



# Existence and convergence theorems for the split quasi variational inequality problems on proximally smooth sets

Jittiporn Tangkhawiwetkul<sup>a,\*</sup>, Narin Petrot<sup>b</sup>

<sup>a</sup>Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok, 65000, Thailand.

<sup>b</sup>Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand.

Communicated by Y. J. Cho

---

## Abstract

In this paper, we consider the split quasi variational inequality problems over a class of nonconvex sets, as uniformly prox-regular sets. The sufficient conditions for the existence of solutions of such a problem are provided. Furthermore, an iterative algorithm for finding a solution is constructed and its convergence analysis are considered. The results in this paper improve and extend the variational inequality problems which have been appeared in literature. ©2016 All rights reserved.

*Keywords:* Split quasi variational inequality, proximally smooth set, uniformly prox-regular set, Lipschitzian mapping, strongly monotone mapping.

*2010 MSC:* 49J40, 47J20.

---

## 1. Introduction

A well known problem, which was studied and interesting for many researchers, is the variational inequality problem. The variational inequality problem is a problem of finding  $x^* \in K$  such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in K, \quad (1.1)$$

where  $T$  is a nonlinear operator on  $H$ ,  $K$  is a nonempty closed and convex subset of a Hilbert space  $H$ . This problem was introduced by Stampacch[31] in 1960s, and it is a power tool which has been used in branches

---

\*Corresponding author

Email addresses: [j\\_suwannawit@hotmail.com](mailto:j_suwannawit@hotmail.com) (Jittiporn Tangkhawiwetkul), [narinp@nu.ac.th](mailto:narinp@nu.ac.th) (Narin Petrot)

of both pure and applied sciences. Subsequently, the most nature, direct, simple and efficient framework for general treatment of wide range of problems are provided for the variational inequalities. Roughly speaking, many researchers interest to develop several numerical methods for solving variational inequalities and relaxed optimization problems(see [2, 10, 11, 12, 13, 33, 34, 35, 36] and the references therein).

In the early 1970s, Bensoussan et al.[3] developed the concept of variational inequality, by introducing the following concept of quasi-variational inequality problem: find  $x^* \in C(x^*)$  such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in C(x^*), \tag{1.2}$$

where  $C$  is a set-valued mapping on  $H$ . We see that if  $C(x) = K$  for all  $x \in H$ , then the problem (1.2) is reduced to the problem (1.1). Notice that, since in many important problems the considered set also depend upon the solutions explicitly or implicitly, evidently, the problem (1.2) is interesting to study, see [9, 22, 23, 24].

On the other hand, in 2012, Cencer et al. [8] introduced the following concept of split variational inequality problem: let  $H_1, H_2$  be real Hilbert spaces and  $K, Q$  be nonempty closed and convex subsets of  $H_1$  and  $H_2$ , respectively,  $T : H_1 \rightarrow H_1, S : H_2 \rightarrow H_2$  be nonlinear mappings and  $A : H_1 \rightarrow H_2$  be a bounded linear operator then they are interested in finding  $x^* \in K$  such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in K,$$

and such that  $Ax^* \in Q$  solves

$$\langle S(Ax^*), y - Ax^* \rangle \geq 0, \text{ for all } y \in Q. \tag{1.3}$$

This problem extends and permits the split minimization between two spaces so the image of minimizer of a given function, under a bounded linear operator, is a minimizer at another function. Furthermore, the split zero problem and split feasibility problem which was studied and used in a model of intensity-modulated radiation therapy treatment planning are contained as special cases of this problem, see [6, 7, 17]. This formulation is also at core of the modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see [5, 17, 21].

By the way, in the early period of these development, it should be pointed out that almost all the results regarding the existence and iterative schemes for solving those variational inequality problems are being considered in the convexity setting. This is because, perhaps, they need the convexity assumption for guaranteeing the well definedness of the proposed iterative algorithm, which almost depends on the projection properties. However, in fact, the convexity assumption may not be required, because the algorithm may be well defined even if the considered set is nonconvex (e.g., when the considered set is a closed subset of a finite dimensional space or a compact subset of Hilbert space, etc.) see [1, 4, 20, 25, 26, 27]. While, it may be from the practical point of view, one may see that the nonconvex problems are more useful and general than convex case, subsequently, now many researchers are convinced and paid attention to many nonconvex cases. Here, we are focusing on the following case, which was presented in 2013 by K. R. Kazmi[19]: let  $T_i : H_i \rightarrow H_i, A : H_1 \rightarrow H_2$  be nonlinear mappings for  $i = 1, 2$  and  $K_r, Q_s$  are uniformly prox-regular subsets of  $H_1$  and  $H_2$ , respectively, with  $r, s \in (0, \infty)$  for finding  $(x^*, y^*) \in K_r \times Q_s$ , where  $y^* = Ax^*$  such that

$$\begin{aligned} 0 &\in \rho T_1(x^*) + N_{K_r}^P(x^*), \\ 0 &\in \lambda T_2(y^*) + N_{Q_s}^P(y^*), \end{aligned} \tag{1.4}$$

where  $\rho, \lambda$  are parameters with positive values and  $N_K^P(x)$  is the proximal normal cone of  $K$  at  $x$ .

In this paper, base on above literatures, we are interested to study split quasi variational inequality of nonconvex type problem. The existence theorems and an algorithm for finding such solution will be considered and introduced, respectively. Our results represent an improvement and refinement of the literature results for the variational inequality problem.

## 2. Preliminaries

In this section, we will recall some basic concepts and useful results which will be used in this paper.

Let  $H$  be a real Hilbert space whose inner product and norm are denoted by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$ , respectively. Let  $2^H$  be denoted for the class of all nonempty subsets of  $H$ , and  $K$  be a closed subset of  $H$ . For each  $K \subseteq H$ , we denote by  $d(\cdot, K)$  for the usual distance function on  $H$  to  $K$ , that is,  $d(u, K) = \inf_{v \in K} \|u - v\|$ , for all  $u \in H$ .

For each  $K \subseteq H$  and  $u \in H$ . A point  $v \in K$  is called the closest point or the projection of  $u$  onto  $K$  if  $d(u, K) = \|u - v\|$ . The set of all such closest points is denoted by  $Proj_K(u)$ , that is,  $Proj_K(u) = \{v \in K : d(u, K) = \|u - v\|\}$ . The proximal normal cone to  $K$  at  $u$  is given by

$$N_K^P(u) = \{v \in H : \exists \rho > 0 \text{ such that } u \in Proj_K(u + \rho v)\}.$$

The following characterization of  $N_K^P(u)$  can be found in [15].

