\left[1+(1+v^2)\frac{\arctan(v)}{v}\right]-\sqrt[3]{1+v^2}}{(1+\frac{1}{3}v^2)-\sqrt[3]{1+v^2}} = \frac{3\pi+6-12\sqrt[3]{2}}{16-12\sqrt[3]{2}}. \tag{3.11}$$

$$\begin{aligned} &N_{AQ}(a,b) - \left[p\left(\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right) + (1-p)C^{1/3}(a,b)A^{2/3}(a,b)\right] \\ =&A(a,b)\left[\frac{1}{2}\left(1+(1+v^2)\frac{\arctan(v)}{v}\right) - \frac{1}{3}p(3+v^2) - (1-p)\sqrt[3]{1+v^2}\right] \\ =&\frac{A(a,b)(1+v^2)}{2v}G(x), \end{aligned} \tag{3.12}$$

where

$$G(x) = \arctan\left(\sqrt{x^3-1}\right) - \frac{\sqrt{x^3-1}\left(2px^3+6(1-p)x+4p-3\right)}{3x^3},$$

$$G(1) = 0, \tag{3.13}$$

$$G(\sqrt[3]{2}) = \frac{3\pi+6-12\sqrt[3]{2}-(16-12\sqrt[3]{2})p}{12}, \tag{3.14}$$

$$G'(x) = -\frac{(x-1)^2}{x^4\sqrt{x^3-1}}f(x), \tag{3.15}$$

where $f(x)$ is defined as in Lemma 2.1.

We divide the proof into two cases.

Case 2.1 ($p = (3\pi+6-12\sqrt[3]{2})/(16-12\sqrt[3]{2}) = 0.3470\cdots$). Then from Lemma 2.1 (3) together with (3.14) and (3.15) we clearly see that there exists $\xi_1 \in (1, \sqrt[3]{2})$ such that $G(x)$ is strictly increasing on $(1, \xi_1]$ and strictly decreasing on $[\xi_1, \sqrt[3]{2})$, and

$$G(\sqrt[3]{2}) = 0. \tag{3.16}$$

Therefore,

$$N_{AQ}(a,b) > \left(\frac{3\pi+6-12\sqrt[3]{2}}{16-12\sqrt[3]{2}}\right)\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right] + \left(1-\frac{3\pi+6-12\sqrt[3]{2}}{16-12\sqrt[3]{2}}\right)C^{1/3}(a,b)A^{2/3}(a,b) \tag{3.17}$$

for all $a, b > 0$ with $a \neq b$ follows easily from (3.12), (3.13) and (3.16) together with the piecewise monotonicity of G.

Case 2.2 ($p = 2/5$). Then from Lemma 2.1 (1) and (3.15) we know that G is strictly decreasing on $(1, \sqrt[3]{2})$. It follows from (3.12) and (3.13) together with the monotonicity of G that

$$N_{AQ}(a,b) < \frac{2}{5}\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right] + \frac{3}{5}C^{1/3}(a,b)A^{2/3}(a,b) \tag{3.18}$$

for all $a, b > 0$ with $a \neq b$.

Therefore, Theorem 3.2 follows from (3.9)-(3.11), (3.17) and (3.18). □

Theorem 3.3. *The double inequality*

$$\lambda_3[H(a,b)/6+5A(a,b)/6] + (1-\lambda_3)\,H^{1/6}(a,b)A^{5/6}(a,b) < N_{GA}(a,b)$$
$$< \mu_3[H(a,b)/6+5A(a,b)/6] + (1-\mu_3)\,H^{1/6}(a,b)A^{5/6}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_3 \leq 16/25$ and $\mu_3 \geq 3\pi/10$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $v = (a-b)/(a+b) \in (0,1)$, $x = \sqrt[6]{1-v^2} \in (0,1)$ and $q \in (0,1)$. Then simple computations lead to

$$H^{1/6}(a,b)A^{5/6}(a,b) = A(a,b)\sqrt[6]{1-v^2},$$
$$\frac{1}{6}H(a,b)+\frac{5}{6}A(a,b) = A(a,b)\left(1-\frac{1}{6}v^2\right),$$

$$\frac{N_{GA}(a,b)-H^{1/6}(a,b)A^{5/6}(a,b)}{H(a,b)/6+5A(a,b)/6-H^{1/6}(a,b)A^{5/6}(a,b)} = \frac{\frac{1}{2}\left[\sqrt{1-v^2}+\frac{\arcsin(v)}{v}\right]-\sqrt[6]{1-v^2}}{\left(1-\frac{1}{6}v^2\right)-\sqrt[6]{1-v^2}}, \tag{3.19}$$

