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Oscillation properties for solutions of a kind of partial fractional differential equations with damping term

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Abstract

The aim of the present paper is to obtain sufficient conditions for oscillation of solutions of partial fractional differential equations with the damping term of the form

$$D_{+,t}^{1+\alpha}u(x,t) + p(t)D_{+,t}^{\alpha}u(x,t) = a(t)\Delta u(x,t) + \sum_{i=1}^{m} a_i(t)\Delta u(x, t-\tau_i)$$
$$- q(x,t)\int_0^t (t-\xi)^{-\alpha}u(x,\xi)d\xi, \ (x,t) \in \Omega \times \mathbb{R}_+ \equiv G.$$

Two examples are given to illustrate the main results. ©2016 All rights reserved.

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1. Introduction

Fractional differential equations are generalizations of classical differential equations to an arbitrary order and have been widely applied in various areas of engineering, physics, mechanics, chemistry, aerodynamics, finance, applied mathematics, bioengineering, nonlinear control and so on. There has been a significant

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development in fractional differential equations in recent years. For example, see the monographs [1, 4, 7, 9, 14].

Recently, the research on the theory of partial fractional differential equations is a very interesting topic and some results are established. Jafari et al.[6] presented the iterative Laplace transform method to derive exact and approximate analytical solutions of fractional partial differential equations. Chen and Jiang[2] exploited the techniques of Lie group analysis to solve n–order linear fractional partial differential equation and nonlinear fractional reaction diffusion convection equation. Zheng [12] applied the (G'/G) method to seek exact solutions for several fractional partial differential equations including the space–time fractional $(2+1)$ dimensional dispersive long wave equations, the $(2+1)$–dimensional space time fractional Nizhnik–Novikov–Veselov system and the time fractional fifth–order Sawada–Kotera equation. Based on a fractional complex transformation, Zheng and Feng[13] obtained Jacobi elliptic function solutions to fractional partial differential equations.

At the same time, a few papers [5, 8, 10, 11] studied the oscillation of fractional partial differential equations which involve the Riemann–Liouville fractional partial derivative. In [11], Prakash et al. considered the oscillation of the fractional partial differential equation

$$\frac{\partial}{\partial t}(r(t)D_{+,t}^\alpha u(x,t)) + q(x,t)f\left(\int_0^t (t-\nu)^{-\alpha}u(x,\nu)d\nu\right) = a(t)\Delta u(x,t), \tag{1.1}$$

where $(x,t) \in \Omega \times \mathbb{R}_+ \equiv G$.

Harikrishnan et al.[5] studied the oscillatory behavior of the fractional partial differential equation of the form

$$D_{+,t}^\alpha\left(r(t)D_{+,t}^\alpha u(x,t)\right) + q(x,t)f(u(x,t)) = a(t)\Delta u(x,t) + g(x,t), \ (x,t) \in G. \tag{1.2}$$

Prakash et al.[10] established the oscillation of a nonlinear fractional partial differential equation with damping and forced term of the form

$$D_{+,t}^\alpha(r(t)D_{+,t}^\alpha u(x,t)) + p(t)D_{+,t}^\alpha u(x,t) + q(x,t)f(u(x,t)) = a(t)\Delta u(x,t) + g(x,t), \ (x,t) \in G. \tag{1.3}$$

In [8], Li investigated the forced oscillation of fractional partial differential equations with the damping term of the form

$$\frac{\partial}{\partial t}\left(D_{+,t}^\alpha u(x,t)\right) + p(t)D_{+,t}^\alpha u(x,t) = a(t)\Delta u(x,t) - q(x,t)u(x,t) + f(x,t), \ (x,t) \in G. \tag{1.4}$$

2. Formulation of the Problems

In this paper, we are concerned with establishing criteria for oscillation of solutions of partial fractional differential equations with the damping term of the form

$$D_{+,t}^{1+\alpha}u(x,t) + p(t)D_{+,t}^\alpha u(x,t) = a(t)\Delta u(x,t) + \sum_{i=1}^m a_i(t)\Delta u(x,t-\tau_i)$$
$$-q(x,t)\int_0^t (t-\xi)^{-\alpha}u(x,\xi)d\xi, \ (x,t) \in \Omega \times \mathbb{R}_+ \equiv G, \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\Omega$, $\alpha \in (0,1)$ is a constant and $D_{+,t}^\alpha u(x,t)$ is the Riemann–Liouville fractional derivative of order α of u with respect to t, Δ is the Laplacian in $\mathbb{R}^n$.

