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Fixed point theorem and nonlinear complementarity problem in Hilbert spaces

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Abstract

In this paper, the concept of the strongly monotone type mapping is introduced, which contains the strongly monotone mapping and firmly type nonexpansive mapping as special cases. We show the equivalence between the fixed point problem and the complementarity problem of strongly monotone type mapping. Furthermore, it is obtained that an iteration sequence strongly converges to a unique solution of such a nonlinear complementarity problem on the proper conditions. The error estimation of such an iteration is discussed.

Keywords: Fixed point, strongly monotone type, complementarity problem, iteration.

2010 MSC: 47H10, 54H25, 49J40, 47H05, 47H04, 65J15, 47H10.

1. Introduction

The well-known Banach Contraction Principle says that if T is a **contraction** from a complete metric space (X, d) to itself, i.e.,

$$d(Tx, Ty) \le \beta d(x, y) \text{ for all } x, y \in X \text{ and some } \beta \text{ with } 0 \le \beta < 1, \tag{1.1}$$

then T has unique fixed point $x^* \in X$ $(x^* = Tx^*)$ and $\lim\limits_{k\to\infty} T^k x = x^*$ for all $x \in X$. Furthermore,

$$d(T^k x, x^*) \le \frac{\beta^k}{1-\beta} d(x, Tx). \tag{1.2}$$

Edelstein [2] relaxed the strict contraction condition (1.1) by permitting $\beta = 1$ and obtained the following result which is referred to as the *Edelstein Contraction Theorem*.

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Theorem 1.1 (The Edelstein Contraction Theorem)**.** *Let (X, d) be a metric space and $T : X \to X$ be a strict contraction, that is*

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X \text{ with } x \neq y. \tag{1.3}$$

If there exist $x, x^* \in X$ such that

$$\lim_{i\to\infty} d(T^{k_i}x, x^*) = 0 \text{ for some } \{T^{k_i}x\} \subset \{T^k x\}, \tag{1.4}$$

then T has a unique fixed point $x^* \in X$ and for each $x \in X$, $\lim_{k\to\infty} T^k x = x^*$.

Clearly, the condition (1.4) is replaced by the fact that metric space X is compact or the range $R(T) = T(X)$ is relatively compact, then the conclusions still hold.

In 2009, Song and Chai [15] introduced the concept of firmly type nonexpansive mapping and showed the convergence theorems of Halpern iteration defined by Halpern [6] for such a mapping. Song and Li [17] showed the convergence results of Ishikawa iteration given by Ishikawa [8, 9] and Krasnoselskii-Mann iteration given by Mann [11] and Krasnoselskii [10] for quasi-firmly type nonexpansive mapping. Recently, Song and Huang [16] obtained the following existence of fixed point of firmly type nonexpansive mapping in Banach space as well as the corresponding convergence conclusions. For more details and examples, see [15, 16, 17].

Theorem 1.2 ([16], Theorem 2.3)**.** *Let K be a weakly compact convex subsets of a Banach space E. If $T : K \to K$ is a firmly type nonexpansive mapping, i.e., for some $\gamma \in (0, +\infty)$,*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \gamma\|(x - Tx) - (y - Ty)\|^2 \text{ for all } x, y \in K, \tag{1.5}$$

then T has a fixed point.

Our main aim of this paper is to apply these fixed point theorems to discuss the existence and uniqueness of the nonlinear complementarity problem as well as its iteration convergence.

