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Existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations with fractional integral boundary conditions

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Abstract

This paper investigates the existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations with Riemann-Liouville fractional integral boundary conditions. By applying a variety of fixed point theorems, combining with a new inequality of fractional order form, some sufficient conditions are established. Some examples are given to illustrate our results.

Keywords: Coupled system, fractional integral conditions, Remann-Liouville fractional derivative, fixed point theorem.
2010 MSC: 26A33, 34B18.

1. Introduction

In this paper, we consider the following coupled system of nonlinear fractional differential equations(FDE for short of the form) with Riemann-Liouville fractional integral boundary conditions

$$\begin{cases} D_{0^+}^{\alpha_1}u(t) = f_1(t,u(t),v(t),D_{0^+}^{\rho_1}u(t),D_{0^+}^{\rho_2}v(t)), \quad t\in(0,1),\\ D_{0^+}^{\alpha_2}v(t) = f_2(t,u(t),v(t),D_{0^+}^{\rho_1}u(t),D_{0^+}^{\rho_2}v(t)), \quad t\in(0,1),\\ u(0)=u'(0)=0, v(0)=v'(0)=0,\\ u(1)=\gamma_1 I_{0^+}^{\beta_1}u(\eta_1), v(1)=\gamma_2 I_{0^+}^{\beta_2}v(\eta_2), \end{cases} \tag{1.1}$$

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where $D_{0+}^{\alpha_i}$ and $D_{0+}^{\rho_i}$ denote the standard Riemann-Liouville fractional derivative, $I_{0+}^{\beta_i}$ denotes the Riemann-Liouville fractional integral, $2 < \alpha_i \leq 3, f_i \in C([0,1] \times \mathbb{R}^4, \mathbb{R}), 0 < \rho_i \leq 1, 0 \leq \eta_i \leq 1, \gamma_i, \beta_i > 0, \Gamma(\alpha_i + \beta_i) \neq \gamma_i \Gamma(\alpha_i) \eta_i^{\alpha_i \beta_i - 1}, i = 1, 2.$

Fractional differential equations are widely used in many engineering and scientific disciplines. Many books on fractional differential equations, fractional calculus have been published (see, for example [15, 16, 18, 21, 32]), and some recent papers on the topic, see [1, 3, 4, 5, 7, 9, 11, 12, 14, 19, 24, 25, 26, 32] and the references therein. At the same time, there are many papers concerned with solvability of coupled systems of nonlinear fractional differential equations, because of their wide applications. For details, see [2, 6, 13, 17, 20, 22, 23, 27, 28, 29, 30, 31, 33] and the references therein.

In [22], by applying the Schauder fixed point theorem, Su established sufficient conditions for the existence of solutions for the following coupled system with two point boundary conditions

$$\begin{cases} D^\alpha u(t) = f(t, v(t), D^p v(t)), \quad t \in (0,1), \\ D^\beta v(t) = g(t, u(t), D^q u(t)), \quad t \in (0,1), \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases} \tag{1.2}$$

where $1 < \alpha, \beta < 2, p, q > 0, \alpha - q \geq 1, \beta - p \geq 1, f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions and D is the standard Riemann-Liouville fractional derivative. B. Ahmad and J. Neito [2] improved the results to a three-point boundary value problem.

In [23], Wang et al. studied the existence and uniqueness of positive solutions to a three-point boundary value problems for the coupled system

$$\begin{cases} D^\alpha u(t) = f(t, v(t)), \quad t \in (0,1), \\ D^\beta v(t) = g(t, u(t)), \quad t \in (0,1), \\ u(0) = v(0) = 0, u(1) = au(\xi), v(1) = bv(\xi), \end{cases} \tag{1.3}$$

where $1 < \alpha, \beta < 2, 0 \leq a, b \leq 1, 0 < \xi < 1, f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are given continuous functions and D is the standard Riemann-Liouville fractional derivative.