We assume throughout this paper that

(A1) $a \in C([0,\infty);[0,\infty))$, $p \in C([0,\infty);[0,\infty))$, $a_i \in C([0,\infty);[0,\infty))$ and $\tau_i \geq 0$ is a constant, $i = 1, 2, \cdots, m$;

(A2) $q \in C(\overline{G};\mathbb{R}_+)$ and $q(t) = \min_{x\in\overline{\Omega}} q(x,t)$.

Consider one of the two following boundary conditions

$$\frac{\partial u(x,t)}{\partial N} + g(x,t)u(x,t) = 0, \ (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{2.2}$$

or

$$u(x,t) = 0, \ (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{2.3}$$

where N is the unit exterior normal vector to $\partial\Omega$ and $g(x,t)$ is a nonnegative continuous function on $\partial\Omega \times \mathbb{R}_+$.

By a solution of the problem (2.1)–(2.2)(or (2.1)–(2.3)), we mean a function $u(x,t)$ which satisfies (2.1) on $\overline{G}$ and the boundary condition (2.2)(or (2.3)).

A solution $u(x,t)$ of the problem (2.1)–(2.2)(or (2.1)–(2.3)) is said to be oscillatory in G if it is neither eventually positive nor eventually negative, otherwise it is non-oscillatory.

Definition 2.1 ([7]). The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \to \mathbb{R}$ on the half–axis $\mathbb{R}_+$ is given by

$$I_+^\alpha y(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} y(\xi) d\xi \quad \text{for} \quad t > 0 \tag{2.4}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where Γ is the gamma function.

Definition 2.2 ([7]). The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $y : \mathbb{R}_+ \to \mathbb{R}$ on the half–axis $\mathbb{R}_+$ is given by

$$\begin{aligned} D_+^\alpha y(t) &:= \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \left(I_+^{\lceil\alpha\rceil-\alpha} y \right)(t) \\ &= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)} \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}} \int_0^t (t-\xi)^{\lceil\alpha\rceil-\alpha-1} y(\xi) d\xi \quad \text{for} \quad t > 0 \end{aligned} \tag{2.5}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$, where $\lceil\alpha\rceil$ is the ceiling function of α.

Definition 2.3 ([7]). The Riemann–Liouville partial fractional derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x,t)$ is given by

$$D_{+,t}^\alpha u(x,t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\xi)^{-\alpha} u(x,\xi) d\xi \tag{2.6}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$.

The following lemmas are very useful in the proof of our main results.

Lemma 2.4 ([7]). *Let $\alpha \geq 0$, $m \in \mathbb{N}$ and $D = \frac{d}{dt}$. If the fractional derivatives $D_+^\alpha y(t)$ and $D_+^{\alpha+m} y(t)$ exist, then*

$$D^m(D_+^\alpha y(t)) = D_+^{\alpha+m} y(t). \tag{2.7}$$

Lemma 2.5 ([11]). *Let*

$$\widetilde{E}(t) =: \int_0^t (t-\xi)^{-\alpha} y(\xi) d\xi \quad \text{for} \quad \alpha \in (0,1) \quad \text{and} \quad t > 0. \tag{2.8}$$

Then $\widetilde{E}'(t) = \Gamma(1-\alpha) D_+^\alpha y(t)$.

3. Oscillation of the Problem (2.1)–(2.2)

Theorem 3.1. *Assume that for some $t_0 > 0$,*

$$\int_{t_0}^{\infty} \exp\left(-\int_{t_0}^{t} p(\xi)d\xi\right)dt = \infty \tag{3.1}$$

and

$$\int_{t_0}^{\infty} q(s)v(s)ds = \infty, \tag{3.2}$$

where

$$v(t) = \exp\left(\int_{t_0}^{t} p(s)ds\right), \ t \geq t_0. \tag{3.3}$$

Then every solution of the problem (2.1)–(2.2) is oscillatory in G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ to the problem (2.1)–(2.2) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality, we may assume that $u(x,t) > 0$ and $u(x, t - \tau_i) > 0$ in $\Omega \times [t_1, \infty)$ for $t_1 \geq t_0, i = 1, 2, \cdots, m$.

