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Tripled coincidence points for mixed comparable mappings in partially ordered cone metric spaces over Banach algebras

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Abstract

Let (X, d) be a complete partially ordered cone metric space, $g : X \to X$ and $F : X \times X \times X \to X$ be two mappings. In this paper, a new concept of F having the mixed comparable property with respect to g is introduced and some tripled coincidence point results of F and g are obtained if F has the mixed comparable property with respect to g and some other natural conditions are satisfied. Moreover, a support example of one of our results is given.

Keywords: Cone metric spaces over Banach algebras, mixed comparable properties, tripled coincidence points, spectral radius.
2010 MSC: 54H25, 47H10.

1. Introduction

One of the important subjects of nonlinear sciences is to study the existence or uniqueness of fixed points for nonlinear mappings. Most recently, Bhaskar and Lakshmikantham [6] introduced the concepts of coupled fixed point and mixed monotone property for contractive operators of the form $F : X \times X \to X$, where X is a partially ordered metric space and established some existence and uniqueness coupled fixed point theorems. Sabetghadam et al. [19] extended the results of Gnana Bhaskar and Lakshmikantham [6] to the setting of complete cone metric spaces. Based on the works of Bhaskar and Lakshmikantham [6]

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and Sabetghadam [19], Berinde and Borcut [1] introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces and obtained existence and uniqueness theorems for contractive type mappings. The results given by V. Berinde and M. Borcut [1] generalized and extended the works of Bhaskar and Lakshmikantham and Sabetghadam.

In 2007, Huang and Zhang [7] introduced the concept of cone metric spaces as a generalization of general metric spaces, in which the distance $d(x, y)$ of x and y is defined to be a vector in an ordered Banach space E and proved that the Banach contraction principle remains true in the setting of cone metric spaces. Since then, many fixed point results of the mappings with certain contractive property on cone metric spaces have been proved on the basis of the work of Huang and Zhang [7] (see [2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20] and the references therein). Among those works, the results of [15] attract much attention since the authors of [15] introduced the concept of cone metric spaces over Banach algebras by replacing Banach spaces with Banach algebras in order to generalize the Banach contraction principle to a more general form. In 2013, Liu and Xu [14] proved some fixed point theorems of generalized Lipschitz mappings with weaker and natural conditions on the generalized Lipschitz constant k by means of spectral radius in the setting of cone metric spaces over Banach algebras. But the proofs of the main results in [14] strongly depend on the condition that the underlying solid cone is normal. In 2014, without the assumption of normality of the cone involved, Xu and Stojan [20] proved the conclusions of [14] remain valid by means of some properties of spectral radius.

Let (X, d) be a complete partially ordered cone metric space, $g : X \to X$ and $F : X \times X \times X \to X$ be two mappings. In this paper, on the basis of [1], [15] and [20], we introduce the concept of F having the mixed comparable property with respect to g and prove some tripled coincidence point results of F and g provided F has the mixed comparable property with respect to g and some other natural conditions are satisfied. Moreover, we give some support examples of part of our results.

2. Preliminaries

The following definitions and results come from Huang and Zhang [7] and Liu and Xu [14], which are needed in the sequel.

Let $\mathcal{A}$ always be a real Banach algebra, that is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all $x, y, z \in \mathcal{A}$, $\alpha \in \mathbb{R}$):

(1) $(xy)z = x(yz)$;
(2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(4) $\|xy\| \leq \|x\|\|y\|$.

Here and subsequently, we assume that a Banach algebra has a unit (*i.e.*, a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. $x \in \mathcal{A}$ is said to be invertible if there is $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. We refer the reader to [14] for more details.

A non-empty closed convex subset P of a Banach algebra $\mathcal{A}$ is called a cone if

(i) $\{\theta, e\} \subset P$;
(ii) $\alpha P + \beta P \subset P$;
(iii) $P^2 = PP \subset P$;
(iv) $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra $\mathcal{A}$.

Fix a cone $P \subset \mathcal{A}$, a partial ordering '$\preceq$' with respect to P can be defined by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ stands for $x \preceq y$ and $x \neq y$. $x \ll y$ stands for $y - x \in \text{int}(P)$, here $\text{int}(P)$ denotes the interior of P. P is called a solid cone if $\text{int}(P) \neq \emptyset$.

Definition 2.1 ([14]). Let X be a non-empty set and $\mathcal{A}$ be a real Banach algebra. Suppose that the mapping $d : X \times X \to \mathcal{A}$ satisfies:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space over the Banach algebra $\mathcal{A}$.

One can refer to [14] for some examples of cone metric spaces over Banach algebras.