**Lemma 2.1.** *Let  $K$  be a closed subset of a Hilbert space  $H$ . Then*

$$v \in N_K^P(u) \Leftrightarrow \exists \sigma > 0 \text{ such that } \langle v, z - u \rangle \leq \sigma \|z - u\|^2, \text{ for all } z \in K. \tag{2.1}$$

The inequality (2.1) is called the proximal normal inequality.

We recall also that the Clarke normal cone is given by

$$N(K, x) = \overline{\text{co}}[N_K^P(x)],$$

where  $\overline{\text{co}}[S]$  means the closure of the convex hull of  $S$  (see [14]). It is clear that one always has  $N_K^P(x) \subset N(K, x)$ , but the converse is not true in general. Note that  $N(K, x)$  is always a closed and convex cone and that  $N_K^P(x)$  is always a convex cone but may be nonclosed (see [14, 15]). Also, in 1995, Clarke et al. [16] introduced a new class of nonconvex sets, which is called proximally smooth sets, and it has played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. Subsequently, in recent years, Bounkhel et al. [4], Cho et al. [12], Noor [25, 26], Petrot [28] and J. Suwannait and N. Petrot [26, 29, 32] have considered both variational inequalities and equilibrium problems in the context of proximally smooth sets. They suggested and analyzed some projection type iterative algorithms by using the prox-regular technique and auxiliary principle technique. Note that the original definition of proximally smooth set was given in terms of the differentiability of the distance function (see [16, 30]), while here, we will take the following characterization, which was proved in [15], as the definition of proximally smooth sets.

**Definition 2.2.** For a given  $r \in (0, +\infty]$ , a subset  $K$  of  $H$  is said to be *uniformly prox-regular with respect to  $r$* , say, *uniformly  $r$ -prox-regular set*, if for all  $\bar{x} \in K$  and for all  $0 \neq z \in N_K^P(\bar{x})$ , one has

$$\left\langle \frac{z}{\|z\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r} \|x - \bar{x}\|^2, \text{ for all } x \in K.$$

For the case of  $r = \infty$ , the uniform  $r$ -prox-regularity  $K$  is equivalent to the convexity of  $K$  (see [16]). Moreover, it is known that the class of uniformly prox-regular sets is sufficiently large to include the class  $p$ -convex sets,  $C^{1,1}$  submanifolds (possibly with boundary) of  $H$ , the images under a  $C^{1,1}$  diffeomorphism of convex sets and many other nonconvex sets, see [15, 30].

For the sake of simplicity, from now on, we will make use of the following notation: for each  $r \in (0, +\infty]$ , we write

$$K_r := \{x \in H : d(x, K) < r\},$$

and  $[Cl(H)]_r$  for the class of all uniformly  $r$ -prox regular subsets of  $H$ .

**Lemma 2.3.** *Let  $r \in (0, +\infty]$  and  $K$  be a nonempty closed subset of  $H$ . If  $K$  is a uniformly  $r$ -prox-regular set, then the following holds*

- (i) *For all  $x \in K_r$ ,  $Proj_K(x) \neq \emptyset$ ;*
- (ii) *For all  $s \in (0, r)$ ,  $Proj_K$  is a  $\frac{r}{r-s}$ -Lipschitz on  $K_s$ ;*
- (iii) *The proximal normal cone is closed as a set-valued mapping.*

*Remark 2.4.* If  $K$  is a uniformly  $r$ -prox-regular set, as a direct consequence of Lemma 2.3(iii), we know that  $N(K, x) = N_K^P(x)$ .

In this work, we will consider the following class of mappings.

**Definition 2.5.** A mapping  $T : H \rightarrow H$  is said to be a  $\sigma$ -strongly monotone if there exists  $\sigma > 0$  such that for all  $x, x^* \in K$ ,

$$\langle T(x) - T(x^*), x - x^* \rangle \geq \|x - x^*\|^2.$$

**Definition 2.6.** A mapping  $T : H \rightarrow H$  is said to be a  $\beta$ -Lipschitzian if there exists a real number  $\beta > 0$  such that

$$\|T(x) - T(y)\| \leq \beta\|x - y\|, \text{ for all } x, y \in H.$$

**Definition 2.7.** A multivalued mapping  $C : H \rightarrow 2^H$  is said to be a  $\kappa$ -Lipschitz continuous if there exists a real number  $\kappa > 0$  such that

$$\|d(y, C(x)) - d(y', C(x'))\| \leq \|y - y'\| + \kappa\|x - x'\|, \text{ for all } x, x', y, y' \in H.$$

The following lemma is a very important tool, in order to prove our main results.

**Lemma 2.8** ([4]). *Let  $r \in (0, +\infty]$  and let  $C : H \rightarrow 2^H$  be a  $\kappa$ -Lipschitz continuous multivalued mapping with uniformly  $r$ -prox regular valued then the following closedness property holds: for any  $x_n \rightarrow x^*, y_n \rightarrow y^*$  and  $u_n \rightarrow u^*$  with  $y_n \in C(x_n)$  and  $u_n \in N_{C(x_n)}^P(y_n)$ , one has  $u^* \in N_{C(x^*)}^P(y^*)$ .*

### 3. Main results

Let  $H_1$  and  $H_2$  be real Hilbert spaces,  $T_i : H_i \rightarrow H_i$  be nonlinear mappings,  $C_i : H_i \rightarrow 2^{H_i}$  be nonlinear multivalued mappings for  $i = 1, 2$  and  $A : H_1 \rightarrow H_2$  be a bounded linear operator. In this paper, we are interested in the following problem: find  $x^* \in C_1(x^*)$  such that,  $Ax^* \in C_2(Ax^*)$  and

$$\begin{aligned} -T_1(x^*) &\in N_{C_1(x^*)}^P(x^*), \\ -T_2(Ax^*) &\in N_{C_2(Ax^*)}^P(Ax^*). \end{aligned} \tag{3.1}$$

Notice that, the problem (3.1) can be reformulated as the following: find  $(x^*, z^*) \in C_1(x^*) \times C_2(z^*)$  with  $z^* = Ax^*$  such that

$$\begin{aligned} x^* &= Proj_{C_1(x^*)}(x^* - \rho T_1(x^*)), \\ z^* &= Proj_{C_2(z^*)}(z^* - \lambda T_2(z^*)), \end{aligned} \tag{3.2}$$

for some  $\rho, \lambda > 0$  are constants.

