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Mild solutions of impulsive semilinear neutral evolution equations in Banach spaces

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Abstract

This paper is concerned with the existence of mild solutions for impulsive semilinear neutral functional integro-differential equations in Banach spaces. The existence result is obtained by using fractional power of operators, Mönch fixed point theorem, the piecewise estimation method and semigroup theory. Applications to partial differential systems are also given.

Keywords: Semilinear neutral evolution equations, measure of non-compactness, mild solution, semigroup.
2010 MSC: 47D03, 34K45, 34K40.

1. Introduction

The theory of impulsive differential and partial differential equations in the field of modern applied mathematics has become an important area of investigation in recent years, see the monographs of Lakshmikantham et al. [22], Haddad et al. [16], and Benchohra et al. [7]. Particularly, the theory of impulsive evolution equations has become more important because of its wide applicability in control, mechanics, electrical engineering, biological, and medical fields. There has been a significant development in impulsive evolution equations in Banach spaces. For more details on this theory and its applications, we refer to the references [2, 4–6, 8–11, 14, 17, 19–21, 23, 27].

In this paper, we will study the existence of mild solutions for first order impulsive semilinear neutral functional integro-differential equations of the following form.

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$$\frac{d}{dt}[u(t)-h(t,u_t)] = Au(t)+f(t,u_t,Su(t)), \quad t \in J=[0,a],\ t\neq t_k, \tag{1.1}$$

$$\Delta u|_{t=t_k} = I_k(u(t_k)), \quad k=1,2,\cdots,m, \tag{1.2}$$

$$u(t)=\phi(t), \quad t\in[-r,0], \tag{1.3}$$

where $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t),\ t \geq 0,\ X$ a real Banach space endowed with the norm $|\cdot|$. $f : J \times \mathbb{D} \times X \to X$ and $h : J \times \mathbb{D} \to X$ are given functions, $I_k \in C(X, X)$ $(k = 1, 2, \cdots , m)$, $\phi \in \mathbb{D}$, in which $\mathbb{D} = \{\Psi : [-r, 0] \to X :$ Ψ is continuous everywhere except for a finite number of points $\underline{t}$ at which $\Psi(\underline{t}^-)$ and $\Psi(\underline{t}^+)$ exist and $\Psi(\underline{t}^-) = \Psi(\underline{t})\}$. $r > 0,\ 0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a,\ (m \in \mathbb{N})$. $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

In (1.1), S is a linear operator defined by $(Su)(t) = \int_0^t K(t, s)u(s)ds$ for $u \in X$ and $t \in J$, where $K \in C(F, \mathbb{R}^+)$, in which $F = \{(t, s) \in J \times J : t \geq s\}$ and $\mathbb{R}^+ = [0, +\infty)$.

For any function u defined on $[-r, a] \setminus \{t_1, t_2, \cdots , t_m\}$ and any $t \in J$, we denote by u_t the element of $\mathbb{D}$ defined by $u_t(s) = u(t + s),\ s \in [-r, 0]$. Here, $u_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t. For $w \in \mathbb{D}$, the norm of w is defined by $\|w\|_{\mathbb{D}} = \sup\{|w(s)| : s \in [-r, 0]\}$.

In some papers the existence of mild solutions for various type neutral equations have been studied under the condition that "A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$ in Banach space X such that $|T(t)| \leq M$ for some $M \geq 1$ and $|AT(t)| \leq L,\ L > 0$." From the comments of [12], these conditions imply that the state space X is finite dimensional. Indeed, the identity $T(t) - I = \int_0^t AT(s)ds$ implies that the semigroup $T(t)$ is uniformly continuous and compactness of $T(t)$ implies that $T(0) = I$ is compact and so X is finite dimensional. So some known results hold only in finite dimensional spaces. In this paper, we use the tool of fractional power of operators, Mönch's fixed point theorem combined with the method of piecewise estimation and the semigroup theory to tackle the problem (1.1)-(1.3).

It is well-known that some restrictions on impulse effects are imposed to assure the existence of solutions for impulsive differential equations. Usually, impulsive functions I_k satisfy some Lipschitz conditions or compactness-type conditions. Here we assume that impulsive functions I_k are continuous and locally bounded, the compactness-type and Lipschitz-type restrictions on the impulse terms have been dropped.

In [24, 25], the authors obtained the existence and uniqueness of mild and classical solutions for the following impulsive semilinear evolution equation

$$\begin{cases} u'(t) = Au(t) + f(t,u(t))(\text{or } f(t,u(t),\int_0^t k(t,s)u(s)ds)), \quad 0<t<a,\ t\neq t_k, \\ u(0)=u_0, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k=1,2,\cdots,m,\ 0<t_1<t_2<\cdots<t_m<a, \end{cases}$$

where the impulsive functions I_k are Lipschitz continuous.

Recently, in the special case where $h \equiv 0$ and f does not include Su, Benchohra et al. [3] investigated the existence of mild solutions for the problem (1.1)-(1.3) by the Leray-Schauder nonlinear alternative and the semigroup theory, where the impulsive functions satisfy compactness-type conditions. When f does not include Su, Benchohra et al. [5] obtained the existence of mild solutions for the problem (1.1)-(1.3) by the Schaefer fixed point theorem and the semigroup theory. In [5], the authors assumed that the impulsive functions I_k are continuous and bounded. And [4] dealt with the existence of mild solutions for semilinear neutral functional differential inclusions with impulse effects given by

$$\begin{cases} \frac{d}{dt}[u(t)-h(t,u_t)] \in Au(t)+F(t,u_t), \quad t\in[0,a],\ t\neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k=1,2,\cdots,m, \\ u(t)=\phi(t), \end{cases}$$

where F is a bounded, closed, and convex-valued multivalued map, and I_k are continuous and bounded.