In [28], by applying coincidence degree theory, Zhang et al. established two existence results for a four-point boundary value problems at resonance for the coupled system of nonlinear fractional differential equations

$$\begin{cases} D^\alpha u(t) = f(t, v(t), D^{\beta-1} v(t)), \quad t \in (0,1), \\ D^\beta v(t) = g(t, u(t), D^{\alpha-1} u(t)), \quad t \in (0,1), \\ u(0) = v(0) = 0, u(1) = \sigma_1 u(\eta_1), v(1) = \sigma_2 v(\eta_2), \end{cases} \tag{1.4}$$

In [20], the existence and uniqueness of solutions for the following problem is studied

$$\begin{cases} {}^C D_{0+}^\alpha u(t) = f(t, u(t), v(t)), \quad t \in [0,1], \\ {}^C D_{0+}^\beta v(t) = g(t, u(t), v(t)), \quad t \in [0,1], \\ u(0) = \gamma I_{0+}^p u(\eta), \quad 0 < \eta < 1, \\ v(0) = \delta I_{0+}^q v(\xi), \quad 0 < \xi < 1, \end{cases} \tag{1.5}$$

where ${}^C D_{0+}^\alpha$ and ${}^C D_{0+}^\beta$ denote the Caputo fractional derivative, I_{0+}^p, I_{0+}^q denote Riemann-Liouville fractional integral, $0 < \alpha, \beta < 1, f, g \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$, and $p, q, \gamma, \delta \in \mathbb{R}$.

Motivated by the above-mentioned works, we investigate the existence and uniqueness of the system (1.1). The main features are as follows: First, the system we discuss here is different from [2, 6, 13, 17, 20, 22, 23, 27, 28, 29, 30, 31, 33], where the nonlinear terms involved two unknown functions u, v and the fractional derivative of unknown functions u, v, and the case is more general and difficult than the above papers. Secondly, by using a new inequality of fractional order form, the restrictive conditions $a_{ik}(t)$ in

$(H_1), (H_2), (H_4)$ and (H_6) are applicable for more general problems than the conditions a_{ik} in the mentioned documents.

The paper is organized as follows. In Section 2, we present preliminaries and several lemmas. In Section 3, we establish the existence and uniqueness of nonlinear FDE (1.1) by using a variety of fixed point theorems and give some examples to demonstrate the main results.

2. Preliminaries

For the convenience, we present some definitions and lemmas.

Definition 2.1 ([15, 21])**.** The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dx}\Big)^n \int_0^x (x-t)^{n-\alpha-1} f(t)dt,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2.2 ([15, 21])**.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}_{0+} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt,$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 2.3 ([15, 21])**.** *Assume that $u(t) \in L^1(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $L^1(0, 1)$. Then*

$$I^{\alpha}_{0+} D^{\alpha}_{0+} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R} (i = 1, 2, \cdots , n), n$ is the smallest integer than or equal to α.

Lemma 2.4 ([15, 21])**.**

(1) If $g(t) \in L^1(a, b), p > q > 0$, then

$$I^{p}_{0+} I^{q}_{0+} g(t) = I^{p+q}_{0+} g(t),\ \ D^{q}_{0+} I^{p}_{0+} g(t) = I^{p-q}_{0+} g(t),\ \ D^{p}_{0+} I^{p}_{0+} g(t) = g(t).$$

(2) If $p > q > 0$, then

$$I^{p}_{0+} t^q = \frac{\Gamma(q+1)}{\Gamma(q+1+p)} t^{q+p},\quad D^{q}_{0+} t^p = \frac{\Gamma(p+1)}{\Gamma(p+1-q)} t^{p-q},\quad D^{p}_{0+} t^q = 0.$$

Let $E = C[0, 1]$, set $\Lambda_i = \frac{\Gamma(\alpha_i+\beta_i)}{\Gamma(\alpha_i+\beta_i)-\gamma_i\Gamma(\alpha_i)\eta_i^{\alpha_i+\beta_i-1}}, U_{\alpha_i} = \{u(t)|u(t) \in E \text{ and } D^{\alpha_i} u(t) \in E\}, i = 1, 2.$

