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Existence results to certain functional equations in probabilistic Banach spaces with an application to integral equations

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Abstract

We consider some classes of functional equations posed in PB-spaces, for which we establish existence and uniqueness of solutions that belong to a cone. An application to integral equations is presented.

Keywords: Functional equation, PB-space, normal cone, partial order, integral equation.
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1. Introduction

Many problems in physics can be described via operator equations posed in a certain space. It is very important to know if such equations admit solutions and if a solution exists, it is unique. Many authors studied such questions and considered various classes of operator equations posed in a Banach space (see, for examples, [1, 2, 3, 7, 9, 12, 13, 14, 15, 16]). In [8], we studied a class of operator equations posed in a probabilistic Banach space and involving decreasing and convex operators. In this contribution, we continue our study for other classes of functional equations. Let us start by recalling some basic concepts and results about probabilistic Banach spaces that will be used through this paper. For more details, the reader is invited to consult [3, 4, 5, 6, 7, 8, 10, 11].

We denote by $\mathbb{D}$ the set of distribution mappings. A probabilistic norm on a $\mathbb{R}$-vector space $\mathbb{V}$ w.r.t a T-norm $\mathbb{T}$ is a mapping $N : \mathbb{V} \to \mathbb{D}$ having the following properties:

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(N1) $N(v)(0) = 0$ for every $v \in \mathbb{V}$;
(N2) $N(v)(\lambda) = 1$ for all $\lambda > 0$ iff $v = 0$;
(N3) $N(\delta v)(\lambda) = N(v)\left(\frac{\lambda}{|\delta|}\right)$ for all $v \in \mathbb{V}$ and $\delta \in \mathbb{R}$, $\delta \neq 0$;
(N4) $N(u+v)(\lambda+\mu) \geq \mathbb{T}(N(u)(\lambda), N(v)(\mu))$ for all $u, v \in \mathbb{V}$ and $\lambda, \mu \geq 0$.
In this case, $(\mathbb{V}, N, \mathbb{T})$ is a probabilistic normed space (shortly PN-space).

For topological concepts in PN-spaces, see [8]. A probabilistic Banach space (shortly PB-space) is a PN-space, for which every Cauchy sequence is convergent.

Let $(\mathbb{V}, N, \mathbb{T})$ be a PB-space and Q be a convex subset of $\mathbb{V}$ such that $Q \neq \emptyset$. Under the conditions:
(Q1) $\overline{Q} = Q$;
(Q2) $qQ \subseteq Q$ for every $q \geq 0$;
(Q3) $-Q \cap Q = \{0\}$,
we say that Q is a cone in PB. In this case, the binary relation $\preceq$ defined by

$$z, w \in \mathbb{V}, \quad z \preceq w \Longleftrightarrow w - z \in Q$$

is a partial order in $\mathbb{V}$. For a pair (u, v) that belongs to $\mathbb{V} \times \mathbb{V}$, we use the notation $u \prec v$ to indicate that $v - u \in Q$ and $u \neq v$. In this case, the interval $[u, v]$ is the set of elements $z \in \mathbb{V}$ such that $z - u \in Q$ and $v - z \in Q$. For a given $\eta \in Q$, we denote by Q_η the set of elements $v \in \mathbb{V}$ such that $v - \lambda\eta \in Q$ and $\mu\eta - v \in Q$ for some $\lambda, \mu > 0$. The cone Q is normal if there is some constant $\rho > 0$ such that

$$(u, v) \in \mathbb{V} \times \mathbb{V},\ 0 \preceq u \preceq v \Longrightarrow N(u)(x) \geq N(v)\left(\frac{x}{\rho}\right),\ x \in \mathbb{R}.$$

Now, we are able to present and establish our obtained results.

