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Construction of a common solution of a finite family of variational inequality problems for monotone mappings

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Abstract

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A_i : C \to H$, for $i = 1, 2$, be two L_i-Lipschitz monotone mappings and let $f : C \to C$ be a contraction mapping. It is our purpose in this paper to introduce an iterative process for finding a point in $VI(C, A_1) \cap VI(C, A_2)$ under appropriate conditions. As a consequence, we obtain a convergence theorem for approximating a common solution of a finite family of variational inequality problems for Lipschitz monotone mappings. Our theorems improve and unify most of the results that have been proved for this important class of nonlinear operators.

Keywords: Fixed points of a mapping, monotone mapping, strong convergence, variational inequality.
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1. Introduction

Let C be a nonempty subset of a real Hilbert space H. A mapping $A : C \to H$ is called *L-Lipschitz* if there exits $L \geq 0$ such that

$$||Ax - Ay| \leq L||x - y||, \forall x, y \in C. \tag{1.1}$$

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If $L = 1$, then T is called *nonexpansive* and if $L < 1$, then T is called *a contraction*. It follows that every contraction mapping is nonexpansive and every nonexpansive mapping is Lipschitz.
A mapping $A : C \to H$ is called *η-strongly monotone* if there exists a positive real number η such that

$$\langle Ax - Ay, x - y\rangle \geq \eta||x - y||^2, \text{ for all } x, y \in C. \tag{1.2}$$

A is called *α-inverse strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha||Ax - Ay||^2, \text{ for all } x, y \in C. \tag{1.3}$$

We note that any α-inverse strongly monotone A is Lipschitz, that is $||Ax - Ay|| \leq L||x - y||, \forall x, y \in C$, where $L = \frac{1}{\alpha}$.
A is called *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \text{ for all } x, y \in C. \tag{1.4}$$

Clearly, the class of monotone mappings includes the class of α-inverse strongly monotone and the class of η-strongly monotone mappings.
Let C be a nonempty, closed and convex subset of H and let $A : C \to H$ be a nonlinear mapping. The variational inequality problem for A and C is the problem of finding a point $x^* \in C$ satisfying

$$\langle Ax^*, x - x^*\rangle \geq 0, \forall x \in C. \tag{1.5}$$

We denote the solution set of this problem by $VI(C, A)$. We know that the solution set of $VI(C, A)$ is always closed and convex under the assumption that A is continuous and monotone.
The theory of variational inequality has emerged as a very natural generalization of the theory of boundary value problems and allows us to consider new problems arising from many fields of applied mathematics, such as mechanics, physics, engineering, the theory of convex programming, and the theory of control: See, for instance, [9, 11, 17, 18, 19, 23, 24]. Variational inequalities were introduced and studied by Stampacchia [13] in 1964. Since then variational inequality problems has been extensively studied in the literature, see [7, 11, 14, 20, 25, 26, 28, 30, 31, 32] and the reference therein. There are several iterative methods for solving variational inequality problems. See, e.g., [2, 3, 4, 7, 11, 23, 24, 27]. The basic idea consists of extending the projected gradient method for constrained optimization, i.e., for the problem of minimizing $f(x)$ subject to $x \in C$. For $x_0 \in C$, compute the sequence $\{x_n\}$ in the following manner:

$$x_{n+1} = P_C[x_n - \alpha_n \nabla f(x_n)], n \geq 0, \tag{1.6}$$

where $\{\alpha_n\}$ is a positive real sequence satisfying certain conditions and P_C is the metric projection onto C. See [1] for convergence properties of this method for the case in which $f : \mathbb{R}^2 \to \mathbb{R}$ is convex and differentiable function. An immediate extension of the method (1.6) to $VI(C, A)$ is the iterative procedure given by

$$x_{n+1} = P_C[x_n - \alpha_n Ax_n], n \geq 0. \tag{1.7}$$

Convergence results for this method require some monotonicity properties of A. Note that for the method given by (1.7) there is no chance of relaxing the assumption on A to plain monotonicity. The typical example consists of taking $C = \mathbb{R}^2$ and A, a rotation with a $\frac{\pi}{2}$ angle. A is monotone and the unique solution of $VI(C, A)$ is $x^* = 0$. However, it is easy to check that $||x_{n+1}|| > ||x_n||$ for all $n \geq 0$ and all $\alpha_n > 0$, therefore the sequence generated by (1.7) moves away from the solution, independently of the choice of the sequence α_n. To overcome this weakness of the method defined by (1.7), Korpelevich [8] proposed a modification of the method, called the extragradient algorithm in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex and a mapping A of C into $\mathbb{R}^n$ is monotone and L-Lipschitz continuous,

$$\begin{cases} y_n = P_C[x_n - \lambda Ax_n], \\ x_{n+1} = P_C[x_n - \lambda Ay_n], n \geq 0, \end{cases} \tag{1.8}$$

for all $n \geq 0$, where $\lambda \in (0, \frac{1}{L})$. He proved that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.8), converge to the same point $x^* \in VI(C, A)$. The difference in (1.8) is that A is evaluated twice and the projection is computed twice at each iteration, but the benefit is significant, because the resulting algorithm is applicable to the whole class of variational inequalities for monotone mappings. Korpelevich's method has received great attention by many authors, who improved it in various ways; see, e.g., [4, 6, 7, 9, 10, 15, 23, 30] and the references therein. In 2006, Nadezhkina and Takahashi [10] suggested the following modified Korpelevich's method for a solution of a variational inequality $VI(C, A)$ for L-Lipschitz continuous monotone mapping A in infinite-dimensional Hilbert spaces. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by

