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# Some results on fixed points of nonlinear operators and solutions of equilibrium problems

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**Abstract**

The purpose of this paper is to investigate fixed points of an asymptotically quasi-$\phi$-nonexpansive mapping in the intermediate sense and a bifunction equilibrium problem. We obtain a strong convergence theorem of solutions in the framework of Banach spaces. 

*Keywords:* Asymptotically quasi-$\phi$-nonexpansive mapping, equilibrium problem, fixed point, variational inequality, iterative process.
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## 1. Introduction and Preliminaries

Let $E$ be a real Banach space and let $C$ be a convex closed subset of $E$. Let $B : C \times C \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers, be a bifunction. Recall that the following equilibrium problem in the terminology of Blum and Oettli [4]. Find $\bar{x} \in C$ such that

$$B(\bar{x}\, y) \geq 0, \forall y \in C. \tag{1.1}$$

In this paper, we use $Sol(B)$ to denote the solution set of equilibrium problem (1.1). That is, $Sol(B) = \{x \in C : B(x, y) \geq 0, \forall y \in C\}$.

The following restrictions on bifunction $B$ are essential in this paper.

(Q1) $B(a, a) \equiv 0, \forall a \in C$;

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(Q2) $B(b,a) + B(a,b) \leq 0, \forall a,b \in C$;

(Q3) $B(a,b) \geq \limsup_{t\to 0} B(tc + (1-t)a, b)$, $\forall a,b,c \in C$;

(Q4) $b \mapsto B(a,b)$ is weakly lower semi-continuous and convex, $\forall a \in C$.

Equilibrium problem (1.1), which includes complementarity problems, variational inequality problems and inclusion problems as special cases, provides us a natural and unified framework to study a wide class of problems arising in physics, economics, finance, transportation, network, elasticity and optimization; see [3], [8], [10], [12], [14], [23], [28], and the references therein. Recently, equilibrium problem (1.1) has been extensively investigated based on fixed point algorithms in Banach spaces; see [9], [11], [13], [15]-[18], [24]-[27], [29]-[32] and the references therein.

Let $E^*$ be the dual space of $E$. Let $S^E$ be the unit sphere of $E$. Recall that $E$ is said to be a strictly convex space iff $\|x+y\| < 2$ for all $x, y \in S^E$ and $x \neq y$. Recall that $E$ is said to have a Gâteaux differentiable norm iff $\lim_{t\to 0} \frac{1}{t}(\|x\| - \|x+ty\|)$ exists for each $x, y \in S^E$. In this case, we also say that $E$ is smooth. $E$ is said to have a uniformly Gâteaux differentiable norm if for each $y \in B_E$, the limit is attained uniformly for all $x \in S^E$. $E$ is also said to have a uniformly Fréchet differentiable norm iff the above limit is attained uniformly for $x, y \in S^E$. In this case, we say that $E$ is uniformly smooth.

Recall that the normalized duality mapping $J$ from $E$ to $2^{E^*}$ is defined by

$$Jx = \{y \in E^* : \|x\|^2 = \langle x, y\rangle = \|y\|^2\}.$$

It is known

if $E$ is uniformly smooth, then $J$ is uniformly norm-to-norm continuous on every bounded subset of $E$;

if $E$ is a strictly convex Banach space, then $J$ is strictly monotone;

if $E$ is a smooth Banach space, then $J$ is single-valued and demicontinuous, i.e., continuous from the strong topology of $E$ to the weak star topology of $E$;

if $E$ is a reflexive and strictly convex Banach space with a strictly convex dual $E^*$ and $J^* : E^* \to E$ is the normalized duality mapping in $E^*$, then $J^{-1} = J^*$;

if $E$ is a smooth, strictly convex and reflexive Banach space, then $J$ is single-valued, one-to-one and onto.

From now on, we use $\rightharpoonup$ and $\to$ to stand for the weak convergence and strong convergence, respectively. Recall that $E$ is said to have the Kadec-Klee property (KK property) if $\lim_{n\to\infty} \|x_n - x\| = 0$ as $n \to \infty$, for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$ as $n \to \infty$.