2. Main result and consequences

In the sequel, $(\mathbb{V}, N, \mathbb{T})$ denotes a PB-space with a normal cone Q. Let $\zeta : (\alpha, \beta) \to (0, 1)$ be a surjective function and $\xi : (\alpha, \beta) \times Q \times Q \to (0, \infty)$ be a function having the following properties:
(ξ_1) $\xi(x, v, w) \in (\zeta(x), 1)$ for all $x \in (\alpha, \beta)$, $(v, w) \in Q \times Q$;
(ξ_2) for every $x \in (\alpha, \beta)$, $\xi(x, v, w)$ is increasing in v (w.r.t $\preceq$) for w fixed and decreasing in w for v fixed.
We denote by $\mathcal{F}$ the set of operators $F : Q \times Q \to Q$ such that
(F1) F is mixed monotone;
(F2) for all $(x, v, w) \in (\alpha, \beta) \times Q \times Q$, we have

$$F(\zeta(x)v, [\zeta(x)]^{-1}w) \succeq \xi(x, v, w)F(v, w).$$

We denote by $\mathcal{G}$ the set of operators $G : \mathbb{V} \to \mathbb{V}$ such that
($\mathcal{G}_1$) G is increasing;
($\mathcal{G}_2$) for all $(p, v) \in (0, 1) \times Q$, we have

$$G(pv) \succeq pGv.$$

Our main result in this paper is the following.

Theorem 2.1. *Let $(F, G) \in \mathcal{F} \times \mathcal{G}$. Suppose that there exists $\eta \in Q$, $\eta \neq 0$ such that*

$$\frac{\zeta(x_0)}{\xi(x_0, \eta, \eta)}\eta \preceq GF(\eta, \eta) \preceq \frac{1}{\zeta(x_0)}\eta, \tag{2.1}$$

for some $x_0 \in (\alpha, \beta)$. Then the operator equation

$$GF(v, v) = v \tag{2.2}$$

has a unique solution $v^* \in Q_\eta$.

Proof. Define the operator $H : Q \times Q \to \mathbb{V}$ by

$$H(v, w) = GF(v, w), \quad (v, w) \in Q \times Q.$$

Observe that from $(\mathcal{G}_2)$, we have

$$G0 \succeq \frac{1}{2}G0,$$

which yields

$$G0 \in Q.$$

Sine G is increasing, we get $H(Q \times Q) \subseteq Q$. Then $H : Q \times Q \to Q$. Let $(\phi_1, \omega_1), (\phi_2, \omega_2) \in Q \times Q$ with $\phi_2 - \phi_1 \in Q$ and $\omega_1 - \omega_2 \in Q$. From (F1), we have

$$F(\phi_1, \omega_1) \preceq F(\phi_2, \omega_2).$$

Then from $(\mathcal{G}_1)$, we get

$$GF(\phi_1, \omega_1) \preceq GF(\phi_2, \omega_2),$$

that is,

$$H(\phi_1, \omega_1) \preceq H(\phi_2, \omega_2).$$

This proves that H is mixed monotone. Using (F2), $(\mathcal{G}_1)$ and $(\mathcal{G}_2)$, for all $(x, \phi, \omega) \in (\alpha, \beta) \times Q \times Q$, we obtain

$$\begin{aligned} H(\zeta(x)\phi, [\zeta(x)]^{-1}\omega) &= GF(\zeta(x)\phi, [\zeta(x)]^{-1}\omega) \\ &\succeq G(\xi(x, \phi, \omega)F(\phi, \omega)) \\ &\succeq \xi(x, \phi, \omega)GF(\phi, \omega) \\ &= \xi(x, \phi, \omega)H(\phi, \omega), \end{aligned}$$

that is,

$$H(\zeta(x)\phi, [\zeta(x)]^{-1}\omega) \succeq \xi(x, \phi, \omega)H(\phi, \omega), \quad (x, \phi, \omega) \in (\alpha, \beta) \times Q \times Q. \tag{2.3}$$

Since $\xi(x_0, \eta, \eta) \in (\zeta(x_0), 1)$, there is some positive integer m such that

$$\left(\frac{\xi(x_0, \eta, \eta)}{\zeta(x_0)}\right)^m > [\zeta(x_0)]^{-1}. \tag{2.4}$$