In [1], Abada et al. proved the existence and the controllability of mild and extremal mild solutions for semilinear impulsive differential inclusions:

$$\begin{cases} u'(t) - Au(t) \in F(t, u_t), & 0 \le t \le a,\ t \ne t_k, \\ \Delta u|_{t=t_k} \in I_k(u(t_k)), & k = 1, 2, \cdots, m, \\ u(t) = \phi(t), \end{cases}$$

where F and I_k are multivalued maps with closed, bounded and convex values, the authors assumed that the impulsive functions I_k are Lipschitz continuous with respect to generalized metric.

Motivated by the above works, we consider the first order impulsive semilinear neutral functional integro-differential equations in Banach spaces. In this paper, by using fractional power of operators, Mönch's fixed point theorem, the method of piecewise estimation [28] and the semigroup theory [26], we derive the existence of mild solutions for problem (1.1)-(1.3). We remark that the conditions imposed on the impulse terms are very weak, the compactness-type and Lipschitz-type restrictions on the impulse terms have been dropped and thus the result substantially improves and generalizes some known results. The result presented in this paper is a generalization and a continuation to some results of impulsive differential equations in [3, 5, 24, 25].

The rest of the paper is organized as follows. In Section 2, we give some preliminaries and several lemmas that will be used throughout Section 3. In Section 3, we establish and prove a theorem for the existence of at least one mild solution to problem (1.1)-(1.3). Finally, an application to partial differential equations is given in Section 4.

2. Preliminaries and lemmas

Let X denotes a Banach space, and A be the infinitesimal generator of an analytic semigroup $T(t)$ in X. If A is the infinitesimal generator of an analytic semigroup, then $(A + \nu I)$ is invertible and generates a bounded analytic semigroup for $\nu > 0$ large enough. This allows to reduce the general case in which A is the infinitesimal generator of an analytic semigroup to the case in which semigroup is bounded and the generator invertible. Hence for convenience, we suppose that $|T(t)| \le M$ for $t \ge 0$ and $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. It follows that for $0 < \nu \le 1$, $(-A)^\nu$ can be defined as a closed linear invertible operator with its domain $D(-A)^\nu$ being dense in X. We denote by X_ν the Banach space $D(-A)^\nu$ endowed with norm $\|x\|_\nu = |(-A)^\nu x|$ which is equivalent to the graph norm of $(-A)^\nu$. We have $X_\beta \subset X_\nu$ for $0 < \nu < \beta$ and the embedding is continuous. For more definitions and details of the operator semigroups, we refer to the monographs [26].

Proposition 2.1 ([26])**.**

(1) *If $0 < \nu < \beta \le 1$, then $X_\beta \subset X_\nu$ and the embedding is compact whenever the resolvent operator of A is compact.*

(2) *For every $0 < \nu \le 1$ there exists $\gamma_\nu > 0$ such that*

$$|(-A)^\nu T(t)| \le \frac{\gamma_\nu}{t^\nu}, \quad t > 0.$$

Let

$$J_0 = [0, t_1],\ J_1 = (t_1, t_2], \cdots, J_{m-1} = (t_{m-1}, t_m],\ J_m = (t_m, a].$$

Let $C(J_k, X)$ denotes the Banach space of all continuous mappings $u : J_k \to X$ with norm $\|u_k\| = \|u\|_{J_k} = \max_{t \in J_k} |u(t)|$, where u_k is the restriction of u to J_k, $k = 0, 1, \cdots, m$. Let $PC = \{u : J \to X : u_k \in C(J_k, X)$, $k = 0, 1, 2, \cdots, m,\ u(t)$ is left continuous at $t = t_k$ and its right limit at $t = t_k$ exists for $k = 1, 2, \cdots, m\}$.

Evidently, PC is a Banach space with norm $\|u\|_{PC} = \max\{\|u_k\|,\ k = 0, 1, \cdots, m\}$. Set

$$SPC = \{u : u\ [-r, a] \to\ X : u \in \mathbb{D} \cap PC\},$$

then SPC is a Banach space with norm

$$\|u\|_{SPC} = \sup\{|u(t)| : t \in [-r, a]\}.$$

For any $R > 0$, set $T_R = \{u \in X : |u| \leq R\}$, $O_R = \{u \in \mathbb{D} : \|u\|_{\mathbb{D}} \leq R\}$, and $B_R = \{u \in SPC : \|u\|_{SPC} \leq R\}$. For any $H \subset SPC$ and $t \in J$, let

$$H(t) = \{u(t) : u \in H\} \subset X, \quad H_t = \{u_t : u \in H\} \subset \mathbb{D},$$
$$(SH)(t) = \left\{ \int_0^t K(t,s)u(s)ds : u \in H \right\} \subset X.$$

Without confusion, let $\alpha(\cdot)$ denotes the Kuratowski measure of non-compactness in X and $\mathbb{D}$.

We give the definition of a mild solution of problem (1.1)-(1.3).