$$\lim_{v\to 0^+}\frac{\frac{1}{2}\left[\sqrt{1-v^2}+\frac{\arcsin(v)}{v}\right]-\sqrt[6]{1-v^2}}{\left(1-\frac{1}{6}v^2\right)-\sqrt[6]{1-v^2}} = \frac{16}{25}, \tag{3.20}$$

$$\lim_{v\to 1^-}\frac{\frac{1}{2}\left[\sqrt{1-v^2}+\frac{\arcsin(v)}{v}\right]-\sqrt[6]{1-v^2}}{\left(1-\frac{1}{6}v^2\right)-\sqrt[6]{1-v^2}}=\frac{3\pi}{10}, \tag{3.21}$$

$$\begin{aligned}&N_{GA}(a,b)-\left[q\left(\frac{1}{6}H(a,b)+\frac{5}{6}A(a,b)\right)+(1-q)H^{1/6}(a,b)A^{5/6}(a,b)\right]\\ =&A(a,b)\left[\frac{1}{2}\left(\sqrt{1-v^2}+\frac{\arcsin(v)}{v}\right)-\frac{1}{6}q(6-v^2)-(1-q)\sqrt[6]{1-v^2}\right]\\ =&\frac{A(a,b)}{2v}H(x),\end{aligned} \tag{3.22}$$

where

$$H(x)=\arcsin\left(\sqrt{1-x^6}\right)-\frac{1}{3}\sqrt{1-x^6}\left[qx^6-3x^3-6(1-q)x+5q\right],$$

$$H(0)=\frac{\pi}{2}-\frac{5}{3}q, \tag{3.23}$$

$$H(1)=0, \tag{3.24}$$

$$H'(x)=\frac{(x-1)^2}{\sqrt{1-x^6}}g(x), \tag{3.25}$$

where $g(x)$ is defined as in Lemma 2.2.

We divide the proof into two cases.

Case 3.1 $(q = 16/25)$. Then Lemma 2.2 (1) and (3.25) lead to the conclusion that $H(x)$ is strictly decreasing on $(0, 1)$. Therefore,

$$N_{GA}(a,b)>\frac{16}{25}\left[\frac{1}{6}H(a,b)+\frac{5}{6}A(a,b)\right]+\frac{9}{25}H^{1/6}(a,b)A^{5/6}(a,b) \tag{3.26}$$

for all $a, b > 0$ with $a \neq b$ follows from (3.22) and (3.24) together with the monotonicity of $H(x)$.

Case 3.2$(q = 3\pi/10)$. Then from Lemma 2.2 (2), (3.23) and (3.25) we know that

$$H(0)=0, \tag{3.27}$$

and there exists $\xi_2 \in (0, 1)$ such that $H(x)$ is strictly decreasing on $(0, \xi_2)$ and strictly increasing on $(\xi_2, 1)$. It follows from (3.22), (3.24), (3.27) and the piecewise monotonicity of $H(x)$ that

$$N_{GA}(a,b)<\frac{3\pi}{10}\left[\frac{1}{6}H(a,b)+\frac{5}{6}A(a,b)\right]+\left(1-\frac{3\pi}{10}\right)H^{1/6}(a,b)A^{5/6}(a,b) \tag{3.28}$$

for all $a, b > 0$ with $a \neq b$.

Therefore, Theorem 3.3 follows easily from (3.19)-(3.21), (3.26) and (3.28). □

Theorem 3.4. *The double inequality*

$$\lambda_4[C(a,b)/6+5A(a,b)/6]+(1-\lambda_4)\,C^{1/6}(a,b)A^{5/6}(a,b)<N_{QA}(a,b)$$

$$<\mu_4[C(a,b)/6+5A(a,b)/6]+(1-\mu_4)\,C^{1/6}(a,b)A^{5/6}(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\lambda_4 \leq [3\sqrt{2}+3\log(1+\sqrt{2})-6\sqrt[6]{2}]/(7-6\sqrt[6]{2}) = 0.5730\cdots$ *and* $\mu_4 \geq 16/25$.