Let $T$ be a mapping on $C$. Recall that a point $p$ is said to be a fixed point of $T$ if and only if $p = Tp$. $p$ is said to be an asymptotic fixed point [22] of $T$ if and only if $C$ contains a sequence $\{x_n\}$, where $x_n \rightharpoonup p$ such that $x_n - Tx_n \to 0$. From now on, we use $Fix(T)$ to stand for the fixed point set and $\widetilde{Fix}(T)$ to stand for the asymptotic fixed point set.

Next, we assume that $E$ is a smooth Banach space which means $J$ is single-valued. Study the functional

$$\phi(x,y) := \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x,y \in E.$$

Let $C$ be a closed convex subset of a real Hilbert space $H$. For any $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$, for all $y \in C$. The operator $P_C$ is called the metric projection from $H$ onto $C$. It is known that $P_C$ is firmly nonexpansive. In [2], Alber studied a new mapping $Proj_C$ in a Banach space $E$ which is an analogue of $P_C$, the metric projection, in Hilbert spaces. Recall that the generalized projection $Proj_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of $\phi(x,y)$, which implies from the definition of $\phi$ that

$$(\|y\| + \|x\|)^2 \geq \phi(x,y) \geq (\|x\| - \|y\|)^2, \quad \forall x,y \in E.$$

Recall that $T$ is said to be relatively nonexpansive [6], [7] iff

$$Fix(T) = \widetilde{Fix}(T) \neq \emptyset, \phi(p, Tx) \leq \phi(p,x), \quad \forall x \in C, \forall p \in Fix(T).$$

$T$ is said to be relatively asymptotically nonexpansive [1] iff

$$Fix(T) = \widetilde{Fix}(T) \neq \emptyset, \phi(p, T^n x) \leq (\mu_n + 1)\phi(p, x), \quad \forall x \in C, \forall p \in Fix(T), \forall n \geq 1,$$

where $\{\mu_n\} \subset [0, \infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

$T$ is said to be relatively asymptotically nonexpansive in the intermediate sense iff $Fix(T) = \widetilde{Fix}(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p \in Fix(T), x \in C} \big(\phi(p, T^n x) - \phi(p, x)\big) \leq 0.$$

Putting $\xi_n = \max\{0, \sup_{p \in Fix(T), x \in C} \big(\phi(p, T^n x) - \phi(p, x)\big)\}$, we see $\xi_n \to 0$ as $n \to \infty$.

$T$ is said to be quasi-$\phi$-nonexpansive [19] iff

$$Fix(T) \neq \emptyset, \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in Fix(T).$$

$T$ is said to be asymptotically quasi-$\phi$-nonexpansive [20] iff there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$Fix(T) \neq \emptyset, \phi(p, T^n x) \leq (\mu_n + 1)\phi(p, x), \quad \forall x \in C, \forall p \in Fix(T), \forall n \geq 1.$$

$T$ is said to be asymptotically quasi-$\phi$-nonexpansive in the intermediate sense [21] iff $Fix(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p \in Fix(T), x \in C} \big(\phi(p, T^n x) - \phi(p, x)\big) \leq 0.$$

Putting $\xi_n = \max\{0, \sup_{p \in Fix(T), x \in C} \big(\phi(p, T^n x) - \phi(p, x)\big)\}$, we see $\xi_n \to 0$ as $n \to \infty$.

*Remark* 1.1. The class of relatively asymptotically nonexpansive mappings covers the class of relatively nonexpansive mappings. The class of (asymptotically) quasi-$\phi$-nonexpansive mappings (in the intermediate sense) is more desirable than the class of relatively (asymptotically) nonexpansive mappings (in the intermediate sense) because of restriction $Fix(T) = \widetilde{Fix}(T)$.

*Remark* 1.2. The class of asymptotically quasi-$\phi$-nonexpansive mappings in the intermediate sense is reduced to the class of asymptotically quasi-nonexpansive mappings in the intermediate sense, which was considered in [5] as a non-Lipschitz continuous mappings, in the framework of Hilbert spaces.