Moreover, by using the definition of uniformly prox-regular set, we also see that the problem (3.1) is of finding  $x^* \in C_1(x^*)$ , and  $z^* = Ax^* \in C_2(z^*)$  such that

$$\begin{aligned} \langle T_1(x^*), \hat{x} - x^* \rangle + \frac{\|T_1(x^*)\|}{2r} \|\hat{x} - x^*\|^2 &\geq 0, \forall \hat{x} \in C_1(x^*), \\ \langle T_2(z^*), \hat{z} - z^* \rangle + \frac{\|T_2(z^*)\|}{2s} \|\hat{z} - z^*\|^2 &\geq 0, \forall \hat{z} \in C_2(z^*). \end{aligned} \tag{3.3}$$

In a special case, when  $K$  and  $Q$  are closed subsets of  $H_1$  and  $H_2$ , respectively, and  $C_i : H_i \rightarrow 2^{H_i}$ , for  $i = 1, 2$  are defined by

$$\begin{aligned} C_1(x) &= K, \quad \text{for all } x \in H_1, \\ C_2(y) &= Q, \quad \text{for all } y \in H_2, \end{aligned} \tag{3.4}$$

then the problem (3.1) is reduced to the problem of finding  $(x^*, z^*) \in K \times Q$  with  $z^* = Ax^*$  such that

$$\begin{aligned} -T(x^*) &\in N_K^P(x^*), \\ -S(z^*) &\in N_Q^P(z^*), \end{aligned} \tag{3.5}$$

which was studied by K. R. Kazmi[18].

Now, we introduce an algorithm which will play an important role in our prove.

**Algorithm (A):** Let  $T_i : H_i \rightarrow H_i, C_1 : H_1 \rightarrow [Cl(H_1)]_r$  and  $C_2 : H_2 \rightarrow [Cl(H_2)]_s$  be nonlinear mappings where  $r, s \in (0, +\infty)$  and  $i = 1, 2$ . Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator with its adjoint operator, denoted by  $A^*$ . Given  $x_0 \in H_1$ , compute the algorithm sequences  $\{x_n\}, \{y_n\}$  and  $\{z_n\}$  as the following projection method:

$$\begin{aligned} y_n &\in Proj_{C_1(x_n)}[x_n - \rho T_1(x_n)], \\ z_n &\in Proj_{C_2(Ay_n)}[Ay_n - \lambda T_2(Ay_n)], \\ x_{n+1} &\in Proj_{C_1(y_n)}[y_n + \gamma A^*(z_n - Ay_n)], \end{aligned} \tag{3.6}$$

where  $\rho, \lambda$  and  $\gamma$  are step size positive real numbers.

The following assumption will be proposed, as the sufficient conditions.

**Assumption (C) :** Let  $T_i : H_i \rightarrow H_i, C_1 : H_1 \rightarrow [Cl(H_1)]_r$  and  $C_2 : H_2 \rightarrow [Cl(H_2)]_s$  be nonlinear mappings for  $r, s \in (0, +\infty)$  and  $i = 1, 2$  which are satisfied by the following conditions:

- (i)  $T_i$  is a  $\beta_i$ -Lipschitzian mapping and a  $\sigma_i$ -strongly monotone mapping for  $i = 1, 2$ ;
- (ii)  $C_i$  is a  $\kappa_i$ - Lipschitzian continuous mapping for some  $\kappa_i \in [0, 1)$  and  $i = 1, 2$ ;
- (iii) for each  $i = 1, 2$ , there is  $\omega_i \in [0, 1)$  such that

$$\|Proj_{C_i(x)}(z) - Proj_{C_i(y)}(z)\| \leq \omega_i \|x - y\|, \quad \text{for all } x, y, z \in H_i.$$

Firstly, based on the assumption (C), we notice the following key remark.

*Remark 3.1.* For a real Hilbert space  $H$  and  $r \in (0, \infty)$ . If  $T : H \rightarrow H$  and  $C : H \rightarrow [Cl(H)]_r$  are nonlinear mappings. Then, for each  $x_0 \in H$  with  $d(x_0, C(x_0)) \leq r^* - \rho \|Tx_0\|$ , where  $r^* \in (0, r)$  and  $\rho$  is a positive real number,  $Proj_{C(x_0)}[x_0 - \rho Tx_0] \neq \emptyset$ . Indeed, since  $C$  is a  $\kappa$ -Lipschitz continuous mapping, we have

$$\begin{aligned} d(x_0 - \rho Tx_0, C(x_0)) &\leq d(x_0, C(x_0)) + \rho \|Tx_0\| \\ &\leq r^* - \rho \|Tx_0\| + \rho \|Tx_0\| \\ &< r. \end{aligned}$$

By Lemma 2.3(i), we obtain that  $Proj_{C(x_0)}[x_0 - \rho Tx_0] \neq \emptyset$ .

The following lemma asserts that, under our setting, Algorithm (A) is well-define.

**Lemma 3.2.** *Let  $H_1, H_2$  be real Hilbert spaces. Assume that Assumption (C)(ii) and (iii) hold and there are  $\mu > 1$  and  $x_0 \in H_1$  such that*

- (i)  $d(x_0, C_1(x_0)) \leq r^* - \rho \|T_1 x_0\|$ ,
- (ii)  $0 < \rho < \frac{r^*}{\delta_{T_1}}, 0 < \lambda < \frac{s^* - \Phi}{\delta_{T_2}}$  and  $0 < \gamma < \frac{r^*}{\delta_{A^*}}$ ,

where  $\delta_{T_1} = \sup\{\|T_1 x_n\| : x_n \in H_1\}$ ,  $\delta_{T_2} = \sup\{\|T_2(Ay_n)\| : Ay_n \in H_2\}$ ,  $\delta_{A^*} = \sup\{\|A^*(z_n)\| : z_n \in H_2\}$ ,  $\Phi = \sup\{d(Ay_n, C_2(Ay_n)) \mid y_n \in H_1\}$ ,  $r^* = \frac{r(1-\kappa)}{1+\mu\kappa}$ ,  $s^* = \frac{s(1-\kappa)}{1+\mu\kappa}$ ,  $A : H_1 \rightarrow H_2$  is a bounded linear operator and  $\{x_n\}, \{y_n\}$  are constructed as in Algorithm (A) with the initial vector  $x_0$ . Then, the sequences  $\{x_n\}, \{y_n\}$  and  $\{z_n\}$  which are constructed by Algorithm (A) are well-defined.