Definition 2.2. A function $u \in SPC$ is said to be a mild solution of problem (1.1)-(1.3) if $u(t) = \phi(t)$ on $[-r, 0]$, $\Delta u|_{t=t_k} = I_k(u(t_k))$, $k = 1, 2, \cdots, m$; for each $t \in J$ and $s \in [0, t)$, the function $AT(t-s)h(s, u_s)$ is integrable and u satisfies the integral equation

$$\begin{aligned} u(t) =& T(t)[\phi(0) - h(0,\phi)] + h(t, u_t) + \int_0^t AT(t-s)h(s,u_s)ds \\ &+ \int_0^t T(t-s)f(s, u_s, Su(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k)). \end{aligned}$$

We need the following lemmas in this paper.

Lemma 2.3 ([15, 18])**.** *If $H = \{u_n\} \subset L[J, X]$ and there exists $\rho(t) \in L[J, \mathbb{R}^+]$ such that $|u_n(t)| \leq \rho(t)$ a.e. $t \in J$ for any $u_n \in H$, then $\alpha(H(t)) \in L[J, \mathbb{R}^+]$ and*

$$\alpha\left(\left\{ \int_0^t u_n(s)ds : n \in N \right\}\right) \leq 2\int_0^t \alpha(H(s))ds, \quad t \in J.$$

Lemma 2.4 ([13, Mönch fixed point theorem])**.** *Let X be a Banach space, Ω is bounded open subset of X with $\theta \in \Omega$. Let $A : \Omega \to X$ be a continuous operator satisfying*

(i) $x \neq \lambda Ax$, $\forall \lambda \in [0, 1]$, $x \in \partial\Omega$*;*

(ii) *if $H \subset \overline{\Omega}$ is countable and $H \subset \overline{co}(\{\theta\} \cup A(H))$, then H is relatively compact.*

Then A has a fixed point in Ω.

3. Main results

We now state and prove our existence result for problem (1.1)-(1.3).

Theorem 3.1. *Suppose the following conditions hold:*

(H_1) *A is the infinitesimal generator of a analytic semigroup of bounded linear operators $T(t)$ in X, and there exists $M \geq 1$ such that $|T(t)| \leq M$, $t \in J$;*

(H_2) *for any $R > 0$, f is uniformly continuous on $J \times O_R \times T_R$ and I_k ($k = 1, 2, \cdots, m$) is bounded on T_R;*

(H_3) *h is uniformly continuous on $J \times \mathbb{D}$, there exists $0 < \beta < 1$ such that h is X_β-valued, $(-A)^\beta h$ is continuous, and there exist $0 \leq c_1 < \frac{1}{M_0}$, $c_2 > 0$ such that*

$$|(-A)^\beta h(t,u)| \leq c_1\|u\|_{\mathbb{D}} + c_2, \quad t \in J,\ u \in \mathbb{D},$$

where $M_0 = M_1 + \gamma_{1-\beta}\frac{a^\beta}{\beta}$, $M_1 = |(-A)^{-\beta}|$;

(H_4) *there exist* $p \in L^1(J, \mathbb{R}^+)$ *and continuous nondecreasing functions* $\psi_1 : \mathbb{R}^+ \to (0, \infty)$, $\psi_2 : \mathbb{R}^+ \to \mathbb{R}^+$, ψ_2 *satisfying* $\psi_2(\lambda x) \leq \lambda \psi_2(x)$ *for* $\lambda > 0$, *such that*

$$|f(t, u, v)| \leq p(t) \left[\psi_1(\|u\|_{\mathbb{D}}) + \psi_2(|v|)\right]$$

for $t \in J$ *and every* $u \in \mathbb{D}$, $v \in X$, *and with*

$$\int_{t_{k-1}}^{t_k} P(s) ds < \int_{N_{k-1}}^{\infty} \frac{ds}{s + \sum_{i=1}^{2} \psi_i(s)}, \quad k = 1, 2, \cdots, m+1,$$

where

$$\begin{aligned}
P(s) &= \frac{Mp(s)}{1 - M_0 c_1} \max\{1,\ aK^*\},\ K^* = \max\{K(t,s) : (t,s) \in F\},\\
N_0 &= \frac{1}{1 - M_0 c_1} \left[M(\|\phi\|_{\mathbb{D}} + M_1 c_1 \|\phi\|_{\mathbb{D}} + M_1 c_2) + M_1 c_2 + \gamma_{1-\beta} c_2 \frac{t_1^\beta}{\beta}\right],\\
N_i &= \frac{1}{1 - M_0 c_1} \left\{M(\|\phi\|_{\mathbb{D}} + M_1 c_1 \|\phi\|_{\mathbb{D}} + M_1 c_2) + M_1 c_2 + \frac{\gamma_{1-\beta}}{\beta} c_1 \sum_{j=0}^{i-1} C_j (t_{j+1}^\beta - t_j^\beta)\right.\\
&\quad \left. + \frac{\gamma_{1-\beta}}{\beta} c_2 t_{i+1}^\beta + \sum_{j=0}^{i-1} [M(t_{j+1} - t_j) + M] \beta_j \right\},\\
\beta_{i-1} &= \sup\{|f(t, u_t, Su(t))|,\ |I_i(u)| :\ t \in J_{i-1},\ \|u_t\|_{\mathbb{D}} \leq C_{i-1},\ |u| \leq C_{i-1}\},\\
C_{i-1} &= \Gamma_i^{-1} \left(\int_{t_{i-1}}^{t_i} P(s) ds\right), \quad i = 1, 2, \cdots, m,
\end{aligned}$$

and

$$\Gamma_l(z) = \int_{N_{l-1}}^{z} \frac{ds}{\sum_{i=1}^{2} \psi_i(s)}, \quad z \geq N_{l-1},\ l = 1, 2, \cdots, m+1;$$

(H_5) *there exist constants* $0 \leq L_0 < \frac{1}{M_1 + \frac{\gamma_{1-\beta}}{\beta} a^\beta}$, $L_1 \geq 0$ *and* $L_2 \geq 0$ *such that*

$$\begin{aligned}
\alpha((-A)^\beta h(t, H_t)) &\leq L_0 \alpha(H_t),\\
\alpha(f(t, H_t, (SH)(t))) &\leq L_1 \alpha(H_t) + L_2 \alpha((SH)(t))
\end{aligned}$$

for any bounded set $H \subset SPC$ *and* $t \in J$.