Lemma 2.5. *Assume that $y(t) \in E$ and $2 < \alpha_1 \le 3, 0 \le \eta_1 \le 1, \gamma_1, \beta_1 > 0$. Then the unique solution of the following boundary value problem*

$$\begin{cases} D^{\alpha_1}_{0+} u(t) = x(t),\ t \in [0, 1], \\ u(0) = u'(0) = 0, u(1) = \gamma_1 I^{\beta_1}_{0+} u(\eta_1), \end{cases} \tag{2.1}$$

is given in U_{α_1} by

$$u(t) = \int_0^1 G_1(t, s)x(s)ds, \tag{2.2}$$

where $G_1(t,s)$ is the Green's function given by

$$G_1(t,s)=\begin{cases} G_{11}(t,s), 0\le t\le \eta_1,\\ G_{12}(t,s), \eta_1\le t\le 1,\end{cases} \tag{2.3}$$

$$G_{11}(t,s)=\begin{cases} \dfrac{(t-s)^{\alpha_1-1}-\Lambda_1(t(1-s))^{\alpha_1-1}}{\Gamma(\alpha_1)}+\dfrac{\Lambda_1\gamma_1 t^{\alpha_1-1}(\eta_1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}, 0\le s\le t,\\ \dfrac{\Lambda_1\gamma_1 t^{\alpha_1-1}(\eta_1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}-\dfrac{\Lambda_1(t(1-s))^{\alpha_1-1}}{\Gamma(\alpha_1)}, t<s\le\eta_1,\\ -\dfrac{\Lambda_1(t(1-s))^{\alpha_1-1}}{\Gamma(\alpha_1)}, \eta_1<s\le 1,\end{cases}$$

$$G_{12}(t,s)=\begin{cases} \dfrac{(t-s)^{\alpha_1-1}-\Lambda_1(t(1-s))^{\alpha_1-1}}{\Gamma(\alpha_1)}+\dfrac{\Lambda_1\gamma_1 t^{\alpha_1-1}(\eta_1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}, 0\le s\le \eta_1,\\ \dfrac{(t-s)^{\alpha_1-1}-\Lambda_1(t(1-s))^{\alpha_1-1}}{\Gamma(\alpha_1)}, \eta_1<s\le t,\\ -\dfrac{\Lambda_1(t(1-s))^{\alpha_1-1}}{\Gamma(\alpha_1)}, t<s\le 1.\end{cases}$$

Proof. Using Lemma 2.3, the equation $D_{0^+}^{\alpha_1}u(t)=x(t)$ in (2.1) means

$$u(t)=I_{0^+}^{\alpha_1}x(t)+c_1t^{\alpha_1-1}+c_2t^{\alpha_1-2}+c_3t^{\alpha_1-3}.$$

Applying the condition $u(0)=u'(0)=0$, we obtain $c_3=c_2=0$. So

$$u(t)=I_{0^+}^{\alpha_1}x(t)+c_1t^{\alpha_1-1}. \tag{2.4}$$

Applying Lemma 2.4 and (2.4), we obtain

$$I_{0^+}^{\beta_1}u(t)=I_{0^+}^{\alpha_1+\beta_1}x(t)+c_1\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1+\beta_1)}t^{\alpha_1+\beta_1-1}. \tag{2.5}$$

Now, the boundary value condition $u(1)=\gamma_1 I_{0^+}^{\beta_1}u(\eta_1)$, (2.4) and (2.5) leads to

$$c_1=\Lambda_1[\gamma_1 I_{0^+}^{\alpha_1+\beta_1}x(\eta_1)-I_{0^+}^{\alpha_1}x(1)].$$

Substituting c_1 to (2.4), we have

$$u(t)=I_{0^+}^{\alpha_1}x(t)+\Lambda_1[\gamma_1 I_{0^+}^{\alpha_1+\beta_1}x(\eta_1)-I_{0^+}^{\alpha_1}x(1)]t^{\alpha_1-1}. \tag{2.6}$$