*Proof.* By condition (i) and Remark 3.1, we know that  $Proj_{C_1(x_0)}[x_0 - \rho T_1 x_0] \neq \emptyset$ . Subsequently, we put  $y_0 \in Proj_{C_1(x_0)}[x_0 - \rho T_1 x_0]$ . Next, by the condition (ii), we see that

$$\begin{aligned} d(Ay_0 - \lambda T_2(Ay_0), C_2(Ay_0)) &\leq d(Ay_0, C_2(Ay_0)) + \lambda \|T_2(Ay_0)\| \\ &\leq d(Ay_0, C_2(Ay_0)) + \lambda \|T_2(Ay_0)\| \\ &< d(Ay_0, C_2(Ay_0)) + \left(\frac{s^* - \Phi}{\delta_{T_2}}\right) \|T_2(Ay_0)\| \\ &< d(Ay_0, C_2(Ay_0)) + s^* - \Phi \\ &< s^*. \end{aligned}$$

Thus,  $Proj_{C_2(Ay_0)}[Ay_0 - \lambda T_2(Ay_0)] \neq \emptyset$ . Let  $z_0 \in Proj_{C_2(Ay_0)}[Ay_0 - \lambda T_2(Ay_0)]$ . Notice that, by using the  $\kappa_1$ -Lipschitz continuous mapping of  $C_1$ , we see that

$$\begin{aligned} d(y_0 + \gamma A^*(z_0 - Ay_0), C_1(y_0)) &\leq d(y_0, C_1(y_0)) + \gamma \|A^*(z_0 - Ay_0)\| \\ &= d(y_0, C_1(y_0)) - d(y_0, C_1(x_0)) + \gamma \|A^*(z_0 - Ay_0)\| \\ &\leq \kappa_1 \|y_0 - x_0\| + r^*. \end{aligned} \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} \|y_0 - x_0\| &\leq \|y_0 - (x_0 - \rho T_1(x_0))\| + \|x_0 - \rho T_1(x_0) - x_0\| \\ &= d(x_0 - \rho T_1(x_0), C_1(x_0)) + \rho \|T_1(x_0)\| \\ &\leq r^* + r^* \\ &= 2r^*. \end{aligned} \tag{3.8}$$

Thus, (3.7) and (3.8), give

$$\begin{aligned} d(y_0 + \gamma A^*(z_0 - Ay_0), C_1(y_0)) &\leq 2\kappa_1 r^* + r^* \\ &= r^*(2\kappa_1 + 1) \\ &= r^* \left(\frac{1 + \kappa_1 - 2\kappa_1^2}{1 - \kappa_1}\right) \\ &< r. \end{aligned}$$

This implies that,  $Proj_{C_1(y_0)}[y_0 + \gamma A^*(z_0 - Ay_0)] \neq \emptyset$ . Let  $x_1 \in Proj_{C_1(y_0)}[y_0 + \gamma A^*(z_0 - Ay_0)]$ , and consider

$$\begin{aligned} d(x_1 - \rho T_1 x_1, C_1(x_1)) &\leq d(x_1, C_1(x_1)) + \rho \|T_1 x_1\| \\ &= d(x_1, C_1(x_1)) - d(x_1, C_1(y_0)) + \rho \|T_1 x_1\| \\ &\leq \kappa_1 \|x_1 - y_0\| + r^*. \end{aligned} \tag{3.9}$$

And, since

$$\begin{aligned} \|x_1 - y_0\| &\leq \|x_1 - (y_0 + \gamma A^*(z_0 - Ay_0))\| + \|y_0 + \gamma A^*(z_0 - Ay_0) - y_0\| \\ &= d(y_0 + \gamma A^*(z_0 - Ay_0), C_1(y_0)) + \gamma \|A^*(z_0 - Ay_0)\| \\ &< r^* \left(\frac{1 + \kappa_1 - 2\kappa_1^2}{1 - \kappa_1}\right) + r^*, \end{aligned} \tag{3.10}$$

we obtain

$$\begin{aligned}
 d(x_1 - \rho T_1 x_1, C_1(x_1)) &\leq r^* \kappa_1 \left( \frac{1 + \kappa_1 - 2\kappa_1^2}{1 - \kappa_1} \right) + r^* \kappa_1 + r^* \\
 &= r^* \left( \frac{1 + \kappa_1 - 2\kappa_1^3}{1 - \kappa_1} \right) \\
 &< r.
 \end{aligned}
 \tag{3.11}$$

This implies that,  $Proj_{C_1(x_1)}[x_1 - \rho T_1 x_1] \neq \emptyset$ . Let  $y_1 \in Proj_{C_1(x_1)}[x_1 - \rho T_1(x_1)]$ , and we see that

$$\begin{aligned}
 d(Ay_1 - \lambda T_2(Ay_1), C_2(Ay_1)) &\leq d(Ay_1, C_2(Ay_1)) + \lambda \|T_2(Ay_1)\| \\
 &< d(Ay_1, C_2(Ay_1)) + \left( \frac{s^* - \Phi}{\delta_{T_2}} \right) \|T_2(Ay_1)\| \\
 &\leq d(Ay_1, C_2(Ay_1)) + s^* - \Phi \\
 &\leq s^* \\
 &< s.
 \end{aligned}
 \tag{3.12}$$

Thus,  $Proj_{C_2(Ay_1)}[Ay_1 - \lambda T_2(Ay_1)] \neq \emptyset$ . Let  $z_1 \in Proj_{C_2(Ay_1)}[Ay_1 - \lambda T_2(Ay_1)]$ , and computes

$$\begin{aligned}
 d(y_1 + \gamma A^*(z_1 - Ay_1), C_1(y_1)) &\leq d(y_1, C_1(y_1)) + \gamma \|A^*(z_1 - Ay_1)\| \\
 &\leq d(y_1, C_1(y_1)) - d(y_1, C_1(x_1)) + r^* \\
 &\leq \kappa_1 \|y_1 - x_1\| + r^*.
 \end{aligned}
 \tag{3.13}$$

Since

$$\begin{aligned}
 \|y_1 - x_1\| &\leq \|y_1 - (x_1 - \rho T_1 x_1)\| + \|x_1 - \rho T_1 x_1 - x_1\| \\
 &= d(x_1 - \rho T_1 x_1, C_1(x_1)) + \rho \|T_1 x_1\| \\
 &< r^* \left( \frac{1 + \kappa_1 - 2\kappa_1^3}{1 - \kappa_1} \right) + r^*,
 \end{aligned}
 \tag{3.14}$$

we have

$$\begin{aligned}
 d(y_1 + \gamma A^*(z_1 - Ay_1), C_1(y_1)) &< r^* \kappa_1 \left( \frac{1 + \kappa_1 - 2\kappa_1^3}{1 - \kappa_1} \right) + \kappa_1 r^* + r^* \\
 &= r^* \left( \frac{1 + \kappa_1 - 2\kappa_1^4}{1 - \kappa_1} \right) \\
 &< r.
 \end{aligned}
 \tag{3.15}$$