Then problem (1.1)-(1.3) *has at least one mild solution* $u \in SPC$.

Proof. Define an operator $G : SPC \to SPC$ as follows:

$$(Gu)(t) = \begin{cases} \phi(t), \quad t \in [-r, 0], \\ T(t)[\phi(0) - h(0, \phi)] + h(t, u_t) + \displaystyle\int_0^t AT(t-s) h(s, u_s) ds \\ \quad + \displaystyle\int_0^t T(t-s) f(s, u_s, Su(s)) ds + \sum_{0 < t_k < t} T(t - t_k) I_k(u(t_k)), \quad t \in J. \end{cases}$$

For each $R > 0$, by Proposition 2.1 and the assumption (H_3), the following inequality holds:

$$|AT(t-s)h(s, u_s)| = |(-A)^{1-\beta} T(t-s) (-A)^\beta h(s, u_s)| \leq \frac{\gamma_{1-\beta}}{(t-s)^{1-\beta}} (c_1 R + c_2). \tag{3.1}$$

From the assumptions, we know that both $s \to (-A)^\beta h(s,u_s)$ and $s \to h(s,u_s)$ are continuous. In addition, in view of the fact that T is an analytic semigroup, the operator function $s \to AT(t-s)$ is continuous in the uniform operator topology on $[0,t)$ and, consequently $AT(t-s)h(s,u_s)$ is continuous on $[0,t)$. On the other hand, the estimate (3.1) and the Bochner theorem imply that $|AT(t-s)h(s,u_s)|$ is integrable on $[0,t)$. Thus G is well-defined on B_R. It is well-known that the fixed points of G are mild solutions to problem (1.1)-(1.3).

We first prove that operator $G : SPC \to SPC$ is continuous. In fact, let $v_n, v \in SPC$ and $\|v_n - v\|_{SPC} \to 0$ $(n \to \infty)$, then there exists $R > 0$ such that $\|v_n\|_{SPC} \le R$, $\|v\|_{SPC} \le R$, i.e., $v_n, v \in B_R$. So we have

$$\begin{aligned}|(Gv_n)(t) - (Gv)(t)| \le & |h(t, v_{n_t}) - h(t, v_t)| + \int_0^t |AT(t-s)[h(s, v_{n_s}) - h(s, v_s)]| ds \\ & + \int_0^t |T(t-s)|\, |f(s, v_{n_s}, Sv_n(s)) - f(s, v_s, Sv(s))|\, ds \\ & + \sum_{0<t_k<t} |T(t-t_k)||I_k(v_n(t_k)) - I_k(v(t_k))|, \quad t \in J.\end{aligned}$$

Since

$$\begin{aligned}|f(t, v_{n_t}, Sv_n(t)) - f(t, v_t, Sv(t))| &\le 2p(t)[\psi_1(R) + \psi_2(aK^*R)], \\ |AT(t-s)[h(s, v_{n_s}) - h(s, v_s)]| &\le \frac{2\gamma_{1-\beta}}{(t-s)^{1-\beta}}(c_1 R + c_2), \quad \forall t \in [-r, a].\end{aligned}$$

By (H_2), (H_3), and the Lebesgue dominated convergence theorem, we have

$$\|G(v_n) - G(v)\|_{SPC} \to 0, \quad n \to \infty.$$

Thus, $G : SPC \to SPC$ is continuous.

Next, we shall show that

$$\Omega_0 = \{u \in SPC : u = \lambda Gu \text{ for some } \lambda \in [0,1]\}$$

is bounded. In fact, if $u \in \Omega_0$, then there exists $\lambda_0 \in [0,1]$ such that

$$u(t) = \lambda_0 (Gu)(t), \quad t \in [-r, a].$$

For the case $t \in [-r, t_1]$, we have

$$u(t) = \begin{cases} \lambda_0 \phi(t), \quad t \in [-r, 0], \\ \lambda_0 T(t)[\phi(0) - h(0,\phi)] + \lambda_0 h(t, u_t) + \lambda_0 \int_0^t AT(t-s)h(s, u_s) ds \\ + \lambda_0 \int_0^t T(t-s) f(s, u_s, Su(s)) ds, \quad t \in J_0. \end{cases}$$

From (H_1), (H_3), and (H_4), for any $t \in J_0$, we have

$$\begin{aligned}|u(t)| \le & M[\|\phi\|_{\mathbb{D}} + M_1(c_1\|\phi\|_{\mathbb{D}} + c_2)] + M_1(c_1\|u_t\|_{\mathbb{D}} + c_2) + \gamma_{1-\beta} c_1 \int_0^t \frac{\|u_s\|_{\mathbb{D}}}{(t-s)^{1-\beta}} ds \\ & + \gamma_{1-\beta} c_2 \frac{t_1^\beta}{\beta} + M \int_0^t p(s)\, [\psi_1(\|u_s\|_{\mathbb{D}}) + \psi_2(|(Su)(s)|)]\, ds.\end{aligned}$$