$$\begin{cases} y_n = P_C[x_n - \lambda_n A x_n], \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) P_C[x_n - \lambda_n A y_n], n \geq 0, \end{cases} \tag{1.9}$$

where P_C is a metric projection from H onto C, $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/L)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, they proved that the sequences $\{x_n\}, \{y_n\}$ converge *weakly* to the minimum-norm point of $VI(C, A)$. We remark that Korpelevich's modified method (1.9) has only weak convergence in the infinite-dimensional Hilbert spaces (see Censor *et al.* [5] and [4]). So to obtain strong convergence the original method was modified by several authors. For example in [2, 6, 21] it is proved that some very interesting Korpelevich-type algorithms strongly converge to a solution of $VI(C, A)$. Recently, Yao et al. [21] suggested the following modified Korpelevich's method for a solution of a variational inequality $VI(C, A)$ for α-inverse strongly monotone mapping A in infinite-dimensional Hilbert spaces. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by

$$\begin{cases} y_n = P_C[x_n - \lambda A x_n - \alpha_n x_n], \\ x_{n+1} = P_C[x_n - \lambda A y_n + \mu(y_n - x_n)], n \geq 0, \end{cases} \tag{1.10}$$

where P_C is a metric projection from H onto C, $\lambda \in [a, b] \subset (0, 2\alpha)$, $\mu \in (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfying certain conditions. Then, they proved that the sequence $\{x_n\}$ converges strongly to the minimum-norm point of $VI(C, A)$. One may also see related results in [21]. More recently, Yao *et al.* [22] investigated the problem of finding a solution of variational inequality $VI(C, A)$ for α-inverse strongly monotone mapping A by considering the following iterative algorithm:

$$\begin{cases} y_n = P_C[x_n - \lambda_n A x_n + \alpha_n(f x_n - x_n)], \\ x_{n+1} = P_C[x_n - \mu_n A y_n + \gamma_n(y_n - x_n)], n \geq 0, \end{cases} \tag{1.11}$$

where $f : C \to H$ is a ρ-contractive mapping and $\{\alpha_n\}, \{\lambda_n\}, \{\mu_n\}$ and $\{\gamma_n\}$ are real sequences satisfying certain conditions. Then they proved that the sequence $\{x_n\}$ generated by (1.11) converges strongly to $x^* \in VI(C, A)$.

A natural question arises: can we obtain an iterative scheme which converges strongly to a solution $VI(C, A)$ of a variational inequality problem for a more general class of monotone mappings?

It is our purpose in this paper to propose an extragradient-type method for solving a common solution of two variational inequality problems for Lipschitz monotone mappings. As a consequence, we obtain a convergence theorem for approximating a common solution of a finite family of variational inequality problems for Lipschitz monotone mappings. The results obtained in this paper improve and extend the results of Nadezhkina and Takahashi [10], Yao et al. [21] and Yao *et al.* [22] and some other results in this direction.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space H. We remark that for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, satisfying

$$||x - P_C x|| \leq ||x - y|| \text{ for all } y \in C. \tag{2.1}$$

The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C and is characterized by the following properties (see, *e.g.*, [16]):

$$P_C x \in C \text{ and } \langle x - P_C x, P_C x - y\rangle \geq 0, \text{ for all } x \in H, y \in C \text{ and} \tag{2.2}$$

$$||y - P_C x||^2 \leq ||x - y||^2 - ||x - P_C x||^2, \text{ for all } x \in H, y \in C. \tag{2.3}$$

Let $A : C \to H$ be a monotone mapping. We note that in the context of variational inequality problem we have that

$$x^* \in VI(C, A) \text{ if and only if } x^* = P_C(x^* - \lambda A x^*), \forall \lambda > 0. \tag{2.4}$$

A monotone mapping $B : C \to 2^H$ is called *maximal monotone* if its graph $G(B)$ is not properly contained in the graph of any other monotone mapping. That is, a monotone mapping B is maximal if and only if, for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$, for every $(y, v) \in G(B)$ implies $u \in Bx$. Let A be a monotone and L-Lipschitz mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}.$$

Define

$$Bv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{2.5}$$

Then, B is maximal monotone and $0 \in Bv$ if and only if $v \in VI(C, A)$ (see, *e.g.*, [12]).
In the sequel we shall make use of the following lammas.

Lemma 2.1 ([29]). *Let H be a real Hilbert space. Then for all $x_i \in H$ and $\alpha_i \in [0, 1]$, for $i = 1, 2, 3$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ the following equality holds:*

$$||\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3||^2 = \sum_{i=1}^{3} \alpha_i ||x_i||^2 - \sum_{1 \leq i,j \leq 3} \alpha_i \alpha_j ||x_i - x_j||^2.$$

Lemma 2.2. *Let H be a real Hilbert space. Then for any given $x, y \in H$, the following inequality holds:*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, x + y\rangle.$$

Lemma 2.3 ([18]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n, n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 ([9]). *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$, for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3. Main Result