Set

$$\phi_0 = [\zeta(x_0)]^m\eta \quad \text{and} \quad \omega_0 = [\zeta(x_0)]^{-m}\eta.$$

Observe that

$$\phi_0, \omega_0 \in Q_\eta \quad \text{and} \quad \phi_0 = [\zeta(x_0)]^{2m}\omega_0 \prec \omega_0.$$

Moreover, take $0 < \varepsilon \leq [\zeta(x_0)]^{2m}$, we obtain

$$\phi_0 \succeq \varepsilon\omega_0.$$

Using (ξ_1), (2.1) and (2.3), we obtain

$$\begin{aligned} H(\phi_0, \omega_0) &= H([\zeta(x_0)]^m\eta, [\zeta(x_0)]^{-m}\eta) \\ &= H([\zeta(x_0)][\zeta(x_0)]^{m-1}\eta, [\zeta(x_0)]^{-1}[\zeta(x_0)]^{1-m}\eta) \\ &\succeq \xi(x_0, [\zeta(x_0)]^{m-1}\eta, [\zeta(x_0)]^{1-m}\eta)H([\zeta(x_0)]^{m-1}\eta, [\zeta(x_0)]^{1-m}\eta) \\ &\succeq \xi(x_0, [\zeta(x_0)]^{m-1}\eta, [\zeta(x_0)]^{1-m}\eta)\xi(x_0, [\zeta(x_0)]^{m-2}\eta, [\zeta(x_0)]^{2-m}\eta)H([\zeta(x_0)]^{m-2}\eta, [\zeta(x_0)]^{2-m}\eta) \\ &\ \vdots \\ &\succeq \xi(x_0, [\zeta(x_0)]^{m-1}\eta, [\zeta(x_0)]^{1-m}\eta) \cdots \xi(x_0, \eta, \eta)H(\eta, \eta) \\ &\succeq [\zeta(x_0)]^{m-1}\xi(x_0, \eta, \eta)H(\eta, \eta) \\ &\succeq [\zeta(x_0)]^m\eta = \phi_0. \end{aligned}$$

On the other hand, from (2.3), we can write

$$H([\zeta(x)]^{-1}\phi,\zeta(x)\omega) \preceq \frac{1}{\xi(x,[\zeta(x)]^{-1}\phi,\zeta(x)\omega)}H(\phi,\omega), \quad (x,\phi,\omega)\in(\alpha,\beta)\times Q\times Q. \tag{2.5}$$

Using (ξ_2), (2.1), (2.4) and (2.5), we get

$$\begin{aligned}
H(\phi_0,\omega_0) &= H([\zeta(x_0)]^{-m}\eta,[\zeta(x_0)]^{m}\eta)\\
&= H([\zeta(x_0)]^{-1}[\zeta(x_0)]^{1-m}\eta,[\zeta(x_0)][\zeta(x_0)]^{m-1}\eta)\\
&\preceq \frac{1}{\xi(x_0,[\zeta(x_0)]^{-m}\eta,[\zeta(x_0)]^{m}\eta)}H([\zeta(x_0)]^{1-m}\eta,[\zeta(x_0)]^{m-1}\eta)\\
&\preceq \frac{1}{\xi(x_0,[\zeta(x_0)]^{-m}\eta,[\zeta(x_0)]^{m}\eta)}\frac{1}{\xi(x_0,[\zeta(x_0)]^{1-m}\eta,[\zeta(x_0)]^{m-1}\eta)}H([\zeta(x_0)]^{2-m}\eta,[\zeta(x_0)]^{m-2}\eta)\\
&\ \ \vdots\\
&\preceq \frac{1}{\xi(x_0,[\zeta(x_0)]^{-m}\eta,[\zeta(x_0)]^{m}\eta)}\cdots\frac{1}{\xi(x_0,[\zeta(x_0)]^{-1}\eta,[\zeta(x_0)(x_0)]\eta)}H(\eta,\eta)\\
&\preceq \left(\frac{1}{\xi(x_0,\eta,\eta)}\right)^{m}H(\eta,\eta)\\
&\preceq \left(\frac{1}{\xi(x_0,\eta,\eta)}\right)^{m}\frac{1}{\zeta(x_0)}\eta\\
&= [\zeta(x_0)]^{-m}\eta=\omega_0.
\end{aligned}$$