Let $\mu_0(t) = \sup\{|u(s)| : -r \le s \le t\}$, $t \in J_0$, and let $t^* \in [-r, t]$ be such that $\mu_0(t) = |u(t^*)|$. If $t^* \in J_0$, then by the previous inequality, we have

$$\mu_0(t) \le l_0 + c_1 \left(M_1 + \gamma_{1-\beta} \frac{a^\beta}{\beta} \right) \mu_0(t) + M \int_0^t p(s)[\psi_1(\mu_0(s)) + aK^* \psi_2(\mu_0(s))] ds, \quad t \in J_0,$$

where

$$l_0 = M(\|\phi\|_{\mathbb{D}} + M_1 c_1 \|\phi\|_{\mathbb{D}} + M_1 c_2) + M_1 c_2 + \gamma_{1-\beta} c_2 \frac{t_1^\beta}{\beta},$$

then

$$\mu_0(t) \leq \frac{1}{1 - M_0 c_1} \left\{ l_0 + M \int_0^t p(s)[\psi_1(\mu_0(s)) + aK^* \psi_2(\mu_0(s))]ds \right\}, \quad t \in J_0. \tag{3.2}$$

If $t^* \in [-r, 0]$, then $\mu_0(t) \leq \|\phi\|_{\mathbb{D}}$ and (3.2) holds, since $0 \leq M_0 c_1 < 1$ and $M \geq 1$.

Let us take the right-hand side of the (3.2) as $v_0(t)$, then we have

$$\begin{aligned} v_0(0) &= \frac{l_0}{1 - M_0 c_1}, \quad \mu_0(t) \leq v_0(t), \\ v_0'(t) &= \frac{M p(t) \left[\psi_1(\mu_0(t)) + aK^* \psi_2(\mu_0(t))\right]}{1 - M_0 c_1} \\ &\leq \frac{M p(t) \left[\psi_1(v_0(t)) + aK^* \psi_2(v_0(t))\right]}{1 - M_0 c_1} \leq P(t) \left(\sum_{i=1}^2 \psi_i(v_0(t)) \right), \quad t \in J_0. \end{aligned}$$

This implies for every $t \in J_0$ that

$$\int_{v_0(0)}^{v_0(t)} \frac{ds}{\sum_{i=1}^2 \psi_i(s)} \leq \int_0^{t_1} P(s)ds.$$

In view of (H_4), we obtain

$$v_0(t) \leq \Gamma_1^{-1} \left(\int_0^{t_1} P(s)ds \right) := C_0.$$

Since $\|u_t\|_{\mathbb{D}} \leq \mu_0(t) \leq v_0(t) \leq C_0$ for every $t \in J_0$, we have $\sup_{t \in [-r, t_1]} |u(t)| \leq C_0$. By ($H_2$), there exists $\beta_0 > 0$ such that

$$|f(t, u_t, Su(t))| \leq \beta_0, \ |I_1(u)| \leq \beta_0, \quad \text{for } \|u_t\|_{\mathbb{D}} \leq C_0, \ |u| \leq C_0, \ t \in J_0,$$

then

$$|u(t_1^+)| = |u(t_1) + \lambda_0 I_1(u(t_1))| \leq C_0 + \beta_0.$$

Consider the case $t \in J_1$, let

$$z(t) = \begin{cases} u(t), & t_1 < t \leq t_2, \\ u(t_1^+), & t = t_1, \end{cases} \quad z_t(s) = \begin{cases} z(t+s), & t_1 \leq t+s \leq t_2, \\ u(t+s), & t_1 - r \leq t+s < t_1, \end{cases}$$

then $z(t) \in C([t_1, t_2], X)$ and

$$\begin{aligned} z(t) =& \lambda_0 \left\{ T(t)[\phi(0) - h(0, \phi)] + h(t, z_t) + \int_0^{t_1} AT(t-s)h(s, u_s)ds \right. \\ &+ \int_{t_1}^t AT(t-s)h(s, z_s)ds + \int_0^{t_1} T(t-s)f(s, u_s, Su(s))ds \\ &\left. + \int_{t_1}^t T(t-s)f(s, z_s, Sz(s))ds + T(t-t_1)I_1(u(t_1)) \right\}. \end{aligned}$$

From (H_3) and (H_4), for any $t \in [t_1, t_2]$, we have

$$\begin{aligned} |z(t)| \leq& M[\|\phi\|_{\mathbb{D}} + M_1 c_1 \|\phi\|_{\mathbb{D}} + M_1 c_2] + M_1 c_1 \|z_t\|_{\mathbb{D}} + M_1 c_2 \\ &+ c_1 C_0 \gamma_{1-\beta} \frac{t_1^\beta}{\beta} + c_2 \gamma_{1-\beta} \frac{t_2^\beta}{\beta} + c_1 \int_{t_1}^t \frac{\gamma_{1-\beta}}{(t-s)^{1-\beta}} \|z_s\|_{\mathbb{D}} ds + (Mt_1 + M)\beta_0 \\ &+ M \int_{t_1}^t p(s)[\psi_1(\|z_s\|_{\mathbb{D}}) + \psi_2(|(Sz)(s)|)]ds. \end{aligned}$$

