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# Stability of cubic and quartic $\rho$-functional inequalities in fuzzy normed spaces

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**Abstract**

In this paper, we solve the following cubic $\rho$-functional inequality

$$\begin{aligned} &N(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x) \\ &-\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right),t) \ge \frac{t}{t+\varphi(x,y)} \end{aligned} \tag{1}$$

and the following quartic $\rho$-functional inequality

$$\begin{aligned} &N(f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y) \\ &-\rho\left(8f\left(x+\frac{y}{2}\right)+8f\left(x-\frac{y}{2}\right)-2f(x+y)-2f(x-y)-12f(x)+3f(y)\right),t) \ge \frac{t}{t+\varphi(x,y)} \end{aligned} \tag{2}$$

in fuzzy normed spaces, where $\rho$ is a fixed real number with $\rho \neq 2$.

Using the direct method, we prove the Hyers-Ulam stability of the cubic $\rho$-functional inequality (1) and the quartic $\rho$-functional inequality (2) in fuzzy Banach spaces. 

*Keywords:* fuzzy Banach space, cubic $\rho$-functional inequality, quartic $\rho$-functional inequality, Hyers-Ulam stability.
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## 1. Introduction and preliminaries

Katsaras [11] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure

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on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [6, 15, 33]. In particular, Bag and Samanta [2], following Cheng and Mordeson [5], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [14]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 18, 19] to investigate the Hyers-Ulam stability of cubic $\rho$-functional inequalities and quartic $\rho$-functional inequalities in fuzzy Banach spaces.

**Definition 1.1** ([2, 18, 19, 20])**.** Let $X$ be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a *fuzzy norm* on $X$ if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

$(N_1)$ $N(x,t) = 0$ for $t \leq 0$;
$(N_2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
$(N_3)$ $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
$(N_4)$ $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$;
$(N_5)$ $N(x,\cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$;
$(N_6)$ for $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

The pair $(X,N)$ is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [18].

**Definition 1.2** ([2, 18, 19, 20])**.** Let $(X,N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is said to *be convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N$-$\lim_{n\to\infty} x_n = x$.

**Definition 1.3** ([2, 18, 19, 20])**.** Let $(X,N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces $X$ and $Y$ is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to $x_0$ in $X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on $X$ (see [3]).

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th. M. Rassias [26] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th. M. Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4, 10, 12, 13, 17, 23, 24, 25, 27, 28, 29, 30, 31]).

In [9], Jun and Kim considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \tag{1.1}$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [16], Lee et al. considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.2}$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Park [21, 22] defined additive $\rho$-functional inequalities and proved the Hyers-Ulam stability of the additive $\rho$-functional inequalities in Banach spaces and non-Archimedean Banach spaces.

In Section 2, we solve the cubic $\rho$-functional inequality (1) and prove the Hyers-Ulam stability of the cubic $\rho$-functional inequality (1) in fuzzy Banach spaces by using the direct method.

In Section 3, we solve the quartic $\rho$-functional inequality (2) and prove the Hyers-Ulam stability of the quartic $\rho$-functional inequality (2) in fuzzy Banach spaces by using the direct method.

Throughout this paper, assume that $\rho$ is a fixed real number with $\rho \neq 2$.

## 2. Cubic $\rho$-functional inequality (1)

**Lemma 2.1.** *Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned} &f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x) \\ &\quad = \rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right) \end{aligned} \tag{2.1}$$

*for all $x, y \in X$. Then $f : X \to Y$ is cubic.*

*Proof.* Letting $y = 0$ in (2.1), we get $2f(2x) - 16f(x) = 0$ and so $f(2x) = 8f(x)$ for all $x \in X$. Thus

$$\begin{aligned} &f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x) \\ &= \rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right) \\ &= \frac{\rho}{2}(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)) \end{aligned}$$

and so $f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x) = 0$ for all $x, y \in X$, as desired. □

We prove the Hyers-Ulam stability of the cubic $\rho$-functional inequality (1) in fuzzy Banach spaces.