**Lemma 1.3** ([2])**.** *Let $E$ be a strictly convex, reflexive, and smooth Banach space and let $C$ be a closed and convex subset of $E$. Let $x \in E$. Then*

$$\phi(y, x) - \phi(\Pi_C x, x) \geq \phi(y, \Pi_C x), \quad \forall y \in C,$$

$\langle y - x_0, Jx - Jx_0 \rangle \leq 0$, $\forall y \in C$ *if and only if* $x_0 = \Pi_C x$.

**Lemma 1.4** ([24])**.** *Let $E$ be a strictly convex, smooth, and reflexive Banach space and let $C$ be a closed convex subset of $E$. Let $B$ be a function with restrictions (Q1), (Q2), (Q3) and (Q4). Let $x \in E$ and let $r > 0$. Then there exists $z \in C$ such that $rB(z, y) + \langle z - y, Jz - Jx \rangle \leq 0$, $\forall y \in C$ Define a mapping $W^{B,r}$ by*

$$W^{B,r} x = \{z \in C : rB(z, y) + \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C\}.$$

*The following conclusions hold:*

(1) *$W^{B,r}$ is single-valued quasi-$\phi$-nonexpansive.*

(2) *$Sol(B) = Fix(W^{B,r})$ is closed and convex.*

**Lemma 1.5** ([21])**.** *Let $E$ be a strictly convex, smooth and reflexive Banach space such that both $E^*$ and $E$ have the KK property. Let $C$ be a convex and closed subset of $E$ and let $T$ be an asymptotically quasi-$\phi$-nonexpansive mapping in the intermediate sense on $C$. Then $Fix(T)$ is convex.*

## 2. Main results

**Theorem 2.1.** *Let $E$ be a smooth, strictly convex, and reflexive Banach space such that both $E$ and $E^*$ have the KK property and let $C$ be a convex and closed subset of $E$. Let $B$ be a bifunction satisfying (Q1), (Q2), (Q3) and (Q4) and let $T$ be an asymptotically quasi-$\phi$-nonexpansive mapping in the intermediate sense on $C$. Assume that $T$ is uniformly asymptotically regular and closed and $Fix(T) \cap Sol(B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = Proj_{C_1} x_0, \\ r_n B(u_n, \mu) \geq \langle u_n - \mu, Ju_n - Jx_n \rangle, \mu \in C, \\ Jy_n = \alpha_n JT^n u_n + (1-\alpha_n) Jx_n, \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) + \xi_n \geq \phi(z, y_n)\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

*where $\xi_n = \max\{\sup_{p\in Fix(T), x\in C} \big(\phi(p, T^n x) - \phi(p, x)\big), 0\}$, $\{\alpha_n\}$ is a real sequence in $[a, 1]$, where $a \in (0, 1)$ is a real number, and $\{r_n\} \subset [r, \infty)$ is a real sequence, where $r$ is some positive real number. Then $\{x_n\}$ converges strongly to $Proj_{Fix(T)\cap Sol(B)} x_1$.*

*Proof.* The proof is split into seven steps.

**Step 1.** Prove $Sol(B) \cap Fix(T)$ is convex and closed.

Using Lemma 1.4 and Lemma 1.5, we find that $Sol(B)$ is convex and closed and $Fix(T)$ is convex. Since $T$ is closed, one has $Fix(T)$ is also closed. So, $Sol(B) \cap Fix(T)$ is convex and closed. $Proj_{Sol(B)\cap Fix(T)} x$ is well defined, for any element $x$ in $E$.