Let

$$\mu_1(t) = \max\left\{\sup_{t_1-r\le s<t_1}|u(s)|,\ \sup_{t_1\le s\le t}|z(s)|\right\},\quad t\in[t_1,t_2],$$

then for $t\in[t_1,t_2]$, we have

$$\mu_1(t)\le l_1 + c_1\left(M_1+\gamma_{1-\beta}\frac{a^\beta}{\beta}\right)\mu_1(t) + M\int_{t_1}^t p(s)[\psi_1(\mu_1(s))+aK^*\psi_2(\mu_1(s))]ds,$$

where

$$l_1 = M(\|\phi\|_{\mathbb{D}}+M_1c_1\|\phi\|_{\mathbb{D}}+M_1c_2)+M_1c_2+M\beta_0(t_1+1)+\frac{\gamma_{1-\beta}}{\beta}(c_1C_0t_1^\beta+c_2t_2^\beta),$$

then

$$\mu_1(t)\le\frac{1}{1-M_0c_1}\left\{l_1+M\int_{t_1}^t p(s)[\psi_1(\mu_1(s))+aK^*\psi_2(\mu_1(s))]ds\right\},\quad t\in J_1.$$

Let us take the right-hand side of the above inequality as $v_1(t)$, then we have

$$v_1(t_1)=\frac{l_1}{1-M_0c_1},\quad \mu_1(t)\le v_1(t),\quad t\in[t_1,t_2],$$

and

$$\begin{aligned} v_1'(t) =& \frac{Mp(t)[\psi_1(\mu_1(t))+aK^*\psi_2(\mu_1(t)]}{1-M_0c_1}\\ \le& \frac{Mp(t)[\psi_1(v_1(t))+aK^*\psi_2(v_1(t))]}{1-M_0c_1}\le P(t)\left(\sum_{i=1}^2\psi_i(v_1(t))\right),\quad t\in[t_1,t_2]. \end{aligned}$$

This implies for every $t\in[t_1,t_2]$ that

$$\int_{v_1(t_1)}^{v_1(t)}\frac{ds}{\sum_{i=1}^2\psi_i(s)}\le\int_{t_1}^{t_2}P(s)ds.$$

In view of (H_4), we obtain

$$v_1(t)\le\Gamma_2^{-1}\left(\int_{t_1}^{t_2}P(s)ds\right):=C_1,\quad t\in[t_1,t_2].$$

Since $\|z_t\|_{\mathbb{D}}\le\mu_1(t)$ for every $t\in J_1$, we have

$$\sup_{t\in[t_1,t_2]}|z(t)|\le C_1,$$

and thus $|u(t)|\le C_1,\ t\in J_1$.

We continue this process, and establish that there exists $C_k>0$ such that $|u(t)|\le C_k$ for $t\in J_k$ ($k=2,3,\cdots,m$). Let $C=\max\{C_i:0\le i\le m\}$, then $|u(t)|\le C,\ t\in[-r,a]$ and $\|u\|_{SPC}\le C$ for any $u\in\Omega_0$, i.e., Ω_0 is bounded.

Next, we verify that conditions (i) and (ii) in Lemma 2.4 hold. Choose $R_0>C$ and set

$$\Omega=\{u\in SPC:\|u\|_{SPC}\le R_0\},$$

then Ω is a bounded open subset of SPC with $\theta\in\Omega$, and for any $\lambda\in[0,1]$ and $u\in\partial\Omega,\ u\ne\lambda Gu$. Then, condition (i) of Lemma 2.4 holds.

Suppose $H\subset\overline{\Omega}$ is countable and $H\subset\overline{co}(\{\theta\}\cup(GH))$. By (H_1)-(H_3), $G(H)$ is equicontinuous on $[-r,0]$ and on each J_k ($k=0,1,\cdots,m$), and thus H is equicontinuous on $[-r,0]$ and on each J_k ($k=0,1,\cdots,m$). It follows from the property of the Kuratowski measure of non-compactness, Lemma 2.3, and (H_5) that

$$\alpha(H(t))\le\alpha((GH)(t))=\alpha(\{\phi(t)\})=0,\quad t\in[-r,0],$$

and for $t \in J_0$,

$$\begin{aligned}
\alpha(H(t)) \leq & \alpha((GH)(t)) \\
\leq & \alpha\left(\left\{h(t,u_t)+\int_0^t AT(t-s)h(s,u_s)ds+\int_0^t T(t-s)f(s,u_s,Su(s))ds : u\in H\right\}\right) \\
\leq & M_1\alpha((-A)^\beta h(t,H_t))+\alpha\left(\int_0^t \frac{\gamma_{1-\beta}}{(t-s)^{1-\beta}}(-A)^\beta h(s,H_s)ds\right)+M\alpha\left(\int_0^t f(s,H_s,(SH)(s))ds\right) \\
\leq & M_1L_0\alpha(H_t)+2\left(\int_0^t \frac{\gamma_{1-\beta}}{(t-s)^{1-\beta}}\alpha((-A)^\beta h(s,H_s))ds\right)+2M\int_0^t [L_1\alpha(H_s)+L_2\alpha((SH)(s))]ds \\
\leq & M_1L_0\alpha(H_t)+2L_0\int_0^t \frac{\gamma_{1-\beta}}{(t-s)^{1-\beta}}\alpha(H_s)ds+2M\int_0^t [L_1\alpha(H_s)+L_2K^*\alpha(H(s))]ds.
\end{aligned}$$

Let $m_0(t)=\sup_{-r\leq s\leq t}\alpha(H(s)),\ t\in J_0$, then by the previous inequality, we have

$$m_0(t)\leq M_1L_0m_0(t)+2L_0m_0(t)\frac{\gamma_{1-\beta}a^\beta}{\beta}+2M(L_1+L_2K^*)\int_0^t m_0(s)ds,$$

then

$$m_0(t)\leq \frac{2M(L_1+L_2K^*)}{1-L_0(M_1+\frac{\gamma_{1-\beta}}{\beta}a^\beta)}\int_0^t m_0(s)ds, \quad t\in J_0.$$

By the Gronwall inequality, we have $m_0(t)=0$, so $\alpha(H(t))=0,\ t\in J_0$, hence H is a relatively compact set in $C(J_0,X)$.