Thus,  $Proj_{C_1(y_1)}[y_1 + \gamma A^*(z_1 - Ay_1)] \neq \emptyset$ . Let  $x_2 \in Proj_{C_1(y_1)}[y_1 + \gamma A^*(z_1 - Ay_1)]$ . In the same way of (3.9), (3.10) and (3.11), we have

$$\begin{aligned}
 d(x_2 - \rho T_1(x_2), C_1(x_2)) &\leq \kappa_1 \|x_2 - y_1\| + r^* \\
 \|x_2 - y_1\| &< r^* \left( \frac{1 + \kappa_1 - 2\kappa_1^4}{1 - \kappa_1} \right) + r^*
 \end{aligned}$$

and

$$d(x_2 - \rho T_1 x_2, C_1(x_2)) \leq r^* \left( \frac{1 + \kappa_1 - 2\kappa_1^5}{1 - \kappa_1} \right).$$

Thus,  $Proj_{C_1(x_2)}[x_2 - \rho T_1 x_2] \neq \emptyset$ . Let  $y_2 \in Proj_{C_1(x_2)}[x_2 - \rho T_1 x_2]$ . In similar way (3.12), we obtain

$$d(Ay_2 - \lambda T_2(Ay_2), C_2(Ay_2)) \leq s^*.$$

Thus,  $Proj_{C_2(Ay_2)}[Ay_2 - \lambda T_2(Ay_2)] \neq \emptyset$ . Let  $z_2 \in Proj_{C_2(Ay_2)}[Ay_2 - \lambda T_2(Ay_2)]$ . In the same way as obtaining (3.13), (3.14) and (3.15), we have

$$d(y_2 + \gamma A^*(z_2 - Ay_2), C_1(y_2)) \leq \kappa_1 \|y_2 - x_2\| + r^*$$

$$\|y_2 - x_2\| < r^* \left( \frac{1 + \kappa_1 - 2\kappa_1^5}{1 - \kappa_1} \right) + r^*,$$

and

$$d(y_2 + \gamma A^*(z_2 - Ay_2), C_1(y_2)) < r^* \left( \frac{1 + \kappa_1 - 2\kappa_1^6}{1 - \kappa_1} \right).$$

Thus,  $Proj_{C_1(y_2)}[y_2 + \gamma A^*(z_2 - Ay_2)] \neq \emptyset$ , and we then put  $x_3 \in Proj_{C_1(y_2)}[y_2 + \gamma A^*(z_2 - Ay_2)]$ .

By using this process, we can construct the sequences  $\{x_n\}, \{y_n\}$  in  $H_1$  and  $\{z_n\}$  in  $H_2$  such that

$$\begin{aligned} x_n - \rho T_1(x_n) &\in [C_1(x_n)]_{\frac{r(1+\kappa_1)}{1+\mu\kappa_1}} \\ y_n &\in Proj_{C_1(x_n)}[x_n - \rho T_1(x_n)] \\ Ay_n - \lambda S(Ay_n) &\in [C_2(Ay_n)]_{s^*} \\ z_n &\in Proj_{C_2(Ay_n)}[Ay_n - \lambda T_2(Ay_n)] \\ y_n + \gamma A^*(z_n - Ay_n) &\in [C_1(y_n)]_{\frac{r(1+\kappa_1)}{1+\mu\kappa_1}} \\ x_{n+1} &\in Proj_{C_1(y_n)}[y_n + \gamma A^*(z_n - Ay_n)], \end{aligned} \tag{3.16}$$

which is, in fact, the Algorithm (A). □

*Remark 3.3.* Let us consider the proposed assumptions of Lemma 3.2. In the application point of view, one may ask for the best choice of the real number  $\mu$ , and hence  $r^*$  and  $s^*$ . We would like to notice here that, the real number  $\mu = \frac{\kappa\Delta-1}{\kappa(1-\Delta)}$ , where  $\Delta = \frac{\beta(1-\omega)}{\sqrt{\beta^2-\sigma^2}}$ , should provide the answer. This is because, by the following observation:

- the domain of function  $f$  is  $\frac{\beta(1-\omega)}{\sqrt{\beta^2-\sigma^2}}$ ,
- $r^* = \frac{r(1-\kappa)}{1+\mu\kappa} \Leftrightarrow t_r = \frac{1+\mu\kappa}{\kappa(1-\mu)}$  and  $s^* = \frac{s(1-\kappa)}{1+\mu\kappa} \Leftrightarrow t_s = \frac{1+\mu\kappa}{\kappa(1-\mu)}$ ,
- the function  $\mu \mapsto \frac{1+\mu\kappa}{\kappa(1+\mu)}$  is an increasing function on its domain,
- $\frac{1+\mu\kappa}{\kappa(1+\mu)} = \Delta \Leftrightarrow \mu = \frac{\kappa\Delta-1}{\kappa(1-\Delta)}$ , where  $\Delta = \frac{\beta(1-\omega)}{\sqrt{\beta^2-\sigma^2}}$ .

The following theorem shows that the sequences  $\{x_n\}, \{y_n\}$  and  $\{z_n\}$ , which are considered in Lemma 3.2 are all convergent sequences.

**Theorem 3.4.** *Let  $H_1, H_2$  be real Hilbert spaces. Assume that Assumption (C) and all of assumptions in Lemma 3.2 hold. If  $\gamma < \min\{\frac{2}{\|A\|^2}, \frac{1-\omega_1-\varphi}{\varphi\theta_2\|A\|^2}\}$ , where  $\varphi = \frac{1+\mu\kappa_1}{\kappa_1(\mu-1)}, \theta_2 = t_{s^*} \sqrt{1 - 2\lambda\sigma_2 + \lambda^2\beta_2^2} + \omega_2$  and  $t_{s^*} = \frac{s}{s-s^*}$ , then  $\{x_n\}, \{y_n\}$  and  $\{z_n\}$ , which are constructed in Algorithm (A), are convergent sequences.*

*Proof.* Using the definition of sequence  $\{x_n\}, \{y_n\}$  and  $\{z_n\}$  in Algorithm (A), we have

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|Proj_{C_1(x_{n+1})}[x_{n+1} - \rho T_1(x_{n+1})] - Proj_{C_1(x_n)}[x_n - \rho T_1(x_n)]\| \\ &\leq \|Proj_{C_1(x_{n+1})}[x_{n+1} - \rho T_1(x_{n+1})] - Proj_{C_1(x_{n+1})}[x_n - \rho T_1(x_n)]\| \\ &\quad + \|Proj_{C_1(x_{n+1})}[x_n - \rho T_1(x_n)] - Proj_{C_1(x_n)}[x_n - \rho T_1(x_n)]\| \\ &\leq \frac{1 + \mu\kappa_1}{\kappa_1(\mu - 1)} \|x_{n+1} - \rho T_1(x_{n+1}) - x_n + \rho T_1(x_n)\| + \omega_1 \|x_{n+1} - x_n\|. \end{aligned} \tag{3.17}$$