Integrating (2.1) with respect to x over the domain Ω, we have

$$\int_{\Omega} D_{+,t}^{1+\alpha} u(x,t)dx + p(t) \int_{\Omega} D_{+,t}^{\alpha} u(x,t)dx = a(t) \int_{\Omega} \Delta u(x,t)dx + \sum_{i=1}^{m} \int_{\Omega} a_i(t)\Delta u(x, t - \tau_i)dx$$
$$- \int_{\Omega} q(x,t)\left(\int_0^t (t-\xi)^{-\alpha} u(x,\xi)d\xi\right)dx, \ t \geq t_1. \tag{3.4}$$

Green's formula and (2.2) yield

$$\int_{\Omega} \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u(x,t)}{\partial N}dS = -\int_{\partial\Omega} g(x,t)u(x,t)dS \leq 0, \ \ t \geq t_1 \tag{3.5}$$

and

$$\int_{\Omega} \Delta u(x, t - \tau_i)dx = \int_{\partial\Omega} \frac{\partial u(x, t - \tau_i)}{\partial N}dS$$
$$= -\int_{\partial\Omega} g(x, t - \tau_i)u(x, t - \tau_i)dS \leq 0, \ t \geq t_1, \ i = 1, 2, \cdots, m, \tag{3.6}$$

where dS is the surface element on $\partial\Omega$. Noting the assumption (A2), we have

$$\int_{\Omega} q(x,t)\left(\int_0^t (t-\xi)^{-\alpha} u(x,\xi)d\xi\right)dx \geq q(t) \int_{\Omega}\left(\int_0^t (t-\xi)^{-\alpha} u(x,\xi)d\xi\right)dx$$
$$= q(t) \int_0^t (t-\xi)^{-\alpha}\left(\int_{\Omega} u(x,\xi)dx\right)d\xi, \ \ t \geq t_1. \tag{3.7}$$

Let

$$V(t) = \int_{\Omega} u(x,t)dx.$$

Combining (3.4)–(3.7), we obtain

$$D_{+}^{1+\alpha}V(t) + p(t)D_{+}^{\alpha}V(t) \leq -q(t)E(t), \ \ t \geq t_1, \tag{3.8}$$

where

$$E(t) = \int_0^t (t-\xi)^{-\alpha}V(\xi)d\xi > 0. \tag{3.9}$$

Using Lemma 2.4, it follows from (3.3), (3.8) and (3.9) that

$$\left((D_+^\alpha V(t))v(t)\right)' = (D_+^{1+\alpha}V(t))v(t) + (D_+^\alpha V(t))p(t)v(t)$$
$$\leq -q(t)E(t)v(t) < 0, \ t \geq t_1. \tag{3.10}$$

Hence, $(D_+^\alpha V(t))v(t)$ is strictly decreasing in $[t_1, \infty)$ and we can claim that $D_+^\alpha V(t) > 0$ for $t \geq t_1$. In fact, if $D_+^\alpha V(t) \leq 0$ for $t \geq t_1$, then there exists a $T \geq t_1$ such that

$$(D_+^\alpha V(t))v(t) < (D_+^\alpha V(T))v(T) = C_1 < 0, \ t > T. \tag{3.11}$$

By Lemma 2.5, from (3.11) we have

$$\frac{E'(t)}{\Gamma(1-\alpha)} = D_+^\alpha V(t) < \frac{C_1}{v(t)} = C_1 \exp\left(-\int_{t_0}^t p(s)ds\right), \ t \geq T. \tag{3.12}$$

Integrating (3.12) from T to t, we obtain

$$E(t) < E(T) + \Gamma(1-\alpha)C_1 \int_T^t \exp\left(-\int_{t_0}^t p(\xi)d\xi\right)ds, \ t > T. \tag{3.13}$$

Letting $t \to \infty$ in (3.13), we have $\lim_{t\to\infty} E(t) = -\infty$, which contradicts with the fact $E(t) > 0$.