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A_i : C \to H$ be a finite family of L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2$. Let $f : C \to C$ be a contraction mapping. Assume that $\mathcal{F} = \cap_{i=1}^{2} VI(C, A_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A_2 x_n], \\ y_n = P_C[x_n - \gamma_n A_1 x_n], \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\big(a_n x_n + b_n P_C[x_n - \gamma_n A_1 y_n] + c_n P_C[x_n - \gamma_n A_2 z_n]\big), \end{cases} \tag{3.1}$$

where P_C is a metric projection from H onto C, $\gamma_n \subset [a, b] \subset (0, \frac{1}{L})$, for $L := \max\{L_1, L_2\}$, $\{a_n\}, \{b_n\}, \{c_n\} \subset [e, 1) \subset (0, 1)$, such that $a_n + b_n + c_n = 1$ and $\{\alpha_n\} \subset (0, c] \subset (0, 1)$ for all $n \geq 0$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$ which is the unique solution of the variational inequality $\langle (I - f)(x^*), x - x^* \rangle \geq 0$ for all $x \in \mathcal{F}$.

Proof. Let $p \in \mathcal{F}$, $u_n = P_C(x_n - \gamma_n A_1 y_n)$ and $v_n = P_C(x_n - \gamma_n A_2 z_n)$ for all $n \geq 0$. Then, from (2.3) we have

$$\begin{aligned} ||u_n - p||^2 &\leq ||x_n - \gamma_n A_1 y_n - p||^2 - ||x_n - \gamma_n A_1 y_n - u_n||^2 \\ &= ||x_n - p||^2 - ||x_n - u_n||^2 + 2\gamma_n \langle A_1 y_n, p - u_n\rangle \\ &= ||x_n - p||^2 - ||x_n - u_n||^2 + 2\gamma_n \big(\langle A_1 y_n - A_1 p, p - y_n\rangle \\ &\quad + \langle A_1 p, p - y_n\rangle + \langle A_1 y_n, y_n - u_n\rangle\big) \\ &\leq ||x_n - p||^2 - ||x_n - u_n||^2 + 2\gamma_n \langle A_1 y_n, y_n - u_n\rangle \\ &= ||x_n - p||^2 - ||x_n - y_n||^2 - 2\langle x_n - y_n, y_n - u_n\rangle \\ &\quad - ||y_n - u_n||^2 + 2\gamma_n \langle A_1 y_n, y_n - u_n\rangle \\ &= ||x_n - p||^2 - ||x_n - y_n||^2 - ||y_n - u_n||^2 \\ &\quad + 2\langle x_n - \gamma_n A_1 y_n - y_n, u_n - y_n\rangle \end{aligned} \tag{3.2}$$

and from (2.2), we obtain

$$\begin{aligned} \langle x_n - \gamma_n A_1 y_n - y_n, u_n - y_n\rangle &= \langle x_n - \gamma_n A_1 x_n - y_n, u_n - y_n\rangle \\ &\quad + \langle \gamma_n A_1 x_n - \gamma_n A_1 y_n, u_n - y_n\rangle \\ &\leq \langle \gamma_n A_1 x_n - \gamma_n A_1 y_n, u_n - y_n\rangle \\ &\leq \gamma_n L ||x_n - y_n|| ||u_n - y_n||. \end{aligned} \tag{3.3}$$

Thus, from (3.2) and (3.3) we get

$$\begin{aligned} ||u_n - p||^2 &\leq ||x_n - p||^2 - ||x_n - y_n||^2 - ||y_n - u_n||^2 \\ &\quad + 2\gamma_n L ||x_n - y_n|| ||u_n - y_n|| \\ &\leq ||x_n - p||^2 - ||x_n - y_n||^2 - ||y_n - u_n||^2 \\ &\quad + \gamma_n L(||x_n - y_n||^2 + ||y_n - u_n||^2) \\ &\leq ||x_n - p||^2 + (\gamma_n L - 1)||x_n - y_n||^2 + (\gamma_n L - 1)||y_n - u_n||^2. \end{aligned} \tag{3.4}$$

Likewise, we obtain that

$$||v_n - p||^2 \leq ||x_n - p||^2 + (\gamma_n L - 1)||x_n - z_n||^2 + (\gamma_n L - 1)||z_n - v_n||^2. \tag{3.5}$$

Thus, from (3.1), Lemma 2.1, (3.4) and (3.5) we have the following:

$$||x_{n+1} - p||^2 = ||\alpha_n f(x_n) + (1 - \alpha_n)(a_n x_n + b_n u_n + c_n v_n) - p||^2$$

$$
\begin{aligned}
&\leq \alpha_n||f(x_n)-p||^2+(1-\alpha_n)||a_n(x_n-p)+b_n(u_n-p)\\
&\quad +c_n(v_n-p)||^2\\
&\leq \alpha_n||f(x_n)-p||^2+(1-\alpha_n)\big[a_n||x_n-p||^2+b_n||u_n-p||^2\\
&\quad +c_n||v_n-p||^2\big]\\
&\leq \alpha_n||f(x_n)-p||^2+(1-\alpha_n)a_n||x_n-p||^2+(1-\alpha_n)b_n\big[||x_n-p||^2\\
&\quad +(\gamma_nL-1)||x_n-y_n||^2+(\gamma_nL-1)||y_n-u_n||^2\big]\\
&\quad +(1-\alpha_n)c_n\big[||x_n-p||^2+(\gamma_nL-1)||x_n-z_n||^2\\
&\quad +(\gamma_nL-1)||z_n-v_n||^2\big]\\
&= \alpha_n||f(x_n)-p||^2+(1-\alpha_n)||x_n-p||^2+(1-\alpha_n)b_n\\
&\quad \times(\gamma_nL-1)||x_n-y_n||^2+(1-\alpha_n)b_n(\gamma_nL-1)\\
&\quad \times||y_n-u_n||^2+(1-\alpha_n)c_n(\gamma_nL-1)||x_n-z_n||^2\\
&\quad +(1-\alpha_n)c_n(\gamma_nL-1)||z_n-v_n||^2.
\end{aligned}
\tag{3.6}
$$