As consequence, we have

$$\phi_0 \preceq H(\phi_0,\omega_0)\preceq H(\omega_0,\phi_0)\preceq \omega_0.$$

Let

$$\phi_{n+1}=H(\phi_n,\omega_n),\ \omega_{n+1}=H(\omega_n,\phi_n),\quad n=0,1,2,\cdots.$$

Then we have

$$\phi_0\preceq\phi_1\preceq\omega_1\preceq\omega_0.$$

By induction, we obtain easily

$$\phi_0\preceq\phi_1\preceq\cdots\preceq\phi_n\preceq\cdots\preceq\omega_n\preceq\cdots\preceq\omega_1\preceq\omega_0. \tag{2.6}$$

Set

$$s_n=\sup\{s>0:\ \phi_n\succeq s\omega_n\},\quad n=0,1,2,\cdots.$$

Then we have

$$\phi_n\succeq s_n\omega_n,\quad n=0,1,2,\cdots,$$

which implies from (2.6) that

$$\phi_{n+1}\succeq\phi_n\succeq s_n\omega_n\succeq s_n\omega_{n+1},\quad n=0,1,2,\cdots,$$

which yields

$$0<s_0\le s_1\le\cdots\le s_n\le s_{n+1}\le\cdots\le 1.$$

Then there exists some $s\in(0,1]$ such that

$$\lim_{n\to\infty}s_n=s. \tag{2.7}$$

Suppose $s\in(0,1)$. We distinguish two cases.
Case 1. $s_N=s$ for some positive integer N.
In this case, we get

$$s_n=s,\quad n\ge N,$$

which yields

$$\phi_n \succeq s\omega_n, \quad n \geq N.$$

Since $s \in (0,1)$, there is some $x_s \in (\alpha, \beta)$ such that $\zeta(x_s) = s$. Using (2.3), (ξ_1) and (2.6), for all $n \geq N$, we get

$$\begin{aligned}\phi_{n+1} &= H(\phi_n, \omega_n)\\ &\succeq H(s\omega_n, s^{-1}\phi_n)\\ &= H(\zeta(x_s)\omega_n, [\zeta(x_s)]^{-1}\phi_n)\\ &\succeq \xi(x_s, \omega_n, \phi_n)H(\omega_n, \phi_n)\\ &\succeq \xi(x_s, \phi_0, \omega_0)\omega_{n+1}.\end{aligned}$$

The above inequality with (ξ_1) yield

$$s = \alpha(x_s) < \xi(tx_s, \phi_0, \omega_0) \leq s_{n+1} = s, \quad n \geq N,$$

which is a contradiction.
Case 2. $s_n < s$ for every n.
In this case, we have

$$0 < \frac{s_n}{s} < 1, \quad n = 1, 2, 3, \cdots .$$

Then

$$\frac{s_n}{s} = \zeta(q_n), \quad n = 1, 2, 3, \cdots$$

for some $q_n \in (\alpha, \beta)$. Then

$$\begin{aligned}\phi_{n+1} &= H(\phi_n, \omega_n)\\ &\succeq H(s_n\omega_n, s_n^{-1}\phi_n)\\ &= H(\zeta(q_n)s\omega_n, [\zeta(q_n)]^{-1}s^{-1}\phi_n)\\ &\succeq \xi(q_n, s\omega_n, s^{-1}\phi_n)A(s\omega_n, s^{-1}\phi_n)\\ &\succeq \xi(q_n, s\phi_0, s^{-1}\omega_0)A(s\omega_n, s^{-1}\phi_n)\\ &\succeq \xi(q_n, s\phi_0, s^{-1}\omega_0)H(\zeta(x_s)\omega_n, [\zeta(x_s)]^{-1}\phi_n)\\ &\succeq \xi(q_n, s\phi_0, s^{-1}\omega_0)\xi(x_s, \omega_n, \phi_n)H(\omega_n, \phi_n)\\ &\succeq \xi(q_n, s\phi_0, s^{-1}\omega_0)\xi(x_s, \phi_0, \omega_0)\omega_{n+1}.\end{aligned}$$