Definition 2.2 ([14])**.** Let (X,d) be a cone metric space over the Banach algebra $\mathcal{A}$, $x \in X$ and let $\{x_n\}$ be a sequence in X. Then

(1) $\{x_n\}$ converges to x if for each $c \in \mathcal{A}$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x) \ll c$ for all $n > N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(2) $\{x_n\}$ is a Cauchy sequence if for each $c \in \mathcal{A}$ with $\theta \ll c$, there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m > N$.

(3) (X,d) is a complete cone metric space if every Cauchy sequence is convergent.

Definition 2.3 ([11])**.** Let P be a solid cone in a Banach space E. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$.

Definition 2.4 ([21])**.** Let $(X, \leqslant)$ be a partial ordering set and $\mathcal{A}$ be a Banach algebra, $d : X \times X \to \mathcal{A}$ be a cone metric on X such that (X,d) is a cone metric space over the Banach algebra $\mathcal{A}$. We say that $x, y \in X$ are comparable if $x \leqslant y$ or $y \leqslant x$ holds.

In the rest of this section, we always assume that $\mathcal{A}$ is a real Banach algebra, (X,d) is a complete partial ordering cone metric space over $\mathcal{A}$ with the partial ordering '$\leqslant$' and P is a solid cone of $\mathcal{A}$ which gives the partial ordering '$\preceq$' in $\mathcal{A}$.

Lemma 2.5 ([17])**.** *If E is a real Banach space with a solid cone P and if $||x_n|| \to 0 (n \to \infty)$, then for any $\theta \ll c$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $x_n \ll c$.*

Lemma 2.6 ([17])**.** *If E is a real Banach space with a solid cone P and if $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

Lemma 2.7 ([20])**.** *Let x, y be vectors in $\mathcal{A}$. If x and y commute, then the spectral radius r satisfies the following properties:*

(i) $r(xy) \leq r(x)r(y)$;

(ii) $r(x+y) \leq r(x) + r(y)$;

(iii) $|r(x) - r(y)| \leq r(x-y)$.

Lemma 2.8 ([17])**.** *Let $\mathcal{A}$ be a Banach algebra with a unit e, P be a cone in $\mathcal{A}$ and $\preceq$ be the semi-order generated by P. Let $\lambda \in P$. If the spectral radius $r(\lambda)$ of λ is less than 1, then the following assertions hold.*

(i) Suppose that x is invertible and that $x^{-1} \succ \theta$ implies $x \succ \theta$, then for any integer $n \geq 1$, we have $\lambda^n \preceq \lambda \preceq e$.

(ii) For any $u \succ \theta$, we have $u \not\preceq \lambda u$, i.e., $\lambda u - u \notin P$.

(iii) If $\lambda \succeq \theta$, then we have $(e-\lambda)^{-1} \succeq \theta$.

Lemma 2.9 ([17])**.** *Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{x_n\}$ be a sequence in P. Then the following conditions are equivalent:*

(1) $\{x_n\}$ is a c-sequence.

(2) For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $x_n \prec c$ for $n \geq n_0$.

(3) For each $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $x_n \preceq c$ for $n \geq n_1$.

Lemma 2.10 ([17])**.** *Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{u_n\}$ be a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.*

3. Main results

In this section, we will present the main results and their proofs. For simplicity, we always assume that $\mathcal{A}$ is a real Banach algebra and (X,d) is a complete partial ordering cone metric space over $\mathcal{A}$ with the partial ordering '$\leqslant$'. Let P be a solid cone of $\mathcal{A}$ which gives the partial ordering '$\preceq$' in $\mathcal{A}$. Consider on the product space $X \times X \times X$ the following partial ordering: for $(x,y,z),(u,v,w) \in X \times X \times X$,

$$(u,v,w) \leq (x,y,z) \Leftrightarrow x \geqslant u, y \leqslant v, z \geqslant w.$$

Let $F : X \times X \times X \to X$ and $g : X \to X$ be two mappings. Following the basic concepts and results established in [1] and as generalizations, we introduce the new concepts of g-continuous mapping and mixed comparable property with respect to g and obtain some tripled coincidence point results for g and F, where F has the mixed comparable property with respect to g. Next, we introduce some concepts.

Definition 3.1. We say that F has the mixed comparable property with respect to g if $F(x,y,z)$ and $F(u,v,w)$ are comparable for any pair (x,y,z) and (u,v,w) in $X \times X \times X$ for which $g(x)$ and $g(u)$, $g(y)$ and $g(v)$ and $g(z)$ and $g(w)$ are comparable.