Proof. Without loss of generality, we assume that $a > b > 0$. Let $v = (a-b)/(a+b) \in (0, 1)$, $x = \sqrt[6]{1+v^2} \in (1, \sqrt[6]{2})$ and $q \in (0, 1)$. Then simple computations lead to

$$C^{1/6}(a,b)A^{5/6}(a,b) = A(a,b)\sqrt[6]{1+v^2},$$
$$\frac{1}{6}C(a,b) + \frac{5}{6}A(a,b) = A(a,b)\left(1+\frac{1}{6}v^2\right),$$

$$\frac{N_{QA}(a,b) - C^{1/6}(a,b)A^{5/6}(a,b)}{C(a,b)/6 + 5A(a,b)/6 - C^{1/6}(a,b)A^{5/6}(a,b)} = \frac{\frac{1}{2}\left[\sqrt{1+v^2} + \frac{\sinh^{-1}(v)}{v}\right] - \sqrt[6]{1+v^2}}{(1+\frac{1}{6}v^2) - \sqrt[6]{1+v^2}}, \tag{3.29}$$

$$\lim_{v\to 0^+} \frac{\frac{1}{2}\left[\sqrt{1+v^2} + \frac{\sinh^{-1}(v)}{v}\right] - \sqrt[6]{1+v^2}}{\left(1+\frac{1}{6}v^2\right) - \sqrt[6]{1+v^2}} = \frac{16}{25}, \tag{3.30}$$

$$\lim_{v\to 1^-} \frac{\frac{1}{2}\left[\sqrt{1+v^2} + \frac{\sinh^{-1}(v)}{v}\right] - \sqrt[6]{1+v^2}}{\left(1+\frac{1}{6}v^2\right) - \sqrt[6]{1+v^2}} = \frac{3\sqrt{2} + 3\log(1+\sqrt{2}) - 6\sqrt[6]{2}}{7 - 6\sqrt[6]{2}}, \tag{3.31}$$

$$\begin{aligned} &N_{QA}(a,b) - \left[q\left(\frac{1}{6}C(a,b) + \frac{5}{6}A(a,b)\right) + (1-q)C^{1/6}(a,b)A^{5/6}(a,b)\right] \\ =&A(a,b)\left[\frac{1}{2}\left(\sqrt{1+v^2} + \frac{\sinh^{-1}(v)}{v}\right) - \frac{1}{6}q(6+v^2) - (1-q)\sqrt[6]{1+v^2}\right] \\ =&\frac{A(a,b)}{2v}J(x), \end{aligned} \tag{3.32}$$

where

$$J(x) = \sinh^{-1}\left(\sqrt{x^6-1}\right) - \frac{1}{3}\sqrt{x^6-1}\left[qx^6 - 3x^3 + 6(1-q)x + 5q\right],$$
$$J(1) = 0, \tag{3.33}$$
$$J(\sqrt[6]{2}) = \log(1+\sqrt{2}) + \sqrt{2} - 2\sqrt[6]{2} - (7/3 - 2\sqrt[6]{2})q, \tag{3.34}$$
$$J'(x) = -\frac{(x-1)^2}{\sqrt{x^6-1}}g(x), \tag{3.35}$$

where $g(x)$is defined as in Lemma 2.2.

We divide the proof into two cases.

Case 4.1 ($q = \left[3\sqrt{2} + 3\log(1+\sqrt{2}) - 6\sqrt[6]{2}\right]/\left(7 - 6\sqrt[6]{2}\right) = 0.5730\cdots$). Then Lemma 2.2 (3), (3.34) and (3.35) lead to the conclusion that there exists $\xi_3 \in (1, \sqrt[6]{2})$ such that $J(x)$ is strictly increasing on $(1, \xi_3]$ and strictly decreasing on $[\xi_3, \sqrt[6]{2})$, and

$$J(\sqrt[6]{2}) = 0. \tag{3.36}$$

Therefore,

$$\begin{aligned} N_{QA}(a,b) >& \left(\frac{3\sqrt{2} + 3\log(1+\sqrt{2}) - 6\sqrt[6]{2}}{7 - 6\sqrt[6]{2}}\right)\left[\frac{1}{6}C(a,b) + \frac{5}{6}A(a,b)\right] \\ &+ \left(1 - \frac{3\sqrt{2} + 3\log(1+\sqrt{2}) - 6\sqrt[6]{2}}{7 - 6\sqrt[6]{2}}\right)C^{1/6}(a,b)A^{5/6}(a,b) \end{aligned} \tag{3.37}$$

for all $a, b > 0$ with $a \neq b$ follows easily from (3.32), (3.33) and (3.36) together with the piecewise monotonicity of $J(x)$.