This yields

$$\frac{s_n}{s}\xi(x_s, \phi_0, \omega_0) < \xi(q_n, s\phi_0, s^{-1}\omega_0)\xi(x_s, \phi_0, \omega_0) \leq s_{n+1}.$$

Passing to the limit as $n \to \infty$ and using (2.7), we obtain

$$\zeta(x_s) < \xi(x_s, \phi_0, \omega_0) \leq s = \zeta(x_s),$$

which is a contradiction.

Hence, we proved that

$$\lim_{n\to\infty} s_n = 1. \tag{2.8}$$

Now, for any positive integer q, we have

$$0 \preceq \phi_{n+q} - \phi_n \preceq \omega_n - \phi_n \preceq \omega_n - s_n\omega_n = (1 - s_n)\omega_n \preceq (1 - s_n)\omega_0$$

and

$$0 \preceq \omega_n - \omega_{n+q} \preceq \omega_n - \phi_n \preceq (1 - s_n)\omega_0.$$

Using (2.8) and Lemma 2.3 in [8], we deduce that $\{\phi_n\}$ and $\{\omega_n\}$ are Cauchy sequences. Then there exist $\phi^*, \omega^* \in \mathbb{V}$ such that $\{\phi_n\}$ converges to ϕ^* and $\{\omega_n\}$ converges to ω^*. By (2.6), we have

$$\phi_0 \preceq \phi_n \preceq \phi^* \preceq \omega^* \preceq \omega_n \preceq \omega_0, \quad n = 0, 1, 2, \cdots . \tag{2.9}$$

Then we have

$$o \preceq \omega^* - \phi^* \preceq \omega_n - \phi_n \preceq (1 - s_n)\omega_0.$$

Passing to the limit as $n \to \infty$, using (2.8) and Lemma 2.4 in [8], we obtain $\phi^* = \omega^*$. Set

$$v^* = \phi^* = \omega^*.$$

Then we have $v^* \in [\phi_0, \omega_0]$. Since H is mixed monotone, using (2.9), we obtain

$$\phi_{n+1} = H(\phi_n, \omega_n) \preceq H(v^*, v^*) \preceq H(\omega_n, \phi_n) = \omega_{n+1}.$$

Passing to the limit as $n \to \infty$, we get

$$v^* \preceq H(x^*, x^*) \preceq x^*,$$

which yields

$$GF(v^*, v^*) = H(v^*, v^*) = v^*.$$

Let us prove now that v^* is the unique point in Q_η satisfying the above equality. Suppose that $w^* \in Q_\eta$ is such that

$$H(w^*, w^*) = w^*.$$

Let

$$\rho = \sup\{0 < c \leq 1 : cw^* \preceq v^* \preceq c^{-1}w^*\}.$$

Observe that $\rho \in (0, 1]$ and

$$\rho w^* \preceq v^* \preceq \rho^{-1}v^*. \tag{2.10}$$

As in the proof of $s = 1$, arguing by contradiction, we can prove that $\rho = 1$. Then from (2.10), we have

$$v^* = w^*.$$

Thus we proved that v^* is the unique solution in Q_η to the operator equation (2.2). □

Now, we present some results that can be deduced from Theorem 2.1.

Let $\xi_i : (\alpha, \beta) \times Q \to (0, \infty)$, $i = 1, 2$ be two functions with the following properties:
(P1) for all $i = 1, 2$, we have $\xi(x, v) \in (\zeta(x), 1)$ for all $(x, v) \in (\alpha, \beta) \times Q$;
(P2) for any $x \in (\alpha, \beta)$, $\xi(x, u, v) = \min\{\xi_1(x, u), \xi_2(x, v)\}$ is increasing in u for fixed v and decreasing in v for fixed u.