On the other hand, we have

$$\begin{aligned} &\|x_{n+1} - \rho T_1(x_{n+1}) - x_n + \rho T_1(x_n)\|^2 \\ &\leq \|x_{n+1} - x_n\|^2 - 2\rho \langle T_1(x_{n+1}) - T_1(x_n), x_{n+1} - x_n \rangle + \rho^2 \|T_1(x_{n+1}) - T_1(x_n)\|^2 \\ &\leq \|x_{n+1} - x_n\|^2 - 2\rho\sigma_1 \|x_{n+1} - x_n\|^2 + \rho^2 \beta_1^2 \|x_{n+1} - x_n\|^2 \\ &= (1 - 2\rho\sigma_1 + \rho^2 \beta_1^2) \|x_{n+1} - x_n\|^2. \end{aligned} \tag{3.18}$$



From (3.17) and (3.18), we get

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \frac{1 + \mu\kappa_1}{\kappa_1(\mu - 1)} \sqrt{1 - 2\rho\sigma_1 + \rho^2\beta_1^2} \|x_{n+1} - x_n\| + \omega_1 \|x_{n+1} - x_n\| \\ &= \left( \frac{1 + \mu\kappa_1}{\kappa_1(\mu - 1)} \sqrt{1 - 2\rho\sigma_1 + \rho^2\beta_1^2} + \omega_1 \right) \|x_{n+1} - x_n\| \\ &= \theta_1 \|x_{n+1} - x_n\|, \end{aligned} \tag{3.19}$$

where  $\theta_1 = \frac{1 + \mu\kappa_1}{\kappa_1(\mu - 1)} \sqrt{1 - 2\rho\sigma_1 + \rho^2\beta_1^2} + \omega_1$ . Observe that, by the choice of  $\rho, \mu$  and  $\kappa_1$ , we have  $\theta_1 < 1$ . Next, by the definition of  $\{z_n\}$ , we have

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|Proj_{C_2(Ay_{n+1})}[Ay_{n+1} - \lambda T_2(Ay_{n+1})] - Proj_{C_2(Ay_n)}[Ay_n - \lambda T_2(Ay_n)]\| \\ &\leq \|Proj_{C_2(Ay_{n+1})}[Ay_{n+1} - \lambda T_2(Ay_{n+1})] - Proj_{C_2(Ay_{n+1})}[Ay_n - \lambda T_2(Ay_n)]\| \\ &\quad + \|Proj_{C_2(Ay_{n+1})}[Ay_n - \lambda T_2(Ay_n)] - Proj_{C_2(Ay_n)}[Ay_n - \lambda T_2(Ay_n)]\| \\ &\leq t_{s^*} \|(Ay_{n+1} - Ay_n) - \lambda(T_2(Ay_{n+1}) - T_2(Ay_n))\| + \omega_2 \|Ay_{n+1} - Ay_n\|. \end{aligned} \tag{3.20}$$

And, since

$$\begin{aligned} &\|(Ay_{n+1} - Ay_n) - \lambda(T_2(Ay_{n+1}) - T_2(Ay_n))\|^2 \\ &\leq \|Ay_{n+1} - Ay_n\|^2 - 2\lambda \langle T_2(Ay_{n+1}) - T_2(Ay_n), Ay_{n+1} - Ay_n \rangle + \lambda \|T_2(Ay_{n+1}) - T_2(Ay_n)\|^2 \\ &\leq \|A\|^2 \|y_{n+1} - y_n\|^2 - 2\lambda\sigma_2 \|Ay_{n+1} - Ay_n\|^2 + \lambda^2\beta_2^2 \|Ay_{n+1} - Ay_n\|^2 \\ &= (1 - 2\lambda\sigma_2 + \lambda^2\beta_2^2) \|A\|^2 \|y_{n+1} - y_n\|^2, \end{aligned}$$

we obtain

$$\begin{aligned} \|z_{n+1} - z_n\| &\leq t_{s^*} \sqrt{1 - 2\lambda\sigma_2 + \lambda^2\beta_2^2} \|A\| \|y_{n+1} - y_n\| + \omega_2 \|A\| \|y_{n+1} - y_n\| \\ &= \left( t_{s^*} \sqrt{1 - 2\lambda\sigma_2 + \lambda^2\beta_2^2} + \omega_2 \right) \|A\| \|y_{n+1} - y_n\| \\ &= \theta_2 \|A\| \|y_{n+1} - y_n\|. \end{aligned} \tag{3.21}$$

Note that, by the choice of  $\lambda$ , we have  $\theta_2 < 1$ . Next, we consider

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \|Proj_{C_1(y_n)}[y_n + \gamma A^*(z_n - Ay_n)] - Proj_{C_1(y_{n+1})}[y_{n+1} + \gamma A^*(z_{n-1} - Ay_{n-1})]\| \\ &\leq \|Proj_{C_1(y_n)}[y_n + \gamma A^*(z_n - Ay_n)] - Proj_{C_1(y_{n-1})}[y_n + \gamma A^*(z_n - Ay_n)]\| \\ &\quad + \|Proj_{C_1(y_{n-1})}[y_n + \gamma A^*(z_n - Ay_n)] - Proj_{C_1(y_{n-1})}[y_{n-1} + \gamma A^*(z_{n-1} - Ay_{n-1})]\| \\ &\leq \omega_1 \|y_n - y_{n-1}\| + \varphi \|y_n - y_{n-1} - \gamma(A^*(z_{n-1} - Ay_{n-1}) - A^*(z_n - Ay_n))\| \\ &\leq \omega_1 \|y_n - y_{n-1}\| + \varphi \|y_n - y_{n-1} - \gamma(A^*(Ay_n - Ay_{n-1}))\| + \varphi\gamma \|A^*(z_n) - A^*(z_{n-1})\|. \end{aligned} \tag{3.22}$$