**Step 2.** Prove $C_n$ is convex and closed.

It is obvious that $C_1 = C$ is convex and closed. Assume that $C_m$ is convex and closed for some $m \geq 1$. Let $p_1, p_2 \in C_{m+1}$. It follows that $p = sp_1 + (1-s)p_2 \in C_m$, where $s \in (0, 1)$. Notice that $\phi(p_1, y_m) - \phi(p_1, x_m) \leq \xi_m$, and $\phi(p_2, y_m) - \phi(p_2, x_m) \leq \xi_m$. Hence, one has

$$\xi_m + \|x_m\|^2 - \|y_m\|^2 \geq 2\langle p_1, Jx_m - Jy_m \rangle,$$

and

$$\xi_m + \|x_m\|^2 - \|y_m\|^2 \geq 2\langle p_2, Jx_m - Jy_m \rangle.$$

Using the above two inequalities, one has $\phi(p, x_m) + \xi_m \geq \phi(z, y_m)$. This shows that $C_{m+1}$ is closed and convex. Hence, $C_n$ is a convex and closed set. This proves that $Proj_{C_{n+1}} x_1$ is well defined.

**Step 3.** Prove $Sol(B) \cap Fix(T) \subset C_n$.

Note that $Sol(B) \cap Fix(T) \subset C_1 = C$ is clear. Suppose that $Sol(B) \cap Fix(T) \subset C_m$ for some positive integer $m$. For any $w \in Sol(B) \cap Fix(T) \subset C_m$, we see that

$$\begin{aligned} \phi(w, y_m) &= \|(1-\alpha_m)Jx_m + \alpha_m JT^m u_m\|^2 + \|w\|^2 \\ &\quad - 2\langle w, (1-\alpha_m)Jx_m + \alpha_m JT^m u_m \rangle \\ &\leq \|w\|^2 - 2\alpha_m \langle w, JT^m u_m \rangle - 2(1-\alpha_m)\langle w, Jx_m \rangle \\ &\quad + \alpha_m \|T^m u_m\|^2 + (1-\alpha_m)\|x_m\|^2 \\ &\leq \alpha_m \phi(w, u_m) + \alpha_m \xi_m + (1-\alpha_m)\phi(w, x_m) \\ &\leq \phi(w, x_m) + \xi_m, \end{aligned}$$

where $\xi_m = \max\{\sup_{p\in Fix(T), x\in C} \big(\phi(p, T^m x) - \phi(p, x)\big), 0\}$. This shows that $w \in C_{m+1}$. This implies that $Sol(B) \cap Fix(T) \subset C_n$.

**Step 4.** Prove $\{x_n\}$ is bounded.

Using Lemma 1.3, one has $\langle z - x_n, Jx_1 - Jx_n\rangle \le 0$, for any $z \in C_n$. It follows that

$$0 \ge \langle w - x_n, Jx_1 - Jx_n\rangle, \forall w \in Sol(B) \cap Fix(T) \subset C_n.$$

Using Lemma 1.3 yields that

$$\phi(\Pi_{Fix(T)\cap Sol(B)}x_1, x_1) \ge \phi(x_n, x_1) \ge 0,$$

which implies that $\{\phi(x_n, x_1)\}$. Hence $\{x_n\}$ is also a bounded sequence. Without loss of generality, we may assume $x_n \rightharpoonup \bar{x}$. Since $C_n$ is convex and closed, we see $\bar{x} \in C_n$.

**Step 5.** Prove $\bar{x} \in Fix(T)$.

Using the fact $\phi(x_n, x_1) \le \phi(\bar{x}, x_1)$, one has

$$\phi(\bar{x}, x_1) \ge \limsup_{n\to\infty} \phi(x_n, x_1) \ge \liminf_{n\to\infty} \phi(x_n, x_1) = \liminf_{n\to\infty}(\|x_n\|^2 + \|x_1\|^2 - 2\langle x_n, Jx_1\rangle) \ge \phi(\bar{x}, x_1).$$

It follows that $\lim_{n\to\infty} \phi(x_n, x_1) = \phi(\bar{x}, x_1)$. Hence, we have

$$\phi(x_{n+1}, x_1) - \phi(x_n, x_1) \ge \phi(x_{n+1}, x_n) \ge 0.$$

Therefore, we have $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. Since $x_{n+1} \in C_{n+1}$, one sees that

$$\phi(x_{n+1}, x_n) + \xi_n \ge \phi(x_{n+1}, y_n) \ge 0.$$

It follows that $\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0$. Hence, one has $\lim_{n\to\infty}(\|y_n\| - \|x_{n+1}\|) = 0$. This implies that

$$\|\bar{x}\| = \|J\bar{x}\| = \lim_{n\to\infty} \|Jy_n\| = \lim_{n\to\infty} \|y_n\|.$$