We denote by $\mathcal{T}$ the set of operators $T : Q \to Q$ such that
(T1) T is increasing;
(T2) for all $(x, v) \in (\alpha, \beta) \times Q$, we have

$$T(\zeta(x)v) \succeq \xi_1(x, v)Tv.$$

We denote by $\mathcal{S}$ the set of operators $S : Q \to Q$ such that
(S1) S is decreasing;
(S2) for all $(x, v) \in (\alpha, \beta) \times Q$, we have

$$S([\zeta(x)]^{-1}v) \succeq \xi_2(x, v)Sv.$$

Corollary 2.2. *Let $(T, S, G) \in \mathcal{T} \times \mathcal{S} \times G$. Suppose that there exists $\eta \in Q$, $\eta \neq 0$ such that*

$$\frac{\zeta(x_0)}{\xi(x_0, \eta, \eta)}\eta \preceq G(T\eta + S\eta) \preceq \frac{1}{\zeta(x_0)}\eta$$

for some $x_0 \in (\alpha, \beta)$. Then the operator equation

$$G(Tv + Sv) = v$$

has a unique solution $v^* \in Q_\eta$.

Proof. Observe that the operator

$$F(v, w) = Tu + Sw, , \quad (v, w) \in Q \times Q$$

belongs to the set $\mathcal{F}$. From Theorem 2.1, the operator equation

$$GF(v, v) = v,$$

which is equivalent to

$$G(Tv + Sv) = v$$

has a unique solution $v^* \in Q_\eta$. □

Let $\xi_1 : (\alpha, \beta) \times Q \to (0, \infty)$ and $\xi_2 : (\alpha, \beta) \times Q \times Q \to (0, \infty)$ be two function satisfying the following assumptions:
(A1) $\xi_1(x, v), \xi_2(x, v, w) \in (\zeta(x), 1)$ for all $(x, v, w) \in (\alpha, \beta) \times Q \times Q$;
(A2) for any $x \in (\alpha, \beta)$, $\xi(x, v, w) = \min\{\xi_1(x, v), \xi_2(x, v, w)\}$ is increasing in v for fixed w and decreasing in w for fixed v.

We denote by $\mathcal{B}$ the set of operators $B : Q \to Q$ such that
(B1) B is increasing;
(B2) $B(\zeta(x)v) \succeq \xi_1(x, v)Bv, \quad (x, v) \in (\alpha, \beta) \times Q$.

We denote by $\mathcal{C}$ the set of operators $C : Q \times Q \to Q$ such that
(C1) C is mixed monotone;
(C2) $C(\zeta(x)v, [\zeta(x)]^{-1}w) \succeq \xi_2(v, w)C(v, w), \quad (x, v, w) \in (\alpha, \beta) \times Q \times Q$.

Corollary 2.3. *Let $(B, C, G) \in \mathcal{B} \times \mathcal{C} \times \mathcal{G}$. Suppose that there exists $\eta \in Q$, $\eta \neq 0$ such that*

$$\frac{\zeta(x_0)}{\xi(x_0, \eta, \eta)}\eta \preceq G(B\eta + C(\eta, \eta)) \preceq \frac{1}{\zeta(x_0)}\eta$$

for some $x_0 \in (\alpha, \beta)$.
Then the operator equation

$$G(Bv + C(v, v)) = v$$

has a unique solution in Q_η.

Proof. We observe that the operator

$$F(v, w) = Bv + C(v, w), \quad (v, w) \in Q \times Q$$

belongs to the set $\mathcal{F}$. From Theorem 2.1, the operator equation

$$GF(v, v) = v,$$

which is equivalent to

$$G(Bv + C(v, v)) = v$$

has a unique solution in Q_η. □

3. An application to integral equations

Consider the integral equation

$$\int_0^x \int_0^1 \vartheta(\tau, y)[f(\tau, v(y)) + g(\tau, v(y), v(y))]\, dy\, d\tau = v(x), \quad x \in [0,1], \tag{3.1}$$

where $\vartheta : [0,1] \times [0,1] \to [0,\infty)$, $f : [0,1] \times [0,\infty) \to [0,\infty)$ and $g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are regular functions.