Case 4.2 ($q = 16/25$). Then Lemma 2.2 (1) and (3.35) lead to the conclusion that $J(x)$ is strictly decreasing on $(1, \sqrt[6]{2})$. It follows from (3.32) and (3.33) together with the monotonicity of $J(x)$ that

$$N_{QA}(a,b) < \frac{16}{25}\left[\frac{1}{6}C(a,b) + \frac{5}{6}A(a,b)\right] + \frac{9}{25}C^{1/6}(a,b)A^{5/6}(a,b) \tag{3.38}$$

for all $a, b > 0$ with $a \neq b$.

Therefore, Theorem 3.4 follows easily from (3.29)-(3.31), (3.37) and (3.38). □

Remark 3.5. Let $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ and $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 1$. Then we clearly see that the inequalities in Theorems 3.1-3.4 become inequalities (1.2)-(1.5). Therefore, Theorems 3.1-3.4 are the refinements and improvements of inequalities (1.2)-(1.5).

From (1.1) we clearly see that

$$N_{AG}(a,b) = \frac{1}{2}\left[A(a,b) + \frac{G^2(a,b)}{L(a,b)}\right], \quad N_{GA}(a,b) = \frac{1}{2}\left[G(a,b) + \frac{A^2(a,b)}{P(a,b)}\right], \tag{3.39}$$

$$N_{QA}(a,b) = \frac{1}{2}\left[Q(a,b) + \frac{A^2(a,b)}{M(a,b)}\right], \quad N_{AQ}(a,b) = \frac{1}{2}\left[A(a,b) + \frac{Q^2(a,b)}{T(a,b)}\right]. \tag{3.40}$$

Theorems 3.1-3.4 together with (3.39) and (3.40) lead to the sharp bounds for the logarithmic mean $L(a,b)$, first Seiffert mean $P(a,b)$, second Seiffert mean $T(a,b)$ and Neuman-Sándor mean $M(a,b)$ in terms of the harmonic mean $H(a,b)$, geometric mean $G(a,b)$, arithmetic mean $A(a,b)$, quadratic mean $Q(a,b)$ and contra-harmonic mean $C(a,b)$.

Theorem 3.6. *The double inequalities*

$$\frac{3G^2(a,b)}{2\mu_1 H(a,b) + (4\mu_1 - 3)A(a,b) + 6(1-\mu_1)H^{1/3}(a,b)A^{2/3}(a,b)} < L(a,b) < \frac{3G^2(a,b)}{2\lambda_1 H(a,b) + (4\lambda_1 - 3)A(a,b) + 6(1-\lambda_1)H^{1/3}(a,b)A^{2/3}(a,b)},$$

$$\frac{3Q^2(a,b)}{2\mu_2 C(a,b) + (4\mu_2 - 3)A(a,b) + 6(1-\mu_2)C^{1/3}(a,b)A^{2/3}(a,b)} < T(a,b) < \frac{3Q^2(a,b)}{2\lambda_2 C(a,b) + (4\lambda_2 - 3)A(a,b) + 6(1-\lambda_2)C^{1/3}(a,b)A^{2/3}(a,b)},$$

$$\frac{3A^2(a,b)}{\mu_3[H(a,b) + 5A(a,b)] + 6(1-\mu_3)H^{1/6}(a,b)A^{5/6}(a,b) - 3G(a,b)} < P(a,b) < \frac{3A^2(a,b)}{\lambda_3[H(a,b) + 5A(a,b)] + 6(1-\lambda_3)H^{1/6}(a,b)A^{5/6}(a,b) - 3G(a,b)},$$

$$\frac{3A^2(a,b)}{\mu_4[C(a,b) + 5A(a,b)] + 6(1-\mu_4)C^{1/6}(a,b)A^{5/6}(a,b) - 3Q(a,b)} < M(a,b) < \frac{3A^2(a,b)}{\lambda_4[C(a,b) + 5A(a,b)] + 6(1-\lambda_4)C^{1/6}(a,b)A^{5/6}(a,b) - 3Q(a,b)}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_1 \leq 2/5$, $\mu_1 \geq 3/4$, $\lambda_2 \leq (3\pi + 6 - 12\sqrt[3]{2})/(16 - 12\sqrt[3]{2}) = 0.3470\cdots$, $\mu_2 \geq 2/5$, $\lambda_3 \leq 16/25$, $\mu_3 \geq 3\pi/10$, $\lambda_4 \leq [3\sqrt{2} + 3\log(1+\sqrt{2}) - 6\sqrt[6]{2}]/(7 - 6\sqrt[6]{2}) = 0.5730\cdots$ and $\mu_4 \geq 16/25$.

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