Since,

$$\begin{aligned} &\|y_n - y_{n-1} - \gamma(A^*(Ay_n - Ay_{n-1}))\|^2 \\ &\leq \|y_n - y_{n-1}\|^2 - 2\gamma \langle y_n - y_{n-1}, A^*(Ay_n - Ay_{n-1}) \rangle + \gamma^2 \|A^*(Ay_n - Ay_{n-1})\|^2 \\ &= \|y_n - y_{n-1}\|^2 - 2\gamma \langle Ay_n - Ay_{n-1}, Ay_n - Ay_{n-1} \rangle + \gamma^2 \|A^*(Ay_n - Ay_{n-1})\|^2 \\ &\leq \|y_n - y_{n-1}\|^2 - 2\gamma \|Ay_n - Ay_{n-1}\|^2 + \gamma^2 \|A\|^2 \|Ay_n - Ay_{n-1}\|^2 \\ &= \|y_n - y_{n-1}\|^2 - (2\gamma - \gamma^2 \|A\|^2) \|Ay_n - Ay_{n-1}\|^2 \\ &= \|y_n - y_{n-1}\|^2 - \gamma(2 - \gamma \|A\|^2) \|Ay_n - Ay_{n-1}\|^2 \\ &\leq \|y_n - y_{n-1}\|^2 \end{aligned} \tag{3.23}$$

and  $\|A^*(z_n) - A^*(z_{n-1})\| \leq \|A\| \|z_n - z_{n-1}\|$ , we get

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \omega_1 \|y_n - y_{n-1}\| + \varphi \|y_n - y_{n-1}\| + \varphi \gamma \|A\| \|z_n - z_{n-1}\| \\ &\leq \omega_1 \|y_n - y_{n-1}\| + \varphi \|y_n - y_{n-1}\| + \varphi \gamma \theta_2 \|A\|^2 \|y_n - y_{n-1}\| \\ &= (\gamma \theta_2 \varphi \|A\|^2 + \varphi + \omega_1) \|y_n - y_{n-1}\| \\ &\leq \theta_1 (\gamma \theta_2 \varphi \|A\|^2 + \varphi + \omega_1) \|x_n - x_{n-1}\| \\ &= \theta_3 \|x_n - x_{n-1}\|, \end{aligned} \tag{3.24}$$

where  $\theta_3 = \theta_1 (\gamma \theta_2 \varphi \|A\|^2 + \varphi + \omega_1)$ . Also, by the choice of  $\gamma$ , we know that  $\theta_3 < 1$ . Hence, for any  $m \geq n > 1$ , we see that

$$\begin{aligned} \|x_m - x_n\| &\leq \sum_{i=n}^{m-1} \|x_{i+1} - x_i\| \\ &\leq \sum_{i=n}^{m-1} \theta_3^i \|x_1 - x_0\| \\ &\leq \|x_1 - x_0\| \sum_{i=n}^{\infty} \theta_3^i \\ &\leq \frac{\theta_3^n}{1 - \theta_3} \|x_1 - x_0\|. \end{aligned} \tag{3.25}$$

Since  $\theta_3 < 1$ , we can conclude that  $\{x_n\}$  is a Cauchy sequence in  $H_1$ . By the completeness of  $H_1$ , we know that  $\{x_n\}$  is a convergent sequence. Also, by (3.19) and the convergence of the sequence  $\{x_n\}$ , we see that  $\{y_n\}$  is a convergent sequence. In similar way, by (3.21) and the convergence of the sequence  $\{y_n\}$ , we obtain that  $\{z_n\}$  is a convergent sequence. This completes the proof.  $\square$

Now, we are in position to present the sufficient condition for existence of solution of problem (3.1) our main theorem.

**Theorem 3.5.** *Let  $H_1, H_2$  be real Hilbert spaces. Let  $T_i : H_i \rightarrow H_i$  be nonlinear mappings for  $i = 1, 2$  and  $C_1 : H_1 \rightarrow [Cl(H_1)]_r$  and  $C_2 : H_2 \rightarrow [Cl(H_2)]_s$  be nonlinear set-valued mappings. Assume that all of the assumptions in Theorem 3.4 hold and if  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$  and  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} z_n$ . Then, the problem (3.1) has a solution.*

*Proof.* Let  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = x^*$  and  $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} z_n = z^*$ . Firstly, we will show that  $-T_1(x^*) \in N_{C_1(x^*)}^P(x^*)$ . Since  $y_n \in Proj_{C_1(x_n)}[x_n - \rho T_1(x_n)]$ , we see that  $x_n - y_n - \rho T_1(x_n) \in N_{C_1(x_n)}^P(y_n)$ . Using this one together with the closedness property of the proximal cone, we obtain that  $-\rho T_1 x^* \in N_{C_1(x^*)}^P(x^*)$ . This means,  $-T_1(x^*) \in N_{C_1(x^*)}^P(x^*)$ .

Next, we want to show that  $-T_2(z^*) \in N_{C_2(z^*)}^P(z^*)$ . Since  $z_n \in Proj_{C_2(Ay_n)}[Ay_n - \lambda T_2(Ay_n)]$ , we have  $Ay_n - z_n - \lambda T_2(Ay_n) \in N_{C_2(Ay_n)}^P(z_n)$ . Again, by using the closedness property of the proximal cone, we have  $Ax^* - z^* - \lambda T_2(Ax^*) \in N_{C_2(Ax^*)}^P(z^*)$ . This is  $-\lambda T_2(Ax^*) \in N_{C_2(z^*)}^P(z^*)$ . Hence,  $-T_2(z^*) \in N_{C_2(z^*)}^P(z^*)$ . The proof is completed.  $\square$

Recall that a multivalued mapping  $C : H \rightarrow 2^H$  is said to be a Hausdorff Lipschitz continuous if there exists a real number  $\kappa > 0$  such that

$$\mathcal{H}(C(x), C(y)) \leq \kappa \|x - y\|, \text{ for all } x, y \in H,$$

where  $\mathcal{H}$  stands for the Hausdorff distance relative to norm associated with the Hilbert space  $H$ , that is

$$\mathcal{H}(A, B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\| \right\},$$

for all  $A, B \in 2^H$ .

It is easy to check that the class of  $\kappa$ -Lipschitz continuous mapping in Definition 2.7 is larger than the class of above Hausdorff Lipschitz continuous mappings. Thus, the following results is followed immediately from Theorem 3.5.

**Corollary 3.6.** *Let  $H_1, H_2$  be real Hilbert spaces. Let  $T_i : H_i \rightarrow H_i$  be nonlinear mappings for  $i = 1, 2$  and  $C_1 : H_1 \rightarrow [Cl(H_1)]_r$  and  $C_2 : H_2 \rightarrow [Cl(H_2)]_s$  be Hausdorff Lipschitz continuous mappings. Assume that Assumption (A)(i) and (iii) and all of assumptions in Theorem 3.5 hold. Then, the problem (3.1) has a solution.*

*Remark 3.7.*

- (i) Corollary 3.6 recovers the result which was presented by K. R. Kazmi[18] as a special case.
- (ii) It is well known that if  $K$  is a closed convex set then it is  $r$ -prox regular set for every  $r > 0$ , so, in this case, the control condition (ii) of Lemma 3.2 can be omitted.