Now, since from the hypotheses, we have $\gamma_n < \frac{1}{L}$ for all $n \geq 1$, the inequality (3.6) implies that

$$||x_{n+1}-p||^2 \leq \alpha_n||f(x_n)-p||^2+(1-\alpha_n)||x_n-p||^2. \tag{3.7}$$

Furthermore, we have that

$$
\begin{aligned}
||f(x_n)-p||^2 &= \big[||f(x_n)-f(p)||+||f(p)-p||\big]^2\\
&\leq \big[\rho||x_n-p||+||f(p)-p||\big]^2\\
&\leq \rho^2||x_n-p||^2+||f(p)-p||^2+2\rho||x_n-p||||f(p)-p||\\
&\leq \rho(1+\rho)||x_n-p||^2+(1+\rho)||f(p)-p||^2,
\end{aligned}
\tag{3.8}
$$

where ρ is a contraction constant of f. Substituting (3.8) into (3.7) we get that

$$||x_{n+1}-p||^2 \leq (1-\alpha_n(1-\rho(1+\rho)))||x_n-p||^2+\alpha_n(1+\rho)||f(p)-p||^2.$$

Therefore, by induction we get that

$$||x_{n+1}-p||^2 \leq \max\{||x_0-p||^2, \frac{1+\rho}{1-\rho(1+\rho)}||f(p)-p||^2\}, \forall n \geq 0,$$

which implies that $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded.
Let $x^* = P_{\mathcal{F}}f(x^*)$. Then, using (3.1), Lemma 2.2, Lemma 2.1, and following the methods used to get (3.6) we obtain

$$
\begin{aligned}
||x_{n+1}-x^*||^2 &= ||\alpha_n(f(x_n)-x^*)+(1-\alpha_n)\big[(a_nx_n+b_nu_n+c_nv_n)-x^*\big]||^2\\
&\leq (1-\alpha_n)||a_nx_n+b_nu_n+c_nv_n-x^*||^2\\
&\quad +2\alpha_n\langle f(x_n)-x^*, x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)a_n||x_n-x^*||^2+(1-\alpha_n)b_n||u_n-x^*||^2\\
&\quad +(1-\alpha_n)c_n||v_n-x^*||^2-(1-\alpha_n)b_na_n||u_n-x_n||^2\\
&\quad -(1-\alpha_n)c_na_n||v_n-x_n||^2+2\alpha_n\langle f(x_n)-x^*, x_{n+1}-x^*\rangle\\
&\leq (1-\alpha_n)a_n||x_n-x^*||^2+(1-\alpha_n)b_n\big[||x_n-x^*||^2\\
&\quad +(\gamma_nL-1)||x_n-y_n||^2+(\gamma_nL-1)||y_n-u_n||^2\big]\\
&\quad +(1-\alpha_n)c_n\big[||x_n-x^*||^2+(\gamma_nL-1)||x_n-z_n||^2
\end{aligned}
$$

$$
\begin{aligned}
&+ (\gamma_n L - 1)||z_n - v_n||^2\big] - (1-\alpha_n)b_n a_n ||u_n - x_n||^2 \\
&- (1-\alpha_n)c_n a_n ||v_n - x_n||^2 + 2\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle \\
=&(1-\alpha_n)||x_n - x^*||^2 + (1-\alpha_n)b_n(\gamma_n L - 1)||x_n - y_n||^2 \\
&+ (1-\alpha_n)b_n(\gamma_n L - 1)||y_n - u_n||^2 + (1-\alpha_n)c_n(\gamma_n L - 1) \\
&\times ||x_n - z_n||^2 + (1-\alpha_n)c_n(\gamma_n L - 1)||z_n - v_n||^2 \\
&- (1-\alpha_n)b_n a_n||u_n - x_n||^2 - (1-\alpha_n)c_n a_n||v_n - x_n||^2 \\
&+ 2\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle && (3.9) \\
\le&(1-\alpha_n)||x_n - x^*||^2 + 2\alpha_n\langle f(x_n) - x^*, x_{n+1} - x^*\rangle. && (3.10)
\end{aligned}
$$

But

$$
\begin{aligned}
\langle f(x_n) - x^*, x_{n+1} - x^*\rangle =&\langle f(x_n) - x^*, x_n - x^*\rangle + \langle f(x_n) - x^*, x_{n+1} - x_n\rangle \\
\le&\langle f(x_n) - f(x^*), x_n - x^*\rangle + \langle f(x^*) - x^*, x_n - x^*\rangle \\
&+ ||x_{n+1} - x_n||||f(x_n) - x^*|| \\
\le&\rho||x_n - x^*||^2 + \langle f(x^*) - x^*, x_n - x^*\rangle \\
&+ ||x_{n+1} - x_n||||f(x_n) - x^*||. && (3.11)
\end{aligned}
$$