This implies that $\{Jy_n\}$ is bounded. Assume that $\{Jy_n\}$ converges weakly to $y^* \in E^*$. In view of the reflexivity of $E$, we see that $J(E) = E^*$. This shows that there exists an element $u \in E$ such that $Jy = y^*$. It follows that $\phi(x_{n+1}, y_n) + 2\langle x_{n+1}, Jy_n\rangle = \|x_{n+1}\|^2 + \|Jy_n\|^2$. Taking $\liminf_{n\to\infty}$, one has $0 \ge \|\bar{x}\|^2 - 2\langle \bar{x}, y^*\rangle + \|y^*\|^2 = \|\bar{x}\|^2 + \|Jy\|^2 - 2\langle \bar{x}, Jy\rangle = \phi(\bar{x}, y) \ge 0$. That is, $\bar{x} = y$, which in turn implies that $J\bar{x} = y^*$. Hence, $Jy_n \rightharpoonup J\bar{x} \in E^*$. Using the KK property, we obtain $\lim_{n\to\infty} Jy_n = J\bar{x}$. Since $J^{-1}$ is demicontinuous and $E$ has the KK property, one gets $y_n \to \bar{x}$, as $n \to \infty$. Using the restriction on $\{\alpha_n\}$, one has $\lim_{n\to\infty} \|Jx_n - JT^n u_n\| = 0$. This implies that $\lim_{n\to\infty} \|JT^n u_n - J\bar{x}\| = 0$. Since $J^{-1}$ is demicontinuous, one has $T^n u_n \rightharpoonup \bar{x}$. Since

$$|\|T^n u_n\| - \|\bar{x}\|| \le \|J(T^n u_n) - J\bar{x}\|,$$

one has $\|T^n u_n\| \to \|\bar{x}\|$, as $n \to \infty$. Since $E$ has the KK property, we obtain $\lim_{n\to\infty} \|\bar{x} - T^n u_n\| = 0$. Since $T$ is also uniformly asymptotically regular, one has $\lim_{n\to\infty} \|\bar{x} - T^{n+1}u_n\| = 0$. That is, $T(T^n u_n) \to \bar{x}$. Using the closedness of $T$, we find $T\bar{x} = \bar{x}$. This proves $\bar{x} \in Fix(T)$.

**Step 6.** Prove $\bar{x} \in Sol(B)$.

Since $\alpha_n\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n) + \xi_n$, one has $\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0$. Hence, one has $\lim_{n\to\infty}(\|u_n\| - \|x_{n+1}\|) = 0$. This implies that

$$\|\bar{x}\| = \|J\bar{x}\| = \lim_{n\to\infty} \|Ju_n\| = \lim_{n\to\infty} \|u_n\|.$$

This implies that $\{Ju_n\}$ is bounded. Assume that $\{Ju_n\}$ converges weakly to $y^* \in E^*$. In view of the reflexivity of $E$, we see that $J(E) = E^*$. This shows that there exists an element $u \in E$ such that $Ju = u^*$. It follows that

$$\phi(x_{n+1}, u_n) + 2\langle x_{n+1}, Ju_n\rangle = \|x_{n+1}\|^2 + \|Ju_n\|^2.$$

Taking $\liminf_{n\to\infty}$, one has

$$0 \ge \|\bar{x}\|^2 - 2\langle \bar{x}, u^*\rangle + \|u^*\|^2 = \|\bar{x}\|^2 + \|Ju\|^2 - 2\langle \bar{x}, Ju\rangle = \phi(\bar{x}, u) \ge 0.$$