**Theorem 2.2.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=1}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{2.2}$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned} &N(f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x) \\ &-\rho\left(4f\left(x+\frac{y}{2}\right)+4f\left(x-\frac{y}{2}\right)-f(x+y)-f(x-y)-6f(x)\right), t) \geq \frac{t}{t+\varphi(x,y)} \end{aligned} \tag{2.3}$$

*for all $x, y \in X$ and all $t > 0$. Then $C(x) := N\text{-}\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$N\left(f(x)-C(x), t\right) \geq \frac{t}{t+\frac{1}{16}\Phi(x,0)} \tag{2.4}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Letting $y = 0$ in (2.3), we get

$$N(2f(2x)-16f(x), t) \geq \frac{t}{t+\varphi(x,0)} \tag{2.5}$$

and so $N\left(f(x)-8f\left(\frac{x}{2}\right), \frac{t}{2}\right) \geq \frac{t}{t+\varphi\left(\frac{x}{2},0\right)}$ for all $x \in X$. Hence

$$N\left(f(x)-8f\left(\frac{x}{2}\right), t\right) \geq \frac{2t}{2t+\varphi\left(\frac{x}{2},0\right)} = \frac{t}{t+\frac{1}{2}\varphi\left(\frac{x}{2},0\right)}$$

for all $x \in X$. Hence

$$\begin{aligned}
&N\left(8^l f\left(\frac{x}{2^l}\right) - 8^m f\left(\frac{x}{2^m}\right), t\right) \qquad (2.6)\\
&\geq \min\left\{N\left(8^l f\left(\frac{x}{2^l}\right) - 8^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, N\left(8^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 8^m f\left(\frac{x}{2^m}\right), t\right)\right\}\\
&= \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 8f\left(\frac{x}{2^{l+1}}\right), \frac{t}{8^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 8f\left(\frac{x}{2^m}\right), \frac{t}{8^{m-1}}\right)\right\}\\
&\geq \min\left\{\frac{\frac{t}{8^l}}{\frac{t}{8^l} + \frac{1}{2}\varphi\left(\frac{x}{2^{l+1}}, 0\right)}, \cdots, \frac{\frac{t}{8^{m-1}}}{\frac{t}{8^{m-1}} + \frac{1}{2}\varphi\left(\frac{x}{2^m}, 0\right)}\right\}\\
&= \min\left\{\frac{t}{t + \frac{8^{l+1}}{16}\varphi\left(\frac{x}{2^{l+1}}, 0\right)}, \cdots, \frac{t}{t + \frac{8^m}{16}\varphi\left(\frac{x}{2^m}, 0\right)}\right\}\\
&\geq \frac{t}{t + \frac{1}{16}\sum_{j=l+1}^{m} 8^j \varphi\left(\frac{x}{2^j}, 0\right)}
\end{aligned}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (2.2) and (2.6) that the sequence $\{8^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{8^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $C : X \to Y$ by

$$C(x) := N\text{-}\lim_{n\to\infty} 8^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

By (2.3),

$$\begin{aligned}
&N\left(8^n\left(f\left(\frac{2x+y}{2^n}\right) + f\left(\frac{2x-y}{2^n}\right) - 2f\left(\frac{x+y}{2^n}\right) - 2f\left(\frac{x-y}{2^n}\right) - 12f\left(\frac{x}{2^n}\right)\right)\right.\\
&\left.-8^n\rho\left(4f\left(\frac{x+\frac{y}{2}}{2^n}\right) + 4f\left(\frac{x-\frac{y}{2}}{2^n}\right) - f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x-y}{2^n}\right) - 6f\left(\frac{x}{2^n}\right)\right), 8^n t\right)\\
&\qquad\geq\geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}
\end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned}
&N\left(8^n\left(f\left(\frac{2x+y}{2^n}\right) + f\left(\frac{2x-y}{2^n}\right) - 2f\left(\frac{x+y}{2^n}\right) - 2f\left(\frac{x-y}{2^n}\right) - 12f\left(\frac{x}{2^n}\right)\right)\right.\\
&\left.-8^n\rho\left(4f\left(\frac{x+\frac{y}{2}}{2^n}\right) + 4f\left(\frac{x-\frac{y}{2}}{2^n}\right) - f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x-y}{2^n}\right) - 6f\left(\frac{x}{2^n}\right)\right), t\right)\\
&\qquad\geq \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = \frac{t}{t + 8^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}.
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{t}{t+8^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\begin{aligned}
&C(2x+y) + C(2x-y) - 2C(x+y) - 2C(x-y) - 12C(x)\\
&\qquad= \rho(4C(x+\frac{y}{2}) + 4C(x-\frac{y}{2}) - C(x+y) - C(x-y) - 6C(x))
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.1, the mapping $C : X \to Y$ is cubic, as desired. □