Thus, substituting (3.11) in (3.10) we obtain that

$$
\begin{aligned}
||x_{n+1} - x^*||^2 \le&(1 - \alpha_n(1-2\rho))||x_n - x^*||^2 + 2\alpha_n\langle f(x^*) - x^*, x_n - x^*\rangle \\
&+ 2\alpha_n||x_{n+1} - x_n||.||f(x_n) - x^*||. && (3.12)
\end{aligned}
$$

Now, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{||x_n - x^*||\}$ is decreasing for all $n \ge n_0$. Then, we get that, $\{||x_n - x^*||)\}$ is convergent. Thus, from (3.9), the fact that $\gamma_n < b < \frac{1}{L}$ for all $n \ge 0$ and $\alpha_n \to 0$ as $n \to \infty$, we have that

$$u_n - x_n \to 0, v_n - x_n \to 0, y_n - x_n \to 0, z_n - x_n \to 0, z_n - v_n \to 0, y_n - u_n \to 0 \text{ as } n \to \infty. \tag{3.13}$$

Moreover, from (3.1) and (3.13) we get that

$$x_{n+1} - x_n =\alpha_n(f(x_n) - x_n) + (1-\alpha_n)\big[b_n(u_n - x_n) + c_n(v_n - x_n)\big] \to 0 \text{ as } n \to \infty. \tag{3.14}$$

Furthermore, since $\{x_n\}$ is bounded subset of H which is reflexive, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$ and $\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{j\to\infty}\langle f(x^*) - x^*, x_{n_j} - x^*\rangle$. This implies from (3.13) that $u_{n_j} \rightharpoonup z$ and $v_{n_j} \rightharpoonup z$.

Now, we show that $z \in \cap_{i=1}^2 VI(C, A)$. But, since A_i, for each $i \in \{1, 2\}$, is Lipschitz continuous, we have

$$||A_1 y_{n_j} - A_1 u_{n_j}|| \to 0 \text{ as } j \to \infty.$$

Let

$$B_1 x = \begin{cases} A_1 x + N_C x, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C, \end{cases} \tag{3.15}$$

where $N_C(x)$ is the normal cone to C at $x \in C$ given by $N_C(x) = \{w \in H : \langle x - u, w\rangle \ge 0 \text{ for all } u \in C\}$. Then, B_1 is maximal monotone and $0 \in B_1 x$ if and only if $x \in VI(C, A_1)$ (see, *e.g.* [12]). Let $(v, w) \in G(B_1)$. Then, we have $w \in B_1 v = A_1 v + N_C v$ and hence $w - A_1 v \in N_C v$. Thus, we get $\langle v-u, w-A_1 v\rangle \ge 0$, for all $u \in C$. On the other hand, since $u_{n_j} = P_C(x_{n_j} - \gamma_{n_j} A_1 y_{n_j})$ and $v \in C$, we have $\langle x_{n_j} - \gamma_{n_j} A_1 y_{n_j} - u_{n_j}, u_{n_j} - v\rangle \ge 0$, and hence, $\langle v - u_{n_j}, (u_{n_j} - x_{n_j})/\gamma_{n_j} + A_1 y_{n_j}\rangle \ge 0$. Therefore, from $w - A_1 v \in N_C v$ and $u_{n_j} \in C$ we get

$$\begin{aligned}\langle v-u_{n_j},w\rangle \geq&\langle v-u_{n_j},A_1v\rangle\\ \geq&\langle v-u_{n_j},A_1v\rangle-\langle v-u_{n_j},(u_{n_j}-x_{n_j})/\gamma_{n_j}+A_1y_{n_j}\rangle\\ =&\langle v-u_{n_j},A_1v-A_1u_{n_j}\rangle+\langle v-u_{n_j},A_1u_{n_j}-A_1y_{n_j}\rangle\\ &-\langle v-u_{n_j},(u_{n_j}-x_{n_j})/\gamma_{n_j}\rangle\\ \geq&\langle v-u_{n_j},A_1u_{n_j}-A_1y_{n_j}\rangle-\langle v-u_{n_j},(u_{n_j}-x_{n_j})/\gamma_{n_j}\rangle.\end{aligned}$$

This implies that $\langle v-z,w\rangle\geq 0$, as $j\to\infty$. Then, maximality of B_1 gives that $z\in B_1^{-1}(0)$. Therefore, $z\in VI(C,A_1)$. Similarly, with the use of $v_{n_j}=P_C(x_{n_j}-\gamma_{n_j}A_2z_{n_j})$ we get that $z\in VI(C,A_2)$ and hence $z\in\cap_{i=1}^2 VI(C,A_i)$. Thus, from 2.2, we immediately obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(x^*)-x^*,x_n-x^*\rangle=&\lim_{j\to\infty}\langle f(x^*)-x^*,x_{n_j}-x^*\rangle\\ =&\langle f(x^*)-x^*,z-x^*\rangle\leq 0.\end{aligned}\tag{3.16}$$