Definition 3.2. An element $(x,y,z) \in X \times X \times X$ is called a tripled coincidence point of g and F if $F(x,y,z) = g(x)$, $F(y,x,y) = g(y)$ and $F(z,y,x) = g(z)$. If there exists $x \in X$ such that $g(x) = F(x,x,x)$, then we say that x is a coincidence point of g and F.

Definition 3.3. If $g(x_n) \to g(x)$ as well as $g(y_n) \to g(y)$ and $g(z_n) \to g(z)$ implies $F(x_n,y_n,z_n) \to F(x,y,z)$ as $n \to \infty$ for any sequences $\{x_n\},\{y_n\}$ and $\{z_n\}$ of X, then F is said to be g-continuous.

Example 3.1. Let $\mathbb{R}$ be the set of all real numbers with the usual metric d, that is, $d(x,y) = |x-y|$ for any $x,y \in \mathbb{R}$. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a mapping defined as follows: for any $x \in \mathbb{R}$, if $x \neq 0$, then $g(x) = \sin\frac{1}{x}$ and $g(x) = 0$ if $x = 0$. Let $F(x,y,z) = (2g(x), g(y), g(z))$ for any $(x,y,z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Obviously, F is g-continuous but not continuous.

Remark 3.4. The concept of g-continuous mapping generalizes continuous mapping, because if g is just taken as the identity mapping on X, then each g-continuous mapping is continuous.

The following theorem is our first main result.

Theorem 3.5. *Let g be a surjection and F be a g-continuous mapping possessing the mixed comparable property with respect to g. Assume that:*

(1) there exist $j,k,l \in \mathcal{A}$ with $r(j+k+l) < 1$ for which

$$d(F(x,y,z),F(u,v,w)) \preceq jd(g(x),g(u)) + kd(g(y),g(v)) + ld(g(z),g(w)) \tag{3.1}$$

for any (x,y,z) and $(u,v,w) \in X \times X \times X$ satisfying that $g(x)$ and $g(u)$, $g(y)$ and $g(v)$ and $g(z)$ and $g(w)$ are comparable;

(2) there exist x_0, y_0, $z_0 \in X$ such that $g(x_0)$ and $F(x_0,y_0,z_0)$, $g(y_0)$ and $F(y_0,x_0,y_0)$ and $g(z_0)$ and $F(z_0,y_0,x_0)$ are comparable.

Then g and F have a tripled coincidence point in X.

Proof. Since g is surjective, there exists $x_1 \in X$ such that $g(x_1) = F(x_0,y_0,z_0)$ and $g(x_1)$ and $g(x_0)$ are comparable. Similarly, there exist $y_1,z_1 \in X$ such that $g(y_1) = F(y_0,x_0,y_0)$ and $g(z_1) = F(z_0,y_0,x_0)$, furthermore, $g(y_1)$ and $g(y_0)$ are comparable and $g(z_1)$ and $g(z_0)$ are comparable. Continuing this process and noting that F has the mixed comparable property with respect to g, for $n \geq 1$, there exist $x_n,y_n,z_n \in X$ such that

$$g(x_n) = F(x_{n-1},y_{n-1},z_{n-1}), \quad g(y_n) = F(y_{n-1},x_{n-1},y_{n-1}) \text{ and } g(z_n) = F(z_{n-1},y_{n-1},x_{n-1}),$$

and $g(x_n)$ and $g(x_{n-1})$, $g(y_n)$ and $g(y_{n-1})$ and $g(z_n)$ and $g(z_{n-1})$ are comparable. To simplify the writing, we denote

$$D_n^x = d(g(x_{n-1}), g(x_n)), \quad D_n^y = d(g(y_{n-1}), g(y_n)) \text{ and } D_n^z = d(g(z_{n-1}), g(z_n)).$$

Then by (3.1), we have

$$\begin{aligned} D_2^x = d(g(x_1), g(x_2)) &= d(F(x_0, y_0, z_0), F(x_1, y_1, z_1)) \\ &\preceq jd(g(x_0), g(x_1)) + kd(g(y_0), g(y_1)) + ld(g(z_0), g(z_1)) \\ &= jD_1^x + kD_1^y + lD_1^z. \end{aligned}$$

Similarly, we have $D_2^y \preceq (j+l)D_1^y + kD_1^x + 0D_1^z$, $D_2^z \preceq jD_1^z + kD_1^y + lD_1^x$ and

$$\begin{aligned} D_3^x &\preceq (j^2+k^2+l^2)D_1^x + (2jk+2kl)D_1^y + 2jlD_1^z, \\ D_3^y &\preceq (kl+2jk)D_1^x + ((j+l)^2+k^2)D_1^y + klD_1^z, \\ D_3^z &\preceq (2jl+k^2)D_1^x + (2kj+2kl)D_1^y + (j^2+l^2)D_1^z. \end{aligned}$$