Since $I_1\in C(X,X),\ \alpha(H(t_1))=0$, it follows that $\alpha(I_1(H(t_1))=0$. Then for any $t\in J_1$, we obtain

$$\begin{aligned}
\alpha(H(t)) \leq & \alpha\left(\left\{h(t,u_t)+\int_0^t AT(t-s)h(s,u_s)ds+\int_0^t T(t-s)f(s,u_s,Su(s))ds\right.\right. \\
& \left.\left. +T(t-t_1)I_1(u(t_1)) : u\in H\right\}\right) \\
\leq & \alpha(h(t,H_t))+\alpha\left(\int_0^t AT(t-s)h(s,H_s)ds\right)+M\alpha\left(\int_0^t f(s,H_s,(SH)(s))ds\right)+M\alpha\left(I_1(H(t_1))\right) \\
\leq & M_1L_0\alpha(H_t)+2L_0\int_0^t \frac{\gamma_{1-\beta}}{(t-s)^{1-\beta}}\alpha(H_s)ds+2M\int_0^t [L_1\alpha(H_s)+L_2\alpha((SH)(s))]ds \\
= & M_1L_0\alpha(H_t)+2L_0\int_{t_1}^t \frac{\gamma_{1-\beta}}{(t-s)^{1-\beta}}\alpha(H_s)ds+2M\int_{t_1}^t [L_1\alpha(H_s)+L_2\alpha((SH)(s))]ds.
\end{aligned}$$

Let $m_1(t)=\sup_{t_1\leq s\leq t}\alpha(H(s)),\ t\in J_1$, then

$$m_1(t)\leq M_1L_0m_1(t)+2L_0\frac{\gamma_{1-\beta}}{\beta}a^\beta m_1(t)+2M(L_1+L_2K^*)\int_{t_1}^t m_1(s)ds,$$

and,

$$m_1(t)\leq \frac{2M(L_1+L_2K^*)}{1-L_0(M_1+\frac{\gamma_{1-\beta}}{\beta}a^\beta)}\int_{t_1}^t m_1(s)ds, \quad t\in J_1.$$

By the Gronwall inequality, we have $m_1(t)=0$, so $\alpha(H(t))=0,\ t\in J_1$, and hence H is a relatively compact set in $C(J_1,X)$.

Similarly, by continuing this process, we establish that H is a relatively compact set in $C(J_k,X)$ $(k=2,3,\cdots,m)$, and thus $H\subset SPC$ is a relatively compact set, i.e., the condition (ii) of Lemma 2.4 is satisfied. As a consequence of Lemma 2.4, we get that G has a fixed point in Ω, which is a mild solution of problem (1.1)-(1.3).

□

4. Applications to partial differential system

As an application of main results, we study the following impulsive partial neutral functional differential system

$$\frac{\partial}{\partial t}\left[z(t,y)-\int_0^\pi U(y,s)z_t(\theta,s)ds\right]=\frac{\partial^2}{\partial y^2}z(t,y)+\frac{\partial}{\partial y}G(t,z_t(\theta,y)),\quad t\in J=[0,a],\ t\neq t_k, \tag{4.1}$$

$$z(t,0)=z(t,\pi)=0,\quad t\in[0,a], \tag{4.2}$$

$$z(t_k^+,y)-z(t_k^-,y)=\int_0^{t_k} q_k(t_k-s)z(s,y)ds,\quad k=1,2,\cdots,m, \tag{4.3}$$

$$z(\theta,y)=(\phi(\theta))(y),\quad \theta\in[-r,0], \tag{4.4}$$

where $a>0,\ y\in[0,\pi],\ 0<t_1<t_2<\cdots<t_m<a$, G is a given function, and $q_k:\mathbb{R}^+\to\mathbb{R}$ are continuous functions for $k=1,2,\cdots,m$. Let $X=L^2[0,\pi]$ with the norm $\|\cdot\|_{L^2[0,\pi]},\ \phi\in\mathbb{D}$, that is, $\phi(\theta)\in X=L^2[0,\pi]$, and $z_t(\theta,y)=z(t+\theta,y),\ t\in[0,a],\ \theta\in[-r,0]$.

We define an operator $A:X\to X$ by $Aw=w''$ with the domain

$$D(A)=\{w\in X: w,w' \text{ are absolutely continuous},\ w''\in X,\ w(0)=w(\pi)=0\}.$$

Then A generates a strongly continuous semigroup $T(t)$ which is compact, analytic and self-adjoint. Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2$, with corresponding normalized eigenvectors $w_n(s)=\sqrt{\frac{2}{\pi}}\sin ns,\ n\in\mathbb{N}$. We also use the following properties:

(a) $\{w_n:n\in\mathbb{N}\}$ is an orthonormal basis of X.