Hence, it follows from (3.12), (3.14), (3.16) and Lemma 2.3 that $||x_n-x^*||\to 0$ as $n\to\infty$. Consequently, $x_n\to x^*=P_{\mathcal{F}}f(x^*)$.
Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$||x_{n_i}-x^*||<||x_{n_i+1}-x^*||$$

for all $i\in\mathbb{N}$. Then, by Lemma 2.4, there exist a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that $m_k\to\infty$, and

$$||x_{m_k}-x^*||\leq||x_{m_k+1}-x^*|| \text{ and } ||x_k-x^*||\leq||x_{m_k+1}-x^*||\tag{3.17}$$

for all $k\in\mathbb{N}$. Now, from (3.9), the fact that $\gamma_n<\frac{1}{L}$ for all $n\geq 0$ and $\alpha_n\to 0$ as $n\to\infty$, we get that $u_{m_k}-x_{m_k}\to 0, v_{m_k}-x_{m_k}\to 0, y_{m_k}-x_{m_k}\to 0, z_{m_k}-x_{m_k}\to 0, z_{m_k}-v_{m_k}\to 0, y_{m_k}-u_{m_k}\to 0$ as $k\to\infty$. Thus, following the method in Case 1, we obtain

$$\limsup_{k\to\infty}\langle f(x^*)-x^*,x_{m_k}-x^*\rangle\leq 0.\tag{3.18}$$

Now, from (3.12) we have that

$$\begin{aligned}||x_{m_k+1}-x^*||^2\leq&(1-\alpha_{m_k}(1-2\rho))||x_{m_k}-x^*||^2+2\alpha_{m_k}\langle f(x^*)-x^*,x_{m_k}-x^*\rangle\\ &+2\alpha_{m_k}||x_{m_k+1}-x_{m_k}||.||f(x_{m_k})-x^*||.\end{aligned}\tag{3.19}$$

and hence (3.17) and (3.19) imply that

$$\begin{aligned}\alpha_{m_k}(1-2\rho)||x_{m_k}-x^*||^2\leq&||x_{m_k}-x^*||^2-||x_{m_k+1}-x^*||^2+2\alpha_{m_k}\langle f(x^*)-x^*,x_{m_k}-x^*\rangle\\ &+2\alpha_{m_k}||x_{m_k+1}-x_{m_k}||.||f(x_{m_k})-x^*||.\end{aligned}$$

But the fact that $\alpha_{m_k}>0$ implies that

$$(1-2\rho)||x_{m_k}-x^*||^2\leq 2\langle f(x^*)-x^*,x_{m_k}-x^*\rangle+2||x_{m_k+1}-x_{m_k}||.||f(x_{m_k})-x^*||.$$

Thus, using (3.18) and (3.14) we get that $||x_{m_k}-x^*||\to 0$ as $k\to\infty$. This together with (3.19) implies that $||x_{m_k+1}-x^*||\to 0$ as $k\to\infty$. But $||x_k-x^*||\leq||x_{m_k+1}-x^*||$ for all $k\in\mathbb{N}$ gives that $x_k\to x^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to a point $x^*=P_{\mathcal{F}}f(x^*)$, which satisfies the variational inequality $\langle(I-f)(x^*),x-x^*\rangle\geq 0$, for all $x\in\mathcal{F}$. The proof is complete. □

If, in Theorem 3.1, we assume that $f(x)=u\in C$, a constant mapping, then we get the following corollary.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A_i : C \to H$ be a finite family of L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2$. Assume that $\mathcal{F} = \cap_{i=1}^{2} VI(C, A_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0, u \in C$ by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A_2 x_n], \\ y_n = P_C[x_n - \gamma_n A_1 x_n], \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)\big(a_n x_n + b_n P_C[x_n - \gamma_n A_1 y_n] + c_n P_C[x_n - \gamma_n A_2 z_n]\big), \end{cases} \tag{3.20}$$

where P_C is a metric projection from H onto C, $\gamma_n \subset [a, b] \subset (0, \frac{1}{L})$, for $L := \max\{L_1, L_2\}$, $\{a_n\}, \{b_n\}$, $\{c_n\} \subset [e, 1) \subset (0, 1)$, such that $a_n + b_n + c_n = 1$ and $\{\alpha_n\} \subset (0, c] \subset (0, 1)$ for all $n \geq 0$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$ which is the unique solution of the variational inequality $\langle x^* - u, x - x^* \rangle \geq 0$ for all $x \in \mathcal{F}$.

Remark 3.3. We note that when $f(x) = u \in C$ we observe that the sequence $\{x_n\}$ converges strongly to the point $x^* \in \mathcal{F}$ which is nearest to u.

If, in Theorem 3.1 we assume only one variational inequality problem for a monotone mapping A, then we obtain the following corollary.

Corollary 3.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be an L-Lipschitz monotone mapping with Lipschitz constant L. Let $f : C \to C$ be a contraction mapping. Assume that $\mathcal{F} = VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} y_n = P_C[x_n - \gamma_n A x_n], \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\big(a_n x_n + (1 - a_n) P_C[x_n - \gamma_n A y_n]\big), \end{cases} \tag{3.21}$$

where P_C is a metric projection from H onto C, $\gamma_n \subset [a, b] \subset (0, \frac{1}{L})$, $\{a_n\} \subset [e, 1) \subset (0, 1)$ and $\{\alpha_n\} \subset (0, c] \subset (0, 1)$ for all $n \geq 0$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality $\langle (I - f)(x^*), x - x^* \rangle \geq 0$ for all $x \in \mathcal{F}$.

If, in Theorem 3.1, we assume that A_i, for $i = 1, 2$, are α_i-inverse strongly monotone mappings, then both are L-Lipschitz with constant $L = \max\{\frac{1}{\alpha_1}, \frac{1}{\alpha_2}\}$ and hence we get the following corollary.