Let H be a real Hilbert space and let $K \subset H$ be a closed convex cone with the vertex at 0 and the dual cone K^*. Let $T : K \to H$ be a nonlinear mapping. Then the **nonlinear complementarity problem**, denoted by NCP(T), is to find a vector $x \in H$ such that

NCP(T) $\qquad x \in K, \quad T(x) \in K^*$ and $\langle x, T(x)\rangle = 0.$

The NCP(T) in $n-$dimensional Euclidean space $\mathbb{R}^n$ was introduced by Cottle in his Ph.D. thesis in 1964. It is well-known that the NCP(T) has a wide range of important applications in operation research, applied science and technology such as optimization, economic equilibrium problems, contact mechanics problems, structural mechanics problem, traffic equilibrium problems, discrete-time optimal control and so on. For more detail, see [3, 7] and references therein. Over a thousand articles and several books have been published on this classical subject, which has developed into a well-established and fruitful discipline in the field of mathematical programming. Nanda and Nanda [12] gave the existence and uniqueness of the solution of NCP(T) for a strong monotone and Lipschitizian mapping F in Hilbert space by means of the Banach Contraction Principle. In 1981, Riddle [13] established the equivalence of complementarity and least-element problems. In 1995, Schaible and Yao [14] proved the equivalence of several classes of complementarity problems for strictly pseudomonotone Z-mapping in Banach lattices. Zeng et al. [19] derived some equivalences of several related complementarity problems under certain regularity and growth conditions. For a different approach to the equivalence problem, see [18, 20] and references therein.

In this paper, we will introduce the concept of the strongly monotone type mapping, which contains the strongly monotone mapping and firmly type nonexpansive mapping as special cases and show the equivalence between the fixed point problem and the complementarity problem of strongly monotone type mapping. By Banach contraction principle, such a nonlinear complementarity problem has a unique solution and the convergent result of the corresponding iteration sequence. The corresponding error estimation of such a iteration is given.

2. Preliminaries and basic results

In the proof of main theorems, we need the following notations, definitions and results. Throughout this work, a Hilbert space H will always be over the real scalar field. We denote its norm by $\|\cdot\|$ and its inner product by $\langle\cdot,\cdot\rangle$. Let $K \subset H$ be a closed convex cone with the vertex 0 and let K^* be the dual cone of K,

$$K^* = \{y \in H; \langle x, y\rangle \geq 0 \text{ for all } x \in P\}. \tag{2.1}$$

Let $\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n)^\top; x_i \in \mathbb{R}, i = 1, 2, \cdots, n\}$ and $\mathbb{R}^n_{++} := \{x \in \mathbb{R}^n; x > 0\}$ and $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n; x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers, $x^\top$ is the transposition of a vector x and $x \geq 0$ ($x > 0$) means $x_i \geq 0$ ($x_i > 0$) for all $i = 1, 2, \cdots, n$.

Definition 2.1. Let T be a mapping with the domain $D(T)$ and range $R(T)$ in Hilbert space H. T is said to be

(i) **strongly monotone type** if for all $x, y \in D(T)$, there exist two real numbers $a, b \in (-\infty, +\infty)$ such that
$$\max\{a\|x-y\|^2 + b\|Tx - Ty\|^2, a\|x-y\|^2, b\|Tx-Ty\|^2\} \leq \langle x-y, Tx-Ty\rangle; \tag{2.2}$$

(ii) **firmly type nonexpansive** (Song and Chai [15]) if for all $x, y \in D(T)$, there exists $\gamma \in (0, +\infty)$ such that
$$\|Tx - Ty\|^2 \leq \|x-y\|^2 - \gamma\|(x - Tx) - (y - Ty)\|^2; \tag{2.3}$$

(iii) **firmly nonexpansive** (Bruck [1]) if for all $x, y \in D(T)$,
$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2, \tag{2.4}$$
or equivalently,
$$\|Tx - Ty\|^2 \leq \|x-y\|^2 - \|(x - Tx) - (y - Ty)\|^2; \tag{2.5}$$

(iv) **nonexpansive** if for all $x, y \in D(T)$, the inequality (2.3) holds for $\gamma = 0$;

(v) **strongly monotone** if there exists a constant $c > 0$ such that
$$c\|x-y\|^2 \leq \langle Tx - Ty, x - y\rangle \text{ for all } x, y \in K. \tag{2.6}$$

Obviously, the firmly type nonexpansive mappings contain the firmly nonexpansive mappings as a special case and they are all nonexpansive. Both the projection operator and the resolvent of monotone operator are two subclasses of the firmly type nonexpansive mappings (see [4, 5]). So, these subclasses of nonexpansive mappings may be looked upon as one of the most important class in nonlinear mappings. There are many examples of such mappings, which are found in the references [15, 16, 17].