Define the function

$$w(t) = \frac{(D_+^\alpha V(t))v(t)}{E(t)}, \ t \geq t_1. \tag{3.14}$$

Then $w(t) > 0$ and

$$w'(t) = \frac{[(D_+^\alpha V(t))v(t)]'}{E(t)} - \frac{(D_+^\alpha V(t))v(t)E'(t)}{E^2(t)}$$
$$= \frac{(D_+^{1+\alpha}V(t))v(t) + (D_+^\alpha V(t))p(t)v(t)}{E(t)} - \frac{\Gamma(1-\alpha)(D_+^\alpha V(t))^2 v(t)}{E^2(t)} \tag{3.15}$$
$$\leq -q(t)v(t), \ t \geq t_1.$$

Integrating (3.15) from t_1 to t, we obtain

$$w(t) \leq w(t_1) - \int_{t_1}^t q(s)v(s)ds. \tag{3.16}$$

Letting $t \to \infty$ in (3.16) and noting the assumption (3.2), we can obtain a contradiction with $w(t) > 0$. The proof of Theorem 3.1 is complete. $\square$

Theorem 3.2. *Assume that* (3.1) *holds and*

$$\int_{t_0}^\infty \left[q(s) - \frac{p^2(s)}{4\Gamma(1-\alpha)}\right]ds = \infty, \ t_0 > 0. \tag{3.17}$$

Then every solution of the problem (2.1)–(2.2) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ to the problem (2.1)–(2.2) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality, we may assume that $u(x,t) > 0$ and $u(x, t - \tau_i) > 0$ in $\Omega \times [t_1, \infty)$ for $t_1 \geq t_0, i = 1, 2, \cdots, m$.

As in the proof of Theorem 3.1, we obtain that (3.7) and $D_+^\alpha V(t) > 0$ for $t \geq t_1$ hold. Defined the function

$$\widetilde{w}(t) = \frac{D_+^\alpha V(t)}{E(t)}, \ t \geq t_1. \tag{3.18}$$

Then $\widetilde{w}(t) > 0$ and

$$
\begin{aligned}
\widetilde{w}'(t) &= \frac{D_+^{1+\alpha}V(t)}{E(t)} - \frac{E'(t)D_+^{\alpha}V(t)}{E^2(t)} \\
&\leq \frac{-p(t)D_+^{\alpha}V(t) - q(t)E(t)}{E(t)} - \frac{\Gamma(1-\alpha)(D_+^{\alpha}V(t))^2}{E^2(t)} \\
&= -q(t) - p(t)\widetilde{w}(t) - \Gamma(1-\alpha)\widetilde{w}^2(t) \\
&= -q(t) - \left[\sqrt{\Gamma(1-\alpha)}\widetilde{w}(t) + \frac{p(t)}{2\sqrt{\Gamma(1-\alpha)}}\right]^2 + \frac{p^2(t)}{4\Gamma(1-\alpha)} \\
&\leq -\left[q(t) - \frac{p^2(t)}{4\Gamma(1-\alpha)}\right], \; t \geq t_1.
\end{aligned}
\tag{3.19}
$$

Integrating (3.19) from t_1 to t, we obtain

$$
\widetilde{w}(t) \leq \widetilde{w}(t_1) - \int_{t_1}^{t}\left[q(s) - \frac{p^2(s)}{4\Gamma(1-\alpha)}\right]ds.
\tag{3.20}
$$

Letting $t \to \infty$ in (3.20), we can obtain a contradiction with $\widetilde{w}(t) > 0$. This completes the proof of Theorem 3.2. $\qquad\square$

4. Oscillation of the Problem (2.1)–(2.3)

The following fact [3] will be used.

The smallest eigenvalue β_0 of the Dirichlet problem

$$
\begin{cases}
\Delta\omega(x) + \beta\omega(x) = 0, \text{ in } \Omega, \\
\omega(x) = 0, \text{ on } \partial\Omega,
\end{cases}
$$

is positive and the corresponding eigenfunction $\varphi(x)$ is positive in Ω.