Corollary 3.5. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A_i : C \to H$ be a finite family of α_i-inverse strongly monotone mappings. Let $f : C \to C$ be a contraction mapping. Assume that $\mathcal{F} = \cap_{i=1}^{2} VI(C, A_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A_2 x_n], \\ y_n = P_C[x_n - \gamma_n A_1 x_n], \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\big(a_n x_n + b_n P_C[x_n - \gamma_n A_1 y_n] + c_n P_C[x_n - \gamma_n A_2 z_n]\big), \end{cases} \tag{3.22}$$

where P_C is a metric projection from H onto C, $\gamma_n \subset [a, b] \subset (0, \frac{1}{L})$, for $L = \max\{\frac{1}{\alpha_1}, \frac{1}{\alpha_2}\}$, $\{a_n\}, \{b_n\}, \{c_n\} \subset [e, 1) \subset (0, 1)$, such that $a_n + b_n + c_n = 1$ and $\{\alpha_n\} \subset (0, c] \subset (0, 1)$ for all $n \geq 0$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality $\langle (I - f)(x^*), x - x^* \rangle \geq 0$ for all $x \in \mathcal{F}$.

We note that the method of proof of Theorem 3.1 provides a convergence theorem for a finite family of Lipschitzian monotone mappings. In fact, we have the following theorem.

Corollary 3.6. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A_i : C \to H$,for $i = 1, 2, ...N$, be a finite family of L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for*

$i = 1, 2, ..., N$. Let $f : C \to C$ be a contraction mapping. Assume that $\mathcal{F} = \cap_{i=1}^{N} VI(C, A_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by

$$\begin{cases} y_{ni} = P_C[x_n - \gamma_n A_i x_n], \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\big(b_{n0}x_n + \sum_{i=1}^{N} b_{ni} P_C[x_n - \gamma_n A_i y_{ni}]\big), \end{cases} \tag{3.23}$$

where P_C is a metric projection from H onto C, $\gamma_n \subset [a, b] \subset (0, \frac{1}{L})$, for $L := \max\{L_i : i = 1, 2, ..., N\}$, $\{b_{ni}\} \subset [e, 1) \subset (0, 1)$, such that $\sum_{i=0}^{N} b_{ni} = 1$ and $\{\alpha_n\} \subset (0, c] \subset (0, 1)$ for all $n \geq 0$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality $\langle (I - f)(x^*), x - x^* \rangle \geq 0$ for all $x \in \mathcal{F}$.

If, in Theorem 3.1, we assume that $C = H$, a real Hilbert space, then P_C becomes identity mapping and $VI(C, A) = A^{-1}(0)$ and hence we get the following corollary.

Corollary 3.7. *Let H be a real Hilbert space. Let $A_i : H \to H$ be a finite family of L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2$. Let $f : H \to H$ be a contraction mapping. Assume that $\mathcal{F} = \cap_{i=1}^{2} A_i^{-1}(0)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} z_n = x_n - \gamma_n A_2 x_n, \\ y_n = x_n - \gamma_n A_1 x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\big(a_n x_n + b_n(x_n - \gamma_n A_1 y_n) + c_n(x_n - \gamma_n A_2 z_n)\big), \end{cases} \tag{3.24}$$

where $\gamma_n \subset [a, b] \subset (0, \frac{1}{L})$, for $L := \max\{L_1, L_2\}$, $\{a_n\}, \{b_n\}, \{c_n\} \subset [e, 1) \subset (0, 1)$, such that $a_n + b_n + c_n = 1$ and $\{\alpha_n\} \subset (0, c] \subset (0, 1)$ for all $n \geq 0$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality $\langle (I - f)(x^*), x - x^* \rangle \geq 0$ for all $x \in \mathcal{F}$.

We note that the method of proof of Theorem 3.1 provides the following theorem for approximating the unique minimum norm common point of solution of two variational inequality problems.

Theorem 3.8. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A_i : C \to H$ be a finite family of L_i-Lipschitz monotone mappings with Lipschitz constants L_i, for $i = 1, 2$. Assume that $\mathcal{F} = \cap_{i=1}^{2} VI(C, A_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} z_n = P_C[x_n - \gamma_n A_2 x_n], \\ y_n = P_C[x_n - \gamma_n A_1 x_n], \\ x_{n+1} = P_C\big[(1-\alpha_n)\big(a_n x_n + b_n P_C[x_n - \gamma_n A_1 y_n] + c_n P_C[x_n - \gamma_n A_2 z_n]\big)\big], \end{cases} \tag{3.25}$$

where P_C is a metric projection from H onto C, $\gamma_n \subset [a, b] \subset (0, \frac{1}{L})$, for $L := \max\{L_1, L_2\}$, $\{a_n\}, \{b_n\}, \{c_n\} \subset [e, 1) \subset (0, 1)$, such that $a_n + b_n + c_n = 1$ and $\{\alpha_n\} \subset (0, c] \subset (0, 1)$ for all $n \geq 0$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a unique minimum norm point x^* of $\mathcal{F}$ which is the unique solution of the variational inequality $\langle x^*, x - x^* \rangle \geq 0$ for all $x \in \mathcal{F}$.

Remark 3.9. Theorem 3.1 provides convergence sequence to a common solution of two variational inequality problems for Lipschitz monotone mappings whereas Corollary 3.6 provides convergence sequence to a common solution point of a finite family of variational inequality problems for Lipschitzian monotone mappings. In addition, Theorem 3.8 provides convergence sequence to a common minimum norm solution of two variational inequality problems for monotone mappings.