Let $(\mathbb{V}, N, \mathbb{T}_m)$ be the PB-space, where $\mathbb{V} = C([0,1])$ is the set of real continuous functions in $[0,1]$ and $N : \mathbb{V} \to \mathbb{D}$ is given by

$$N(u)(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ \frac{x}{x + \max_{0 \leq z \leq 1} |u(z)|} & \text{if} \quad x > 0 \end{cases}, \quad u \in \mathbb{V}.$$

Here $\mathbb{T}_m$ is the T-norm given by $\mathbb{T}_m(\mu, \lambda) = \min\{\mu, \lambda\}$ for $\mu, \lambda \in [0,1]$. We consider the normal cone Q given by

$$Q = \{u \in \mathbb{V} : u(x) \geq 0, \text{ for all } x \in [0,1]\}.$$

Let $\xi : (0,1) \to (0,1)$ be a function such that

$$\xi(x) \in (x, 1), \quad x \in (0,1).$$

Let $\mathbb{F}$ be the set of functions $f : [0,1] \times [0,\infty) \to [0,\infty)$ such that for a fixed $0 \leq s \leq 1$, the function $f(s, \cdot)$ is increasing in $[0,1]$ and

$$f(s, xz) \geq \xi(x) f(s, z), \quad (x, z) \in (0,1) \times [0,\infty).$$

Let $\mathbb{G}$ be the set of functions $g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ such that for a fixed $0 \leq s \leq 1$, the function $g(s, \cdot, \cdot)$ is mixed monotone and

$$g(s, xz, x^{-1}w) \geq \xi(x) g(s, z, w), \quad (x, z, w) \in (0,1) \times [0,\infty) \times [0,\infty).$$

Theorem 3.1. *Let $(f, g) \in \mathbb{F} \times \mathbb{G}$. Suppose that there exists $\eta \in Q$, $\eta \neq 0$ and $x_0 \in (0,1)$ such that for all $x \in [0,1]$,*

$$\frac{x_0}{\xi(x_0)} \eta(x) \leq \int_0^x \int_0^1 \vartheta(\tau, y)[f(\tau, \eta(y)) + g(\tau, \eta(y), \eta(y))]\, dy\, d\tau \leq \frac{1}{x_0} \eta(x). \tag{3.2}$$

Then (3.1) *has a unique solution $v^* \in Q_\eta$.*

Proof. Observe that the operator $G : \mathbb{V} \to \mathbb{V}$ be the operator defined by

$$(Gv)(x) = \int_0^x v(y)\, dy, \quad x \in [0,1]$$

belongs to the set $\mathcal{G}$. Let $B : Q \to Q$ be the operator given by

$$(Bv)(x) = \int_0^1 \vartheta(x, y) f(x, v(y))\, dy, \quad x \in [0,1].$$

It is not difficult to see that $B \in \mathcal{B}$ with $\zeta(x) = x$, $x \in (0,1)$ and $\xi_1(x, v) = \xi(x)$, $(x, v) \in (0,1) \times Q$. Let $C : Q \times Q \to Q$ be the operator defined by

$$C(v, w)(x) = \int_0^1 \vartheta(x, y) g(x, v(y), w(y))\, dy, \quad x \in [0,1].$$

Then $C \in \mathcal{C}$ with $\xi_2(x, v, w) = \xi(x)$, $(x, v, w) \in (0,1) \times Q \times Q$. Moreover, from (3.2), there exists $\eta \in Q$, $\eta \neq 0$ such that

$$\frac{\zeta(x_0)}{\xi(x_0)} \eta \preceq G(B\eta + C(\eta, \eta)) \preceq \frac{1}{\zeta(x_0)} \eta,$$

for some $x_0 \in (0,1)$. Using Corollary 2.3, we obtain that the operator equation

$$G(Bv + C(v, v))) = v,$$

which is equivalent to the integral equation (3.1), has a unique solution in Q_η. □

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