For simplicity, we also consider the matrix

$$\mathbf{A} = \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix} \text{denoted by} \begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & b_1 & h_1 \end{pmatrix}$$

and further denote

$$\mathbf{A^2} = \begin{pmatrix} j^2+k^2+l^2 & 2jk+2kl & 2jl \\ kl+2jk & (j+l)^2+k^2 & kl \\ 2jl+k^2 & 2jk+2kl & j^2+l^2 \end{pmatrix} = \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & b_2 & h_2 \end{pmatrix},$$

where

$$a_2 + b_2 + c_2 = d_2 + e_2 + f_2 = g_2 + b_2 + h_2 = (j+k+l)^2. \tag{3.2}$$

Here the operational laws of matrices are same to those of general matrices.

Now we prove by induction that

$$\mathbf{A^n} = \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix},$$

where

$$a_n + b_n + c_n = d_n + e_n + f_n = g_n + b_n + h_n = (j+k+l)^n. \tag{3.3}$$

In fact, if we assume that (3.3) is true for some $n \geq 1$, then since

$$\begin{aligned} \mathbf{A^{n+1}} = \mathbf{A^n A} &= \begin{pmatrix} a_n & b_n & c_n \\ d_n & e_n & f_n \\ g_n & b_n & h_n \end{pmatrix} \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix} \\ &= \begin{pmatrix} ja_n + kb_n + lc_n & ka_n + (j+l)b_n + kc_n & la_n + jc_n \\ jd_n + ke_n + lf_n & kd_n + (j+l)e_n + kf_n & ld_n + jf_n \\ jg_n + kb_n + lh_n & kg_n + (j+l)b_n + kh_n & lg_n + jh_n \end{pmatrix}, \end{aligned}$$

we have

$$\begin{aligned} a_{n+1} + b_{n+1} + c_{n+1} &= a_n j + b_n k + c_n l + a_n k + b_n j + c_n k + a_n l + b_n l + c_n j \\ &= a_n(j+k+l) + b_n(k+j+l) + c_n(l+k+j) \\ &= (a_n+b_n+c_n)(j+k+l) = (j+k+l)^n(j+k+l) \\ &= (j+k+l)^{n+1}. \end{aligned}$$

Similarly one has

$$d_{n+1} + e_{n+1} + f_{n+1} = g_{n+1} + b_{n+1} + h_{n+1} = (j + k + l)^{n+1}.$$

Therefore, we have

$$\begin{pmatrix} D^x_{n+1} \\ D^y_{n+1} \\ D^z_{n+1} \end{pmatrix} \preceq \begin{pmatrix} j & k & l \\ k & j+l & 0 \\ l & k & j \end{pmatrix}^n \begin{pmatrix} D^x_1 \\ D^y_1 \\ D^z_1 \end{pmatrix}.$$

That is,

$$D^x_{n+1} \preceq a_n D^x_1 + b_n D^y_1 + c_n D^z_1, \tag{3.4}$$

$$D^y_{n+1} \preceq d_n D^x_1 + e_n D^y_1 + f_n D^z_1, \tag{3.5}$$

$$D^z_{n+1} \preceq g_n D^x_1 + b_n D^y_1 + h_n D^z_1. \tag{3.6}$$

Following from (3.4 and 3.6), we can easily show that $\{g(x_n)\}$, $\{g(y_n)\}$ and $\{g(z_n)\}$ are Cauchy sequences. In fact, for $m > n$, we have

$$\begin{aligned} d(g(x_m), g(x_n)) &\preceq d(g(x_m), g(x_{m-1})) + \cdots + d(g(x_{n+1}, g(x_n)) = D^x_m + D^x_{m-1} + \cdots + D^x_{n+1} \\ &\preceq (a_{m-1}D^x_1 + b_{m-1}D^y_1 + c_{m-1}D^z_1) + \cdots + (a_n D^x_1 + b_n D^y_1 + c_n D^z_1) \\ &= (a_n + \cdots + a_{m-1})D^x_1 + (b_n + \cdots + b_{m-1})D^y_1 + (c_n + \cdots + c_{m-1})D^z_1 \\ &\preceq (\alpha^n + \cdots + \alpha^{m-1})D^x_1 + (\alpha^n + \cdots + \alpha^{m-1})D^y_1 + (\alpha^n + \cdots + \alpha^{m-1})D^z_1 \\ &= (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1})(D^x_1 + D^y_1 + D^z_1) \\ &\preceq (\alpha^n (e - \alpha)^{-1})(D^x_1 + D^y_1 + D^z_1), \end{aligned}$$