Remark 3.10. Theorem 3.1 extends Theorem 3.1 of Nadezhkina and Takahashi [10], Yao et al. [21] and Therem 1 of Yao *et al.* [22] in the sense that our scheme provides strong convergence to a common solution of variational inequality problem for a Lipschitz monotone mappings.

4. Numerical example

Now, we give an example of two monotone mappings satisfying Theorem 3.1 and some numerical experiment result to explain the conclusion of the theorem as follows:

Example 4.1. Let $H = \mathbb{R}$ with absolute value norm. Let $C = [-2, 1]$ and $A_1, A_2 : C \to \mathbb{R}$ be defined by

$$A_1x := \begin{cases} -x^2, x \in [-2, 0], \\ 0, \qquad\quad x \in (0, 1], \end{cases} \qquad \text{and} \qquad A_2x := \begin{cases} 0, \qquad\qquad\quad x \in [-2, \frac{1}{2}], \\ 3(x - \frac{1}{2})^2, x \in (\frac{1}{2}, 1]. \end{cases} \tag{4.1}$$

Clearly, $\mathcal{F} = VI(C, A_1) \cap VI(C, A_2) = [0, 1] \cap [-2, \frac{1}{2}] = [0, \frac{1}{2}]$ and A_1 and A_2 are monotone. Next, we show that A_1 is Lipschitz with $L_1 = 5$. If $x, y \in [-2, 0]$, then

$$|A_1x - A_1y| = |x^2 - y^2| = |x + y||x - y| \leq 4|x - y| \leq 5|x - y|. \tag{4.2}$$

If $x, y \in (0, 1]$, then

$$|A_1x - A_1y| = 0 \leq 5|x - y|.$$

If $x \in [-2, 0]$ and $y \in (0, 1]$, then

$$\begin{aligned} |A_1x - A_1y| &= |x^2 - 0| = |x^2 - y^2 + y^2| \\ &\leq |x + y||x - y| + y^2 \leq 4|x - y| + |y - x| \leq 5|x - y|. \end{aligned}$$

Thus, we get that A_1 is Lipschitz monotone mapping with $L_1 = 5$. Similarly, we can show that A_2 is Lipschitz monotone mapping with $L_2 = 9$.
Now, taking $\alpha_n = \frac{1}{10n+100}$, $\gamma_n = \frac{1}{n+100} + 0.065$, $a_n = b_n = \frac{1}{n+10} + 0.01$, $c_n = 1 - \frac{2}{n+10} - 0.02$, and $f(x) = u \in C$, we observe that conditions of Theorem 3.1 are satisfied and scheme (3.1) provides the data in Tables 1 and 2 and Figures 1 and 2.
(i) When $f(x) = u = 0.6$ and $x_0 = 0.9$, we see that the sequence $\{x_n\}$ in (3.1) converges to $x^* = 0.5$ as shown in Table 1 and Figure 1 (see below).

n	0	50	100	300	400	500	700	800
x_n	0.9000	0.5777	0.5457	0.5195	0.5159	0.5136	0.5108	0.5100

Table 1

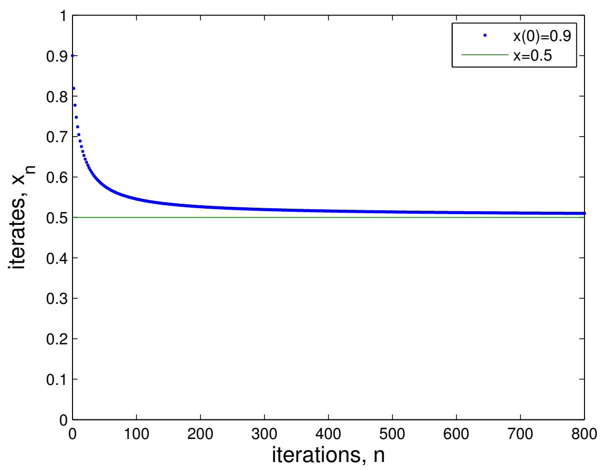

Figure 1

(ii) When $f(x) = u = -1.0$ and $x_0 = 0.8$, we see that the sequence $\{x_n\}$ in (3.1) converges to $x^* = 0$ as shown in Table 2 and Figure 2 (see below).

n	0	50	100	1,000	4000	19000	40000	90000
x_n	0.8000	0.4312	0.3454	0.0765	-0.0604	-0.1001	-0.0712	-0.0455

Table 2

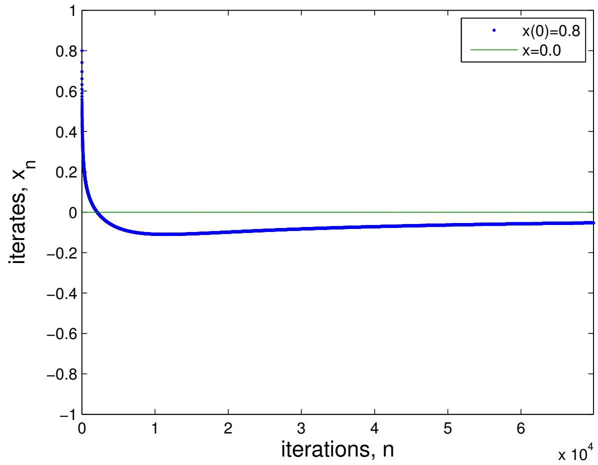

Figure 2

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