Remark 2.2. Each firmly type nonexpansive mapping with $\gamma > 1$ is a strongly monotone type mapping with $a = \frac{1}{2} - \frac{1}{2\gamma}$ and $b = \frac{1}{2} + \frac{1}{2\gamma}$. In fact, it follows from the definition of firmly type nonexpansive mapping that

$$\begin{aligned}\|Tx - Ty\|^2 &\leq \|x-y\|^2 - \gamma\|(x - Tx) - (y - Ty)\|^2 \\ &= \|x-y\|^2 - \gamma(\|x-y\|^2 + \|Tx - Ty\|^2 - 2\langle Tx - Ty, x - y\rangle) \\ &= (1-\gamma)\|x-y\|^2 - \gamma\|Tx - Ty\|^2 + 2\gamma\langle Tx - Ty, x - y\rangle.\end{aligned}$$

Then we have
$$(1 + \frac{1}{\gamma})\|Tx - Ty\|^2 + (1 - \frac{1}{\gamma})\|x-y\|^2 \leq 2\langle Tx - Ty, x - y\rangle. \tag{2.7}$$

So the firmly type nonexpansive mappings with $0 < \gamma \leq 1$ may not be strongly monotone since $\frac{1}{2}(1 - \frac{1}{\gamma}) \leq 0$.

In mathematics, the Hilbert projection theorem is a famous result of convex analysis.

Theorem 2.3 (**Hilbert projection theorem**). *Let $K \subset H$ be closed convex. Then for each $x \in H$, there exists a unique element $x^* \in K$ such that*

$$\|x - x^*\| = \min_{y \in K} \|x - y\|.$$

We denote

$$P_K(x) = \{x^* \in K; \|x - x^*\| \leq \|x - y\| \text{ for all } y \in K\}.$$

Then the Hilbert projection theorem assures that P_K is a mapping from Hilbert space H to K, which is called **metric projection** from H to K. The following conclusions are well-known (for example, see [4, 5]).

Lemma 2.4 ([4, 5]). *Let $K \subset H$ be closed convex. Then*

(i) P_K is metric projection from H to K if and only if

$$\langle x - P_K(x), P_K(x) - y\rangle \geq 0 \text{ for all } y \in K.$$

(ii) P_K is firmly nonexpansive, i.e.,

$$\langle P_K(x) - P_K(y), x - y\rangle \geq \|P_K(x) - P_K(y)\|^2.$$

So P_K is nonexpansive,

$$\|P_K(x) - P_K(y)\| \leq \|x - y\|.$$

3. Main results

In this section, we first show that z is a unique solution of the NCP(T) for a strongly monotone type mapping T using the similar proof technique in Nanda and Nanda [12].

Theorem 3.1. *Let H be a real Hilbert space and let $K \subset H$ be a closed convex cone with vertex 0 and dual cone K^*. Assume that $T : K \to H$ is a strongly monotone type mapping with two constants a and b and for each $x^0 = x \in K$ and $x^{k+1} = P_K(x^k - Tx^k)$.*

(i) If $0 < a \leq \frac{1}{2}$ and $b \geq \frac{1}{2}$, then there exists a unique $z \in K$ such that

$$z \in K, \quad Tz \in K^* \text{ and } \langle z, Tz\rangle = 0. \tag{3.1}$$

Furthermore,

$$\lim_{k\to\infty} x^k = z \text{ and } \|x^k - z\| \leq \frac{(1-2a)^{\frac{k}{2}}}{1-\sqrt{1-2a}}\|Tx\|. \tag{3.2}$$