where $\alpha = j + k + l$. Since $r(\alpha) = r(j + k + l) < 1$, by Remark 2.1 in [20], we get $||\alpha^n|| \to 0$, which together with Lemma 2.5, Lemma 2.9 and Lemma 2.10 shows that $\{g(x_n)\}$ is a Cauchy sequence. Similarly one can verify that $\{g(y_n)\}$ and $\{g(z_n)\}$ are Cauchy sequences too. Since X is complete and $g : X \to X$ is surjective, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} g(x_n) = g(x), \ \lim_{n\to\infty} g(y_n) = g(y) \text{ and } \lim_{n\to\infty} g(z_n) = g(z). \tag{3.7}$$

Finally, we prove $F(x, y, z) = g(x)$, $F(y, x, y) = g(y)$ and $F(z, y, x) = g(z)$. By using the g-continuity of F and noting (3.7), we have, as $n \to \infty$,

$$g(x_{n+1}) = F(x_n, y_n, z_n) \to F(x, y, z). \tag{3.8}$$

Thus $g(x) = F(x, y, z)$. Similarly, we have $g(y) = F(y, x, y)$ and $g(z) = F(z, y, x)$. So (x, y, z) is a tripled coincidence point of g and F. □

Next we replace the g-continuity of F by an additional property. We discuss this in the following theorem.

Theorem 3.6. *Let g be a surjection and F be a mapping with the mixed comparable property with respect to g and the following conditions be satisfied:*

(1) there exist $j, k, l \in \mathcal{A}$ with $r(j + k + l) < 1$ such that (3.1) is satisfied for any (x, y, z) and $(u, v, w) \in X \times X \times X$ for which $g(x)$ and $g(u)$, $g(y)$ and $g(v)$ and $g(z)$ and $g(w)$ are comparable;

(2) for any sequence $\{x_n\} \subset X$ satisfying that $g(x_n)$ and $g(x_{n+1})$ are comparable for all n and $g(x_n)$ converges to $g(x)$, we have that $g(x_n)$ and $g(x)$ are comparable for all n;

(3) there exist x_0, y_0, $z_0 \in X$ such that $g(x_0)$ and $F(x_0, y_0, z_0)$, $g(y_0)$ and $F(y_0, x_0, y_0)$ and $g(z_0)$ and $F(z_0, y_0, x_0)$ are comparable.

Then there exists a tripled coincidence point of g and F in $X \times X \times X$.

Proof. From the proof of Theorem 3.5, we get, for all $n > 0$, $g(x_n)$ and $g(x_{n+1})$ are comparable. The same argument holds for $\{g(y_n)\}$ and $\{g(z_n)\}$ and $g(x_n) \to g(x)$, $g(y_n) \to g(y)$ and $g(z_n) \to g(z)$ as $n \to \infty$. By the given condition (2), we obtain that $g(x_n)$ and $g(x)$ are comparable for all n. Similarly, $g(y_n)$ and $g(y)$ and $g(z_n)$ and $g(z)$ are comparable for all n. Next we only prove that $g(x) = F(x,y,z)$, $g(y) = F(y,x,y)$ and $g(z) = F(z,y,x)$.

For any $\theta \ll c$, since $g(x_n) \to g(x)$, $g(y_n) \to g(y)$ and $g(z_n) \to g(z)$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $d(g(x_n), g(x)) \ll \frac{c}{4}$, $\quad d(g(y_n), g(y)) \ll \frac{c}{4}$ and $d(g(z_n), g(z)) \ll \frac{c}{4}$. Taking $n > N$, we get

$$\begin{aligned} d(F(x,y,z), g(x)) &\preceq d(F(x,y,z), g(x_{n+1})) + d(g(x_{n+1}), g(x)) \\ &= d(F(x,y,z), F(x_n,y_n,z_n)) + d(g(x_{n+1}), g(x)) \\ &\preceq jd(g(x), g(x_n)) + kd(g(y), g(y_n)) + ld(g(z), g(z_n)) + d(g(x_{n+1}), g(x)) \\ &\ll (j+k+l+e)\,\frac{c}{4}. \end{aligned}$$

Lemma 2.6 together with Lemma 2.9 and Lemma 2.10 gives that $F(x,y,z) = g(x)$. Similarly, we can prove $F(y,x,y) = g(y)$ and $F(z,y,x) = g(z)$, that is, $(x,y,z) \in X \times X \times X$ is a tripled coincidence point of g and F. □