Theorem 4.1. *Assume that the conditions of* Theorem 3.1 *hold. Then every solution of the problem* (2.1)–(2.3) *is oscillatory in* G.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ to the problem (2.1)–(2.3) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. Without loss of generality, we may assume that $u(x,t) > 0$ and $u(x, t - \tau_i) > 0$ in $\Omega \times [t_1, \infty)$ for $t_1 \geq t_0, i = 1, 2, \cdots, m$.

Multiplying both sides of (2.1) by $\varphi(x)$ and integrating with respect to x over the domain Ω, we have

$$
\begin{aligned}
\int_{\Omega} D_{+,t}^{1+\alpha}u(x,t)\varphi(x)dx &+ p(t)\int_{\Omega}D_{+,t}^{\alpha}u(x,t)\varphi(x)dx \\
&= a(t)\int_{\Omega}\Delta u(x,t)\varphi(x)dx + \sum_{i=1}^{m}\int_{\Omega}a_i(t)\Delta u(x, t-\tau_i)\varphi(x)dx \\
&\quad - \int_{\Omega}q(x,t)\left(\int_0^t (t-\xi)^{-\alpha}u(x,\xi)d\xi\right)\varphi(x)dx, \quad t \geq t_1.
\end{aligned}
\tag{4.1}
$$

Green's formula and (2.3) yield

$$
\int_{\Omega}\Delta u(x,t)\varphi(x)dx = \int_{\Omega}u(x,t)\Delta\varphi(x)dx = -\beta_0\int_{\Omega}u(x,t)\varphi(x)dx \leq 0, \quad t \geq t_1,
\tag{4.2}
$$

and

$$
\begin{aligned}
\int_{\Omega}\Delta u(x, t-\tau_i)\varphi(x)dx &= \int_{\Omega}u(x, t-\tau_i)\Delta\varphi(x)dx \\
&= -\beta_0\int_{\Omega}u(x, t-\tau_i)\varphi(x)dx \leq 0, \quad t \geq t_1, \; i = 1, 2, \cdots, m.
\end{aligned}
\tag{4.3}
$$

It follows from the assumption (A2) that

$$\int_\Omega q(x,t)\Big(\int_0^t (t-\xi)^{-\alpha}u(x,\xi)d\xi\Big)\varphi(x)dx \geq q(t)\int_\Omega\Big(\int_0^t (t-\xi)^{-\alpha}u(x,\xi)d\xi\Big)\varphi(x)dx$$
$$= q(t)\int_0^t (t-\xi)^{-\alpha}\Big(\int_\Omega u(x,\xi)\varphi(x)dx\Big)d\xi, \quad t \geq t_1. \tag{4.4}$$

Let

$$U(t) = \int_\Omega u(x,t)\varphi(x)dx, \ t \geq t_1.$$

Combining (4.1)–(4.4), we have

$$D_+^{1+\alpha}U(t) + p(t)D_+^\alpha U(t) \leq -q(t)\overline{E}(t), \quad t \geq t_1, \tag{4.5}$$

where

$$\overline{E}(t) = \int_0^t (t-\xi)^{-\alpha}U(\xi)d\xi > 0. \tag{4.6}$$

Using Lemma 2.4, it follows from (4.5) and (4.6) that

$$\Big((D_+^\alpha U(t))v(t)\Big)' = (D_+^{1+\alpha}U(t))v(t) + (D_+^\alpha U(t))p(t)v(t)$$
$$\leq -q(t)\overline{E}(t)v(t) < 0, \ t \geq t_1. \tag{4.7}$$

Hence, $(D_+^\alpha U(t))v(t)$ is strictly decreasing in $[t_1, \infty)$ and $D_+^\alpha U(t) > 0$ for $t \geq t_1$.

The remainder of the proof is similar to that of Theorem 3.1 and we omit it here. The proof of Theorem 4.1 is complete. □

It is not difficult to prove the following theorem.

Theorem 4.2. *Assume that the conditions of* Theorem 3.2 *hold. Then every solution of the problem* (2.1)–(2.3) *is oscillatory in* G.

5. Examples

In this section, we give two examples to illustrate our main results.