(ii) If $b > 0$ and $\frac{1}{2b^2} - \frac{1}{b} < a \leq \frac{1}{2} + \frac{1}{2b^2} - \frac{1}{b}$, then there exists a unique $z \in K$ satisfying (3.1) and

$$\lim_{k\to\infty} x^k = z \text{ and } \|x^k - z\| \leq \frac{(1-2a+\frac{1}{b^2}-\frac{2}{b})^{\frac{k}{2}}}{1-\sqrt{1-2a+\frac{1}{b^2}-\frac{2}{b}}}\|Tx\|. \tag{3.3}$$

(iii) If $a \geq 0$ and $\frac{1}{2} < b$, then there exists a unique $z \in K$ satisfying (3.1) and

$$\lim_{k\to\infty} x^k = z \text{ and } \|x^k - z\| \leq \frac{|1-\frac{1}{b}|^k}{1-|1-\frac{1}{b}|}\|Tx\|. \tag{3.4}$$

(iv) If $b > 0$ and $\frac{1}{2b^2} < a \leq \frac{1}{2} + \frac{1}{2b^2}$, then there exists a unique $z \in K$ satisfying (3.1) and

$$\lim_{k\to\infty} x^k = z \text{ and } \|x^k - z\| \leq \frac{(\frac{1}{b^2} + 1 - 2a)^{\frac{k}{2}}}{1 - \sqrt{\frac{1}{b^2} + 1 - 2a}} \|Tx\|. \tag{3.5}$$

Proof. Let $F(x) = P_K(x - Tx)$ for $x \in K$. Then it follows from the definition of the metric projection and Lemma 2.4 that F is a mapping from K into itself and

$$\|F(x) - F(y)\| = \|P_K(x - Tx) - P_K(y - Ty)\| \leq \|x - Tx - (y - Ty)\|.$$

So, we have

$$\begin{aligned} \|F(x) - F(y)\|^2 &\leq \|x - Tx - (y - Ty)\|^2 \\ &= \|x - y\|^2 + \|Tx - Ty\|^2 - 2\langle Tx - Ty, x - y\rangle. \end{aligned} \tag{3.6}$$

(i) It follows from Definition 2.1 together with (3.6) that

$$\begin{aligned} \|F(x) - F(y)\|^2 &\leq \|x - y\|^2 + \|Tx - Ty\|^2 - 2b\|Tx - Ty\|^2 - 2a\|x - y\|^2 \\ &= (1 - 2a)\|x - y\|^2 + (1 - 2b)\|Tx - Ty\|^2. \end{aligned}$$

Since $0 < a \leq \frac{1}{2}$ and $b \geq \frac{1}{2}$, we have

$$\|F(x) - F(y)\|^2 \leq (1 - 2a)\|x - y\|^2,$$

and so

$$\|F(x) - F(y)\| \leq \sqrt{1 - 2a}\|x - y\|.$$

Thus, $F : K \to K$ is a contraction with a contraction coefficient $\beta = \sqrt{1 - 2a} < 1$. By the Banach contraction principle, there exists a unique $z \in K$ such that

$$z = F(z), \ \lim_{n\to\infty} F^k(x) = z \text{ and } \|F^k(x) - z\| \leq \frac{\beta^k}{1 - \beta}\|x - F(x)\| \text{ for each } x \in K. \tag{3.7}$$

It follows from Lemma 2.4 (i) together with $F(z) = P_K(z - Tz) = z$ that

$$0 \leq \langle z - Tz - P_K(z - Tz), P_K(z - Tz) - y\rangle = \langle -Tz, z - y\rangle \tag{3.8}$$

for all $y \in K$. Since $K \subset H$ is a closed convex cone with vertex 0, we may take $y = 0 \in K$ and $y = 2z \in K$ in (3.8),

$$\langle Tz, z\rangle \leq 0 \text{ and } \langle Tz, z\rangle \geq 0$$

and so

$$\langle Tz, z\rangle = 0.$$

Therefore, from (3.8), it follows

$$0 \leq \langle Tz, y\rangle - \langle Tz, z\rangle = \langle Tz, y\rangle \text{ for all } y \in K,$$

and hence, $Tz \in K^*$. So, z is a solution of the NCP(T), i.e., z satisfies (3.1).