Example 3.2. Let $\mathcal{A} = \mathbb{R}^2$. For each $x = (x_1,x_2) \in \mathcal{A}$, let $\|x\| = |x_1| + |x_2|$. For $x = (x_1,x_2)$ and $y = (y_1,y_2) \in \mathcal{A}$, the multiplication is defined by

$$xy = (x_1,x_2)(y_1,y_2) = (x_1y_1 + x_2y_2, x_2y_1 + x_1y_2).$$

Then it is easy to verify that $\mathcal{A}$ is a Banach algebra with unit $e = (1,0)$. Let $X = \mathbb{R}^2$ and $P = \{(x_1,x_2) \in \mathbb{R}^2 | x_1 \geq 0,\ x_2 \geq 0\}$. Then P is a cone in $\mathcal{A}$. A metric d on X is defined by

$$d(x,y) = d((x_1,x_2),(y_1,y_2)) = (|x_1 - y_1|,\ |x_2 - y_2|) \in P.$$

Then (X,d) is a complete cone metric space over the Banach algebra $\mathcal{A}$.

A mapping $g : X \to X$ is defined as follows: $g(x) = (x_1, 2x_2)$ for each $x = (x_1,x_2) \in X$. Clearly, g is a surjection. Define $F : X \times X \times X \to X$ by $F(x,y,z) = (\frac{1}{8},\frac{1}{8})g(x) + (\frac{1}{16},\frac{1}{16})g(y) + (\frac{1}{12},\frac{1}{12})g(z) + (-2,4)$ for each $x = (x_1,x_2), y = (y_1,y_2)$ and $z = (z_1,z_2)$ in X. It is easy to check that all the conditions of Theorem 3.5 are satisfied for $j = \left(\frac{1}{8},\frac{1}{8}\right)$, $k = \left(\frac{1}{16},\frac{1}{16}\right)$ and $l = \left(\frac{1}{12},\frac{1}{12}\right)$. Moreover by a simple calculation, we can get that $((-2,2),(-2,2),(-2,2)) \in X \times X \times X$ is the tripled coincidence point of g and F.

4. Uniqueness of tripled coincidence point of g and F

In this section, we consider some additional conditions to ensure the uniqueness of the tripled coincidence point of g and F and appropriate conditions to ensure that for the tripled coincidence point (x,y,z) of g and F, we have $x = y = z$.

Theorem 4.1. *Let g be a bijection and F be a g-continuous mapping satisfying the mixed comparable property with respect to g. Assume that the following hold:*

(1) there exist $j,k,l \in \mathcal{A}$ with $r(j+2k+l) < 1$ for which

$$d(F(x,y,z), F(u,v,w)) \preceq jd(g(x),g(u)) + kd(g(y),g(v)) + ld(g(z),g(w)) \tag{4.1}$$

for any (x,y,z) and $(u,v,w) \in X \times X \times X$ satisfying that $g(x)$ and $g(u)$, $g(y)$ and $g(v)$ and $g(z)$ and $g(w)$ are comparable;

(2) there exist x_0, y_0, $z_0 \in X$ such that $g(x_0)$ and $F(x_0,y_0,z_0)$, $g(y_0)$ and $F(y_0,x_0,y_0)$ and $g(z_0)$ and $F(z_0,y_0,x_0)$ are comparable;

(3) *for every* $(a, b, c), (a_1, b_1, c_1) \in X \times X \times X$, *there exists* $(u, v, w) \in X \times X \times X$ *such that* $(g(u), g(v), g(w))$ *is comparable to* $(g(a), g(b), g(c))$ *and* $(g(a_1), g(b_1), g(c_1))$.

Then g *and* F *have a unique tripled coincidence point in* $X \times X \times X$.

Proof. By the proof of Theorem 3.5, we obtain that there exists $(x, y, z) \in X \times X \times X$ such that $g(x) = F(x, y, z)$, $g(y) = F(y, x, y)$ and $g(z) = F(z, y, x)$. If g and F have another tripled fixed point $(u, v, w) \in X \times X \times X$, then we can prove that $g(x) = g(u)$, $g(y) = g(v)$ and $g(z) = g(w)$ as follows.