That is, $\bar{x} = u$, which in turn implies that $u^* = J\bar{x}$. Hence, $Ju_n \rightharpoonup J\bar{x} \in E^*$. Using the KK property, we obtain $\lim_{n\to\infty} Ju_n = J\bar{x}$. Since $J^{-1}$ is demi-continuous and $E$ has the KK property, one gets $u_n \to \bar{x}$, as $n \to \infty$. Since

$$r_n B(y, u_n) + \langle u_n - y, Ju_n - Jy_n\rangle \geq 0, \forall y \in C_n,$$

we see that $B(y, \bar{x}) \leq 0$. Let $0 < t < 1$ and define $y_t = ty + (1-t)\bar{x}$. It follows that $y_t \in C$, which yields that $B(y_t, \bar{x}) \leq 0$. It follows from the (Q1) and (Q4) that

$$0 = B(y_t, y_t) \leq tB(y_t, y) + (1-t)B(y_t, \bar{x}) \leq tB(y_t, y).$$

That is, $B(y_t, y) \geq 0$. It follows from (Q3) that $B(\bar{x}, y) \geq 0$, $\forall y \in C$. This implies that $\bar{x} \in Sol(B)$. This completes the proof that $\bar{x} \in Sol(B) \cap Fix(T)$.

**Step 7.** Prove $\bar{x} = Proj_{Sol(B)\cap Fix(T)} x_1$.

Note the fact $\langle w - x_n, Jx_1 - Jx_n\rangle \leq 0$, $\forall w \in Sol(B) \cap Fix(T)$. It follows that

$$\langle \bar{x} - w, Jx_1 - J\bar{x}\rangle \geq 0, \quad \forall w \in Fix(T) \cap Sol(B).$$

Using Lemma 1.3, we find that that $\bar{x} = Proj_{Fix(T)\cap Sol(B)} x_1$. This completes the proof. □

From Theorem 2.1, the following results are not hard to derive.

**Corollary 2.2.** *Let $E$ be a smooth, strictly convex, and reflexive Banach space such that both $E$ and $E^*$ have the KK property and let $C$ be a convex and closed subset of $E$. Let $B$ be a bifunction satisfying (Q1), (Q2), (Q3) and (Q4). Assume that $Sol(B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = Proj_{C_1} x_0, \\ r_n B(u_n, \mu) \geq \langle u_n - \mu, Ju_n - Jx_n\rangle, \mu \in C, \\ Jy_n = \alpha_n Ju_n + (1-\alpha_n)Jx_n, \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) \geq \phi(z, y_n)\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \end{cases}$$

*where $\{\alpha_n\}$ is a real sequence in $[a, 1]$, $a \in (0, 1]$ is a real number and $\{r_n\} \subset [r, \infty)$ is a real sequence, where $r$ is some positive real number. Then $\{x_n\}$ converges strongly to $Proj_{Sol(B)} x_1$.*

**Corollary 2.3.** *Let $E$ be a Hilbert space and let $C$ be a convex and closed subset of $E$. Let $B$ be a bifunction satisfying (Q1), (Q2), (Q3) and (Q4) and let $T$ be an asymptotically quasi-nonexpansive mapping in the intermediate sense on $C$. Assume that $T$ is uniformly asymptotically regular and closed and $Fix(T) \cap Sol(B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, x_1 = P_{C_1} x_0, \\ r_n B(u_n, \mu) \geq \langle u_n - \mu, u_n - x_n\rangle, \mu \in C, \\ y_n = \alpha_n T^n u_n + (1-\alpha_n)x_n, \\ C_{n+1} = \{z \in C_n : \|z - x_n\|^2 + \xi_n \geq \|z - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases}$$

*where $\xi_n = \max\{\sup_{p\in Fix(T), x\in C} \left(\|p - T^n x\|^2 - \|p - x\|^2\right), 0\}$, $\{\alpha_n\}$ is a real sequence in $[a, 1]$, where $a \in (0, 1]$ is a real number, and $\{r_n\} \subset [r, \infty)$ is a real sequence, where $r$ is some positive real number. Then $\{x_n\}$ converges strongly to $P_{Fix(T)\cap Sol(B)} x_1$.*

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