#### 4. Conclusion

In this work, we introduce and study a type of split quasi variational inequality problem over a class of nonconvex sets. In order to prove the existence theorems, an algorithm is constructed as an important tool. This problem generalizes and extends the variational inequality problems and the split variational inequality problems from the setting of convex sets to nonconvex case. We desire that the results presented here will be useful and valuable for researchers who study the branch of variational inequality and related applications.

#### Acknowledgements

The authors wish to express their gratitude to the referees for careful reading of the manuscript. The first author is supported by Pibulsongkram Rajabhat University.

#### References

- [1] J. Balooee, Y. J. Cho, *Perturbed projection and iterative algorithms for a system of general regularized nonconvex variational inequalities*, J. Inequal. Appl., **2012** (2012), 23 pages.1
- [2] J. Balooee, Y. J. Cho, *Algorithms for solutions of extended general mixed variational inequalities and fixed points*, Optim. Lett., **7** (2013), 1929–1955.1
- [3] A. Bensoussan, M. Goursat, J. L. Lions, *Control impulsinnel et inequations quasi-variationnelles stationaries*, C. R. Acad. Sci., **276** (1973), 1279–1284.1
- [4] M. Bounkhel, L. T. Tadj, A. Hamdi, *Iterative schemes to solve nonconvex variational problems*, J. Inequal. Pure Appl. Math., **4** (2003), 14 pages.1, 2, 2.8
- [5] C. Byrne, *Iterative oblique projection onto convex sets and split feasibility problem*, Inverse Problems, **18** (2002), 441–453.1
- [6] Y. Censor, T. Bortfeld, B. Martin, A. Trofimos, *A unified approach for inversion problems in intensity modulated radiation therapy*, Phys. Med. Biol., **51** (2006), 2353–2365.1
- [7] Y. Censor, T. Elfving, *A multiprojection algorithm using Bergman projections in product space*, Numer. Algorithms, **8** (1994), 221–239.1
- [8] Y. Censor, A. Gibali, S. Reich, *Algorithms for the split variational inequality problem*, Numer. Algorithms, **59** (2012), 301–323.1
- [9] D. Chan, J. S. Pang, *The generalized quasi-variational inequality problem*, Math. Oper. Res., **7** (1982), 211–222.1
- [10] S. S. Chang, *Variational inequality and complementarity problem theory with applications*, Shanghai Scientific and Tech. Literature Publishing House, Shanghai, (1991).1
- [11] Y. J. Cho, Y. P. Fang, N. J. Huang, H. J. Hwang, *Algorithms for systems of nonlinear variational inequalities*, J. Korean Math. Soc., **41** (2004), 489–499.1

- [12] Y. J. Cho, J. K. Kim, R. U. Verma, *A class of nonlinear variational inequalities involving partially relaxed monotone mappings and general auxiliary principle*, Dyn. Sys. Appl., **11** (2002), 333–338.1, 2
- [13] Y. J. Cho, X. Qin, *Systems of generalized nonlinear variational inequalities and its projection methods*, Nonlinear Anal., **69** (2008), 4443–4451.1
- [14] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, (1983).2
- [15] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer-Verlag, New York, (1998).2, 2, 2
- [16] F. H. Clarke, R. J. Stern, P. R. Wolenski, *Proximal smoothness and the lower  $C^2$  property*, J. Convex Anal., **2** (1995), 117–144.2, 2
- [17] P. L. Combettes, *The convex feasibility problem in image recovery*, Adv. Imaging Electron Phys., **95** (1996), 155–270.1
- [18] K. R. Kazmi, *Split nonconvex variational inequality problem*, Math. Sci., **7** (2013), 5 pages.3, 3.7
- [19] K. R. Kazmi, *Split general quasi-variational inequality problem*, Georgian Math. J., **22** (2015), 385–392.1
- [20] A. Moudafi, *Projection methods for a system of nonconvex variational inequalities*, Nonlinear Anal., **71** (2009), 517–520.1
- [21] A. Moudafi, *Split monotone variational inclusions*, J. Optim. Theory Appl., **150** (2011), 275–283.1
- [22] M. A. Noor, *An iterative scheme for a class of quasi variational inequalities*, J. Math. Anal. Appl., **110** (1985), 463–468.1
- [23] M. A. Noor, *Quasi variational inequalities*, Appl. Math. Lett., **1** (1988), 367–370.1
- [24] M. A. Noor, *Generalized set-valued mixed nonlinear quasi variational inequalities*, Korean J. Comput. Appl. Math., **5** (1998), 73–89.1
- [25] M. A. Noor, *Iterative schemes for nonconvex variational inequalities*, J. Optim. Theory Appl., **121** (2004), 385–395.1, 2
- [26] M. A. Noor, N. Petrot, J. Suwannawit, *Existence theorems for multivalued variational inequality problems on uniformly prox-regular sets*, Optim. Lett., **8** (2014), 99–111.1, 2
- [27] L. P. Pang, J. Shen, H. S. Song, *A modified predictor-corrector algorithm for solving nonconvex generalized variational inequalities*, Comput. Math. Appl., **54** (2007), 319–325.1
- [28] N. Petrot, *Some existence theorems for nonconvex variational inequalities problems*, Abstr. Appl. Anal., **2010** (2010), 9 pages.2
- [29] N. Petrot, J. Suwannawit, *Existence theorems for some systems of quasi-variational inequalities problems on uniformly prox-regular sets*, Math. Inequal. Appl., **16** (2013), 1229–1242.2
- [30] R. A. Poliquin, R. T. Rockafellar, L. Thibault, *Local differentiability of distance functions*, Trans. Amer. Math. Soc., **352** (2000), 5231–5249.2, 2
- [31] G. Stampacchia, *Formes bilineaires coercitives sur les ensembles convexes*, C. R. Acad. Sci. Paris, **258** (1964), 4413–4416.1
- [32] J. Suwannawit, N. Petrot, *Existence Theorems for Quasi variational Inequality Problem on Proximally Smooth Sets*, Abstr. Appl. Anal., **2013** (2013), 7 pages.2
- [33] N. X. Tan, *Random quasi-variational inequality*, Math. Nachr., **125** (1986), 319–328.1
- [34] R. U. Verma, *Projection methods, algorithms, and a new system of nonlinear variational inequalities*, Comput. Math. Appl., **41** (2001), 1025–1031.1
- [35] R. U. Verma, *Generalized system for relaxed cocoercive variational inequalities and projection methods*, J. Optim. Theory Appl., **121** (2004), 203–210.1
- [36] Y. H. Yao, Y. C. Liou, J. C. Yao, *An extragradient method for fixed point problems and variational inequality problems*, J. Inequal. Appl., **2007** (2007), 12 pages.1