In fact, by the conditions of Theorem 4.1, there exists $(r, s, t) \in X \times X \times X$ such that $g(r)$ is comparable to $g(x)$ and $g(u)$, $g(s)$ is comparable to $g(y)$ and $g(v)$, and $g(t)$ is comparable to $g(z)$ and $g(w)$. Let $r_0 = r, s_0 = s$ and $t_0 = t$. By a proof similar to Theorem 3.5, we can prove that there exist sequences $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ such that, for all $n \geq 0$,

$$g(r_{n+1}) = F(r_n, s_n, t_n), \quad g(s_{n+1}) = F(s_n, r_n, s_n) \text{ and } g(t_{n+1}) = F(t_n, s_n, r_n).$$

Since F has the mixed comparable property with respect to g and by induction, we can prove easily that $g(r_n)$ is comparable to $g(x)$ for all n. Similarly, $g(s_n)$ is comparable to $g(y)$ and $g(t_n)$ is comparable to $g(z)$ for all n. Set $R_{n+1} = d(g(r_{n+1}), g(x))$, $S_{n+1} = d(g(s_{n+1}), g(y))$ and $T_{n+1} = d(g(t_{n+1}), g(z))$. Then from the given conditions of Theorem 4.1, we have

$$R_{n+1} = d(g(r_{n+1}), g(x)) = d(F(r_n, s_n, t_n), F(x, y, z)) \preceq jR_n + kS_n + lT_n. \tag{4.2}$$

$$S_{n+1} = d(g(s_{n+1}), g(y)) = d(F(s_n, r_n, s_n), F(y, z, y)) \preceq jS_n + k \cdot R_n + lS_n. \tag{4.3}$$

$$T_{n+1} = d(g(t_{n+1}), g(z)) = d(F(t_n, s_n, r_n), F(z, y, x)) \preceq jT_n + kS_n + lR_n. \tag{4.4}$$

From (4.2), (4.3) and (4.4), we obtain that, for all n,

$$R_{n+1} + S_{n+1} + T_{n+1} \preceq \alpha(R_n + S_n + T_n), \tag{4.5}$$

where $\alpha = j + 2k + l$. Hence,

$$\begin{aligned} R_{n+1} + S_{n+1} + T_{n+1} \preceq \alpha(R_n + S_n + T_n) &\preceq \alpha^2(R_{n-1} + S_{n-1} + T_{n-1}) \\ &\preceq \cdots \preceq \alpha^n(R_1 + S_1 + T_1). \end{aligned}$$

Noting that $r(j+2k+l) < 1$ and by Lemma 2.7, Lemma 2.5 and Remark 2.1 in [20], we have $\lim_{n\to\infty} g(r_n) = g(x)$, $\lim_{n\to\infty} g(s_n) = g(y)$ and $\lim_{n\to\infty} g(t_n) = g(z)$. Similarly, we can prove that $\lim_{n\to\infty} g(r_n) = g(u)$, $\lim_{n\to\infty} g(s_n) = g(v)$ and $\lim_{n\to\infty} g(t_n) = g(w)$. So $g(x) = g(u)$, $g(y) = g(v)$ and $g(z) = g(w)$. The injective property of g implies that $x = u$, $y = v$ and $z = w$. Hence g and F have a unique tripled coincidence point. □

Theorem 4.2. *Let* g *be a bijection and* F *be a mapping having the mixed comparable property with respect to* g *and the following conditions be satisfied:*

(1) *there exist* $j, k, l \in \mathcal{A}$ *with* $r(j + 2k + l) < 1$ *such that* (3.1) *is satisfied for any* (x, y, z) *and* $(u, v, w) \in X \times X \times X$ *for which* $g(x)$ *and* $g(u)$, $g(y)$ *and* $g(v)$ *and* $g(z)$ *and* $g(w)$ *are comparable;*

(2) *for any sequence* $\{x_n\} \subset X$ *satisfying that* $g(x_n)$ *and* $g(x_{n+1})$ *are comparable for all* n *and* $g(x_n)$ *converges to* $g(x)$, *we have that* $g(x_n)$ *and* $g(x)$ *are comparable for all* n;

(3) *there exist* x_0, y_0, $z_0 \in X$ *such that* $g(x_0)$ *and* $F(x_0, y_0, z_0)$, $g(y_0)$ *and* $F(y_0, x_0, y_0)$ *and* $g(z_0)$ *and* $F(z_0, y_0, x_0)$ *are comparable;*

(4) *for any* (a, b, c) *and* $(a_1, b_1, c_1) \in X \times X \times X$, *there exists* $(u, v, w) \in X \times X \times X$ *such that* $(g(u), g(v), g(w))$ *is comparable to* $(g(a), g(b), g(c))$ *and* $(g(a_1), g(b_1), g(c_1))$.

Then g and F have a unique tripled coincidence point in $X \times X \times X$.