**Corollary 2.3.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 3$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying*

$$\begin{aligned}
&N\left(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\right. \qquad (2.7)\\
&\left.-\rho\left(4f\left(x+\frac{y}{2}\right) + 4f\left(x-\frac{y}{2}\right) - f(x+y) - f(x-y) - 6f(x)\right), t\right) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}
\end{aligned}$$

*for all $x, y \in X$ and all $t > 0$. Then $C(x) := N\text{-}\lim_{n\to\infty} 8^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$N\left(f(x) - C(x), t\right) \geq \frac{2(2^p - 8)t}{2(2^p - 8)t + \theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. □

**Theorem 2.4.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=0}^{\infty} \frac{1}{8^j} \varphi\left(2^j x, 2^j y\right) < \infty$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.3). Then $C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$N\left(f(x) - C(x), t\right) \geq \frac{t}{t + \frac{1}{16}\Phi(x, 0)}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* It follows from (2.5) that

$$N\left(f(x) - \frac{1}{8}f(2x), \frac{1}{16}t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

and so

$$N\left(f(x) - \frac{1}{8}f(2x), t\right) \geq \frac{16t}{16t + \varphi(x, 0)} = \frac{t}{t + \frac{1}{16}\varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 2.2. □

**Corollary 2.5.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 3$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying (2.7). Then $C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$N\left(f(x) - C(x), t\right) \geq \frac{2(8 - 2^p)t}{2(8 - 2^p)t + \theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. □

## 3. Quartic $\rho$-functional inequality (2)

In this section, we solve and investigate the quartic $\rho$-functional inequality (2) in fuzzy Banach spaces.

**Lemma 3.1.** *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned} &f(2x + y) + f(2x - y) - 4f(x + y) - 4f(x - y) - 24f(x) + 6f(y) \\ &\quad = \rho\left(8f\left(x + \frac{y}{2}\right) + 8f\left(x - \frac{y}{2}\right) - 2f\left(x + y\right) - 2f\left(x - y\right) - 12f\left(x\right) + 3f\left(y\right)\right) \end{aligned} \tag{3.1}$$

*for all $x, y \in X$. Then $f : X \to Y$ is quartic.*

*Proof.* Letting $y = 0$ in (3.1), we get $2f(2x) - 32f(x) = 0$ and so $f(2x) = 16f(x)$ for all $x \in X$. Thus

$$\begin{aligned}
&f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\\
&= \rho\left(8f\left(x+\frac{y}{2}\right) + 8f\left(x-\frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y)\right)\\
&= \frac{\rho}{2}(f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y))
\end{aligned}$$

and so $f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y) = 0$ for all $x, y \in X$. □

We prove the Hyers-Ulam stability of the quartic $\rho$-functional inequality (2) in fuzzy Banach spaces.

**Theorem 3.2.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that*

$$\Phi(x,y) := \sum_{j=1}^{\infty} 16^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty \tag{3.2}$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}
&N\,(f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y) \\
&-\rho\left(8f\left(x+\frac{y}{2}\right) + 8f\left(x-\frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y)\right), t\Big) \geq \frac{t}{t+\varphi(x,y)}
\end{aligned} \tag{3.3}$$