Now we show the uniqueness. Suppose that there is z^* satisfying (3.1). Then we have

$$\langle Tz^*, y\rangle \geq 0 \text{ for all } y \in K \text{ and } \langle Tz^*, z^*\rangle = 0$$

and so,

$$\langle (z^* - Tz^*) - z^*, z^* - y\rangle = \langle Tz^*, y - z^*\rangle \geq 0 \text{ for all } y \in K.$$

Thus, we have

$$F(z^*) = P_K(z^* - Tz^*) = z^*,$$

and so $z = z^*$ by the uniqueness of fixed point of F.

Let $x^{k+1} = F(x^k) = P_K(x^k - Tx^k)$ for $x^0 = x \in K$. Then we have $x = P_K(x)$ and

$$\|x - F(x)\| = \|P_K(x) - P_K(x - Tx)\| \leq \|x - (x - Tx)\| = \|Tx\|.$$

By (3.7) together with $\beta = \sqrt{1-2a}$, we obtain (3.2). This shows (i).

(ii) Similarly, it follows from Definition 2.1 together with (3.6) that

$$\|F(x) - F(y)\|^2 \leq (1-2a)\|x-y\|^2 + (1-2b)\|Tx - Ty\|^2$$

and

$$b\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle \leq \|Tx - Ty\|\|x - y\|.$$

Since $b > 0$ and $\frac{1}{2b^2} - \frac{1}{b} < a \leq \frac{1}{2} + \frac{1}{2b^2} - \frac{1}{b}$, we have

$$\|Tx - Ty\| \leq \frac{1}{b}\|x - y\| \tag{3.9}$$

and so

$$\|F(x) - F(y)\|^2 \leq (1 - 2a + \frac{1}{b^2} - \frac{2}{b})\|x - y\|^2.$$

Thus

$$\|F(x) - F(y)\| \leq \sqrt{1 - 2a + \frac{1}{b^2} - \frac{2}{b}}\|x - y\|.$$

So, $F : K \to K$ is a contraction with a contraction coefficient $\beta = \sqrt{1 - 2a + \frac{1}{b^2} - \frac{2}{b}} < 1$. The remainder of the proof is the same as ones of (i) and we omit it.

(iii) It follows from Definition 2.1 together with (3.6) that

$$\|F(x) - F(y)\|^2 \leq \|x - y\|^2 + \|Tx - Ty\|^2 - 2b\|Tx - Ty\|^2.$$

Since $b > \frac{1}{2}$, using (3.9), we have

$$\|F(x) - F(y)\|^2 \leq (1 + \frac{1}{b^2} - \frac{2}{b})\|x - y\|^2$$

and so

$$\|F(x) - F(y)\| \leq |1 - \frac{1}{b}|\|x - y\|.$$

Thus, $F : K \to K$ is a contraction with a contraction coefficient $\beta = |1 - \frac{1}{b}| < 1$. The remainder of the proof is the same as ones of (i) and we omit it.