(b) If $w\in D(A)$, then $Aw=-\sum_{n=1}^\infty n^2\langle w,w_n\rangle w_n$.

(c) For $w\in X,\ (-A)^{-\frac{1}{2}}w=\sum_{n=1}^\infty \frac{1}{n}\langle w,w_n\rangle w_n$.

(d) The operator $(-A)^{\frac{1}{2}}$ is given as $(-A)^{\frac{1}{2}}w=\sum_{n=1}^\infty n\langle w,w_n\rangle w_n$ on the space $D[(-A)^{\frac{1}{2}}]=\{w\in X:\sum_{n=1}^\infty n\langle w,w_n\rangle w_n\in X\}$.

The system (4.1)-(4.4) can be reformulated as the abstract form

$$\begin{cases}\dfrac{d}{dt}[u(t)-h(t,u_t)]=Au(t)+f(t,u_t),\quad t\in J,\ t\neq t_k,\\ \Delta u|_{t=t_k}=I_k(u(t_k)),\quad k=1,2,\cdots,m,\\ u(t)=\phi(t),\quad t\in[-r,0],\end{cases}$$

where $u_t=z_t(\theta,\cdot)$, that is $(u(t+\theta))(y)=z(t+\theta,y),\ t\in J,\ y\in[0,\pi],\ \theta\in[-r,0]$. The function $h:J\times\mathbb{D}\to X$ is given by

$$(h(t,u_t))(y)=\int_0^\pi U(y,s)z_t(\theta,s)ds.$$

Let $(Bv)(y)=\int_0^\pi U(y,s)v(s)ds$ for $v\in X,\ y\in[0,\pi]$. The function $f:J\times\mathbb{D}\to X$ is given by

$$(f(t,u_t))(y)=\frac{\partial}{\partial y}G(t,z_t(\theta,y)),$$

and the functions $I_k:X\to X$ are given by

$$(I_k(u(t)))(y)=\int_0^t q_k(t-s)z(s,y)ds,\quad t\in J,\ k=1,2,\cdots,m.$$

We can take $(f(t,u_t))(y)=k_0(y)\sin u_t(\theta,y)$. Suppose further that:

(i) The function $U(y,s),\ y,s\in[0,\pi]$ is measurable and

$$\left(\int_0^\pi\int_0^\pi U^2(y,s)dsdy\right)^{\frac{1}{2}}<\infty.$$

(ii) The function $\frac{\partial U(y,s)}{\partial y}$ is measurable, $U(0,s)=U(\pi,s)=0$, and let

$$L=\left[\int_0^\pi\int_0^\pi\left(\frac{\partial}{\partial y}U(y,s)\right)^2dsdy\right]^{\frac{1}{2}}<\infty.$$

(iii) The function $k_0(y)$ is continuous on $[0,\pi]$.

Take $p(t)=\|k_0(y)\|_{L^2[0,\pi]},\ \psi_1(x)=x+1$, then

$$\|f(t,u_t)\|_{L^2[0,\pi]}\le\|k_0(y)\|_{L^2[0,\pi]}\|u_t\|_{\mathbb{D}}\le p(t)\psi_1(\|u_t\|_{\mathbb{D}}).$$

From (i) it is clear that B is a bounded linear operator on X. Furthermore, $B(v)\in D[(-A)^{\frac{1}{2}}]$, and $\|(-A)^{\frac{1}{2}}B\|_{L^2[0,\pi]}\le L$. In fact from the definition of B and (ii) it follows that

$$\begin{aligned}\langle B(v),w_n\rangle=&\int_0^\pi\left(\int_0^\pi U(y,s)v(s)ds\right)w_n(y)dy\\=&\frac{1}{n}\sqrt{\frac{2}{\pi}}\left\langle\int_0^\pi\frac{\partial}{\partial y}U(y,s)v(s)ds,\cos(ny)\right\rangle=\frac{1}{n}\sqrt{\frac{2}{\pi}}\langle B_1(v),\cos(ny)\rangle,\end{aligned}$$

where $B_1(v)=\int_0^\pi\frac{\partial}{\partial y}U(y,s)v(s)ds$. From (ii) we know that $B_1:X\to X$ is a bounded linear operator with $\|B_1\|_{L^2[0,\pi]}\le L$. Hence $\|(-A)^{\frac{1}{2}}B(x)\|_{L^2[0,\pi]}=\|B_1\|_{L^2[0,\pi]}$, which implies the assertion.

On the other hand, for any $u_t^1,\ u_t^2\in\mathbb{D}$, from (ii) and (iii) we have

$$\|(-A)^{\frac{1}{2}}h(t,u_t^1)-(-A)^{\frac{1}{2}}h(t,u_t^2)\|_{L^2[0,\pi]}\le L\|u_t^1-u_t^2\|_{\mathbb{D}},$$

and

$$\|f(t,u_t^1)-f(t,u_t^2)\|_{L^2[0,\pi]}\le\|k_0(y)\|_{L^2[0,\pi]}\|u_t^1-u_t^2\|_{\mathbb{D}}.$$

Then, for any bounded set $H\subset X$ and $t\in J$,

$$\alpha((-A)^{\frac{1}{2}}h(t,H_t))\le L\alpha(H_t),\quad\alpha(f(t,H_t))\le\|k_0(y)\|_{L^2[0,\pi]}\alpha(H_t).$$

Hence, from Theorem 3.1, problem (4.1)-(4.4) admits a mild solution provided that the inequalities of (H_4) hold.

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