Proof. The proof is similar to that of Theorem 4.1, so we omit it. □

Theorem 4.3. *Suppose the hypothesis of* Theorem 3.5 *(resp.* Theorem 3.6*) and the following conditions are satisfied: g is an injection and for the elements x, y and z appearing in the proof of* Theorem 3.5 *(resp.* Theorem 3.6*), $g(x)$, $g(y)$ and $g(z)$ are mutually comparable. Then $x = y = z$, that is, x is a coincidence point of g and F.*

Proof. Suppose $g(x)$, $g(y)$ and $g(z)$ are mutually comparable and g is an injection. By the mixed comparable property with respect to g of F, we have

$$d(g(x), g(z)) = d(F(x, y, z), F(z, y, x)) \preceq (j + l)d(g(x), g(z)). \tag{4.6}$$

Lemma 2.7 gives that $r(j+l) \leq r(k+j+l) < 1$, so by Lemma 2.8, we have $g(x) = g(z)$. As g is an injection, we get $x = z$. By the mixed comparable property with respect to g of F again, we obtain

$$d(g(x), g(y)) = d(F(x, y, z), F(y, x, y)) \preceq jd(g(x), g(y)) + kd(g(y), g(x)) + ld(g(z), g(y)). \tag{4.7}$$

Noting $x = z$ and by (4.7), we have

$$d(g(x), g(y)) = d(F(x, y, z), F(y, x, y)) \preceq (j + k + l)d(g(x), g(y)).$$

Since $r(k+j+l) < 1$, by Lemma 2.8 again, we get $g(x) = g(y)$ which implies $x = y$ by the injective property of g. Thus $x = y = z$, that is, x is a coincidence point of g and F. □

Corollary 4.4. *By adding to the hypothesis of* Theorem 3.5 *(resp.* Theorem 3.6*) the condition: X is a totally ordering set, then g and F have a unique coincidence point, that is, there exists unique $x \in X$ such that $g(x) = F(x, x, x)$.*

Proof. According to Theorem 4.3, it suffices to prove the uniqueness of the coincidence point of g and F. Suppose on the contrary that there exist two elements x and x_1 in X such that $g(x) = F(x, x, x)$ and $g(x_1) = F(x_1, x_1, x_1)$. Since X is a totally ordering set, $g(x)$ and $g(x_1)$ are comparable. By the mixed comparable property with respect to g of F, we have

$$d(g(x), g(x_1)) = d(F(x, x, x), F(x_1, x_1, x_1)) \preceq (j + k + l)d(g(x), g(x_1)). \tag{4.8}$$

Noting $r(j + k + l) < 1$, by Lemma 2.8, we get $g(x) = g(x_1)$ which together with g is an injection implies $x = x_1$. □

Example 4.1. Let $\mathcal{A} = \mathbb{R}^3$. For each $x = (x_1, x_2, x_3) \in \mathcal{A}$, let $\|x\| = |x_1| + |x_2| + |x_3|$. For $x = (x_1, x_2, x_3),\ y = (y_1, y_2, y_3) \in \mathcal{A}$, the multiplication is defined by

$$xy = (x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1y_1 + x_2y_3 + x_3y_2, x_1y_2 + x_2y_1 + x_3y_3, x_1y_3 + x_2y_2 + x_3y_1).$$

Then one can easily verify that $\mathcal{A}$ is a Banach algebra with unit $e = (1, 0, 0)$. Set $X = \mathbb{R}^3$ and $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}$. Obviously, P is a cone in $\mathcal{A}$. A metric d on X is defined by

$$d(x, y) = d((x_1, x_2, x_3), (y_1, y_2, y_3)) = (|x_1 - y_1|,\ |x_2 - y_2|,\ |x_3 - y_3|) \in P.$$

Under the metric d, (X, d) is a complete cone metric space over the Banach algebra $\mathcal{A}$.

A mapping $g : X \to X$ is defined as follows: $g(x) = (x_1, 2x_2, 3x_3)$ for each $x = (x_1, x_2, x_3) \in X$. Then g is a surjection. Define $F : X \times X \times X \to X$ by

$$F(x, y, z) = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right) g(x) + \left(\frac{1}{18}, \frac{1}{18}, \frac{1}{18}\right) g(y) + \left(\frac{1}{24}, \frac{1}{24}, \frac{1}{24}\right) g(z) + (-2, 4, 1)$$

for each $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ in X. It is easy to check that all the conditions of Theorem 4.1 are satisfied for $j = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}\right)$, $k = \left(\frac{1}{18}, \frac{1}{18}, \frac{1}{18}\right)$ and $l = \left(\frac{1}{24}, \frac{1}{24}, \frac{1}{24}\right)$. By Theorem 4.1, there exists a unique tripled coincidence point of g and F in $X \times X \times X$.

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