*for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 16^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{t}{t + \frac{1}{32}\Phi(x,0)} \tag{3.4}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Letting $y = 0$ in (3.3), we get

$$N(2f(2x) - 32f(x), t) = N(32f(x) - 2f(2x), t) \geq \frac{t}{t+\varphi(x,0)} \tag{3.5}$$

and so $N\left(f(x) - 16f\left(\frac{x}{2}\right), \frac{t}{2}\right) \geq \frac{t}{t+\varphi\left(\frac{x}{2},0\right)}$ for all $x \in X$. Hence

$$N\left(f(x) - 16f\left(\frac{x}{2}\right), t\right) \geq \frac{2t}{2t + \varphi\left(\frac{x}{2},0\right)} = \frac{t}{t + \frac{1}{2}\varphi\left(\frac{x}{2},0\right)}$$

for all $x \in X$. Hence

$$\begin{aligned}
&N\left(16^l f\left(\frac{x}{2^l}\right) - 16^m f\left(\frac{x}{2^m}\right), t\right) \\
&\geq \min\left\{N\left(16^l f\left(\frac{x}{2^l}\right) - 16^{l+1} f\left(\frac{x}{2^{l+1}}\right), t\right), \cdots, N\left(16^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 16^m f\left(\frac{x}{2^m}\right), t\right)\right\}\\
&= \min\left\{N\left(f\left(\frac{x}{2^l}\right) - 16f\left(\frac{x}{2^{l+1}}\right), \frac{t}{16^l}\right), \cdots, N\left(f\left(\frac{x}{2^{m-1}}\right) - 16f\left(\frac{x}{2^m}\right), \frac{t}{16^{m-1}}\right)\right\}\\
&\geq \min\left\{\frac{\frac{t}{16^l}}{\frac{t}{16^l} + \frac{1}{2}\varphi\left(\frac{x}{2^{l+1}},0\right)}, \cdots, \frac{\frac{t}{16^{m-1}}}{\frac{t}{16^{m-1}} + \frac{1}{2}\varphi\left(\frac{x}{2^m},0\right)}\right\}\\
&= \min\left\{\frac{t}{t + \frac{16^{l+1}}{32}\varphi\left(\frac{x}{2^{l+1}},0\right)}, \cdots, \frac{t}{t + \frac{16^m}{32}\varphi\left(\frac{x}{2^m},0\right)}\right\}\\
&\geq \frac{t}{t + \frac{1}{32}\sum_{j=l+1}^{m} 16^j \varphi\left(\frac{x}{2^j},0\right)}
\end{aligned} \tag{3.6}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$ and all $t > 0$. It follows from (3.2) and (3.6) that the sequence $\{16^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{16^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := N\text{-}\lim_{n\to\infty} 16^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

By the same method as in the proof of Theorem 2.2, it follows from (3.3) that

$$\begin{aligned}&Q(2x+y) + Q(2x-y) - 4Q(x+y) - 4Q(x-y) - 24Q(x) + 6Q(y)\\ &= \rho\left(8Q\left(x+\frac{y}{2}\right) + 8Q\left(x-\frac{y}{2}\right) - 2Q(x+y) - 2Q(x-y) - 12Q(x) + 3Q(y)\right)\end{aligned}$$

for all $x, y \in X$. By Lemma 3.1, the mapping $Q : X \to Y$ is quartic. □

**Corollary 3.3.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 4$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\begin{aligned}&N(f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\\ &-\rho\left(8f\left(x+\frac{y}{2}\right) + 8f\left(x-\frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y)\right), t)\\ &\qquad\geq \frac{t}{t+\theta(\|x\|^p + \|y\|^p}\end{aligned} \tag{3.7}$$

*for all $x, y \in X$ and all $t > 0$. Then $Q(x) := N\text{-}\lim_{n\to\infty} 16^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{2(2^p - 16)t}{2(2^p-16)t + \theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. □

**Theorem 3.4.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{j=0}^{\infty} \frac{1}{16^j}\varphi\left(2^j x, 2^j y\right) < \infty$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.3). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{16^n} f(2^n x)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{t}{t + \frac{1}{32}\Phi(x, 0)}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* It follows from (3.5) that

$$N\left(f(x) - \frac{1}{16}f(2x), \frac{1}{32}t\right) \geq \frac{t}{t+\varphi(x,0)}$$

and so

$$N\left(f(x) - \frac{1}{16}f(2x), t\right) \geq \frac{32t}{32t+\varphi(x,0)} = \frac{t}{t+\frac{1}{32}\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.2. □

**Corollary 3.5.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 4$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{16^n} f(2^n x)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{2(16-2^p)t}{2(16-2^p)t + \theta\|x\|^p}$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. □

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