(iv) It follows from Definition 2.1 together with (3.6) that

$$\|F(x) - F(y)\|^2 \leq \|x - y\|^2 + \|Tx - Ty\|^2 - 2a\|x - y\|^2.$$

Since $b > 0$ and $\frac{1}{2b^2} < a \leq \frac{1}{2} + \frac{1}{2b^2}$, using (3.9), we have

$$\|F(x) - F(y)\|^2 \leq (1 + \frac{1}{b^2} - 2a)\|x - y\|^2,$$

and so

$$\|F(x) - F(y)\| \leq \sqrt{1 + \frac{1}{b^2} - 2a}\|x - y\|.$$

Thus, $F : K \to K$ is a contraction with a contraction coefficient $\beta = \sqrt{1 + \frac{1}{b^2} - 2a} < 1$. The remainder of the proof is the same as ones of (i) and we omit it.

The desired conclusion is proved. □

Corollary 3.2. *Let H be a real Hilbert space and let $K \subset H$ be a closed convex cone with vertex 0 and dual cone K^*. If $T : K \to H$ is a firmly type nonexpansive mapping with $\gamma > 1$, then there exists a unique $z \in K$ such that*

$$z \in K, \quad Tz \in K^* \text{ and } \langle z, Tz \rangle = 0.$$

Furthermore, for each $x^0 = x \in K$ and $x^{k+1} = P_K(x^k - Tx^k)$,

$$\lim_{k\to\infty} x^k = z \text{ and } \|x^k - z\| \le \frac{1}{\gamma^{\frac{k}{2}} - \gamma^{\frac{k-1}{2}}} \|Tx\|. \tag{3.10}$$

Proof. It follows from (2.7) that T is a strongly monotone type mapping with $a = \frac{1}{2} - \frac{1}{2\gamma}$ and $b = \frac{1}{2} + \frac{1}{2\gamma}$. Then by Theorem 3.1 (i) and (iii), the desired conclusion follows. □

It is obvious that the existence and uniqueness of the NCP(T) in $\mathbb{R}^n$ are easily obtained.

Corollary 3.3. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n$ be a firmly type nonexpansive mapping with $\gamma > 1$. Then there exists a unique $z \in \mathbb{R}^n$ such that*

$$z \ge 0, \quad Tz \ge 0 \text{ and } z^\top (Tz) = 0.$$

Furthermore, for each $x^0 = x \ge 0$ and $x^{k+1} = P_{\mathbb{R}^n_+}(x^k - Tx^k)$,

$$\lim_{k\to\infty} x^k = z \text{ and } \|x^k - z\| \le \frac{1}{\gamma^{\frac{k}{2}} - \gamma^{\frac{k-1}{2}}} \|Tx\|.$$

Remark 3.4. It follows from (2.7) that a firmly type nonexpansive mapping is strongly monotone with $c = \frac{1}{2}(1 - \frac{1}{\gamma}) < \frac{1}{2}$ ($\gamma > 1$). Nanda and Nanda [12] studied strongly monotone and Lipschitzian mapping with a Lipschitzian constant l and $l^2 < 2c < l^2 + 1$ and so $\frac{1}{2} < c < 1$ when $l = 1$. The results of Nanda and Nanda [12] may be known as Theorem 3.1 (iv) with $l = \frac{1}{b} > 0$ and $c = a$. Thus, our conclusions may be referred to as the complementary and development of ones of Nanda and Nanda [12].

Remark 3.5. For a firmly type nonexpansive mapping T with $0 < \gamma \le 1$, whether or not the NCP(T) has a unique solution, which is worth doing further research.

By Theorem 3.1, the following conclusion will be easily obtained.

Corollary 3.6. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n$ be a strongly monotone type mapping with two constants $a, b \in \mathbb{R}$. Assumed that one of the following conditions holds:*

(i) $b > \frac{1}{2}$;

(ii) $0 < a \le \frac{1}{2}$ *and* $b = \frac{1}{2}$;

(iii) $b > 0$ *and* $\frac{1}{2b^2} - \frac{1}{b} < a \le \frac{1}{2} + \frac{1}{2b^2}$.

Then there exists a unique $z \in \mathbb{R}^n$ such that

$$z \ge 0, \quad Tz \ge 0 \text{ and } z^\top (Tz) = 0.$$

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