Example 5.1. Consider the following partial fractional differential equation

$$D_+^{\frac{5}{4}}u(x,t) + \frac{1}{t}D_{+,t}^{\frac{1}{4}}u(x,t) = \frac{1}{8}e^t\Delta u(x,t) + 2t\Delta u(x,t-1)$$
$$- \Big(x^3 + \frac{1}{t^2}\Big)\int_0^t (t-\xi)^{-\frac{1}{4}}u(x,\xi)d\xi, \ (x,t) \in (0,\pi) \times \mathbb{R}_+, \tag{5.1}$$

with the boundary condition

$$u_x(0,t) = u_x(\pi,t) = 0, \ t > 0. \tag{5.2}$$

Here $\alpha = \frac{1}{4}, \Omega = (0,\pi), n = 1, m = 1, a_1(t) = 2t, \tau_1 = 1, p(t) = \frac{1}{t}, a(t) = \frac{1}{8}e^t$ and $q(x,t) = x^3 + \frac{1}{t^2}$. Hence $q(t) = \frac{1}{t^2}$. It is easy to see that

$$\int_{t_0}^\infty \exp\Big(-\int_{t_0}^t p(\xi)d\xi\Big)dt = \int_{t_0}^\infty \exp\Big(-\int_{t_0}^t \frac{1}{\xi}d\xi\Big)dt = \int_{t_0}^\infty \frac{t_0}{t}dt = \infty, \tag{5.3}$$

$$v(t) = \exp\Big(\int_{t_0}^t p(\xi)d\xi\Big) = \exp\Big(\int_{t_0}^t \frac{1}{\xi}d\xi\Big) = \frac{t}{t_0}, \tag{5.4}$$

and

$$\int_{t_0}^{\infty} q(s)v(s) = \int_{t_0}^{\infty} \frac{1}{s^2}\frac{s}{t_0}ds = \int_{t_0}^{\infty} \frac{1}{t_0 s}ds = \infty, \ t_0 > 0. \tag{5.5}$$

Therefore, the conditions of Theorem 3.1 are fulfilled. Then every solution of the problem (5.1)–(5.2) oscillates in $(0, \pi) \times \mathbb{R}_+$.

Example 5.2. Consider the following partial fractional differential equation

$$D_+^{\frac{6}{5}}u(x,t) + \frac{1}{t}D_{+,t}^{\frac{1}{5}}u(x,t) = 3e^{-t}\Delta u(x,t) + \frac{t}{3}\Delta(x, t - \frac{1}{3})$$
$$- \left(2x^2 + \frac{1}{t}\right)\int_0^t (t - \xi)^{-\frac{1}{5}}u(x,\xi)d\xi, \ (x,t) \in (0, \pi) \times \mathbb{R}_+, \tag{5.6}$$

with the boundary condition

$$u(0,t) = u(\pi, t) = 0, \ t > 0. \tag{5.7}$$

Here $\alpha = \frac{1}{5}, \Omega = (0, \pi), n = 1, m = 1, p(t) = \frac{1}{t}, a(t) = 3e^{-t}, a_1(t) = \frac{t}{3}, \tau_1 = \frac{1}{3}$ and $q(x,t) = 2x^2 + \frac{1}{t}$. Hence $q(t) = \frac{1}{t}$.

We easily see that

$$\int_{t_0}^{\infty} \exp\left(-\int_{t_0}^{t} p(\xi)d\xi\right)dt = \int_{t_0}^{\infty} \exp\left(-\int_{t_0}^{t}\frac{1}{\xi}d\xi\right)dt = \int_{t_0}^{\infty}\frac{t_0}{t}dt = \infty \tag{5.8}$$

and

$$\int_{t_0}^{\infty}\left[q(s) - \frac{p^2(s)}{4\Gamma(\frac{4}{5})}\right]ds = \int_{t_0}^{\infty}\left[\frac{1}{s} - \frac{1}{4\Gamma(\frac{4}{5})s^2}\right]ds = \infty, \ t_0 > 0. \tag{5.9}$$

Therefore, the conditions of Theorem 4.2 are satisfied. Then every solution of the problem (5.6)–(5.7) oscillates in $(0, \pi) \times \mathbb{R}_+$.

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