That is

$$\begin{aligned} u(t)&=\int_0^t \frac{(t-s)^{\alpha_1-1}-\Lambda_1(t(1-s))^{\alpha_1-1}}{\Gamma(\alpha_1)}x(s)ds-\int_t^1 \frac{\Lambda_1(t(1-s))^{\alpha_x-1}}{\Gamma(\alpha_1)}x(s)ds\\ &\quad+\int_0^{\eta_1}\frac{\Lambda_1\gamma_1 t^{\alpha_1-1}(\eta_1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}x(s)ds\\ &=\int_0^1 G_1(t,s)x(s)ds.\end{aligned}$$

The proof is completed. □

Similarly, the unique solution of $D_{0^+}^{\alpha_2}v(t)=y(t), v(0)=v'(0)=0, v(1)=\gamma_2 I_{0^+}^{\beta_2}v(\eta_2)$ is

$$v(t)=\int_0^1 G_2(t,s)y(s)ds,$$

where $G_2(t,s)$ can be given from $G_1(t,s)$ by replacing $\alpha_1, \beta_1, \gamma_1, \Lambda_1$ with $\alpha_2, \beta_2, \gamma_2, \Lambda_2$.

Let $X = U_{\rho_1}, \|u\|_X = \max\limits_{t\in J}|u(t)| + \max\limits_{t\in J}|D^{\rho_1}u(t)|$, and $Y = U_{\rho_2}, \|v\|_Y = \max\limits_{t\in J}|v(t)| + \max\limits_{t\in J}|D^{\rho_2}v(t)|$. The product space $(X\times Y, \|(u,v)\|_{X\times Y})$ is a Banach space equipped with norm $\|(u,v)\|_{X\times Y} = \|u\|_X + \|v\|_Y$. We can define operator $T : X\times Y \to X\times Y$ by

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)), \tag{2.7}$$

where

$$T_i(u,v)(t) = \int_0^1 G_i(t,s) f_i(s, u(s), v(s), D_{0^+}^{\rho_1}u(s), D_{0^+}^{\rho_2}v(s))ds, i = 1,2.$$

Lemma 2.6. *Assume that $f_1, f_2 \in C([0,1]\times\mathbb{R}^4, \mathbb{R})$. Then $(u,v)\in U_{\alpha_1}\times U_{\alpha_2}$ is a solution of FDE (1.1) if and only if $(u,v)\in X\times Y$ is a solution of the operator equations $T(u,v) = (u,v)$.*

Proof. The proof is immediate from Lemma 2.5, so we omit. □

Denote that $L^1([0,1],\mathbb{R})$ is the Banach space of Lebesgue integrable functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|y\|_{L^1} = \int_0^1 |y(t)|dt$.

Lemma 2.7 ([10])**.** *Assume that $q > 0, f \in L^1([a,b], R_+)$. Then, we have*

$$I_{0^+}^{q+1}f(t) \le \|I_{0^+}^q f\|_{L^1}, t\in[a,b].$$

Lemma 2.8 ([8])**.** *Let F be a Banach space and Ω be a bounded open subset of $F, 0\in\Omega, T : \overline{\Omega}\to F$ be a completely continuous operator. Then, either there exists $x\in\partial\Omega, \lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^*\in\overline{\Omega}$.*

3. Main results

For convenience, we set

$$\begin{aligned}
A_{ik} &= (1+|\Lambda_i|)||I_{0^+}^{\alpha_i-1}a_{ik}||_{L^1} + |\Lambda_i|\gamma_i||I_{0^+}^{\alpha_i+\beta_i-1}a_{ik}||_{L^1},\ i=1,2, k=1,2,3,4,5,\\
B_{ik} &= ||I_{0^+}^{\alpha_i-\rho_i-1}a_{ik}||_{L^1} + \frac{|\Lambda_i|\gamma_i\Gamma(\alpha_i)}{\Gamma(\alpha_i-\rho_i)}||I_{0^+}^{\alpha_i+\beta_i-1}a_{ik}||_{L^1} + \frac{|\Lambda_i|\Gamma(\alpha_i)}{\Gamma(\alpha_i-\rho_i)}||I_{0^+}^{\alpha_i-1}a_{ik}||_{L^1},\\
&\qquad i=1,2, k=1,2,3,4,5,\\
C_{ik} &= A_{ik} + B_{ik},\ i=1,2, k=1,2,3,4,5,\\
D_i &= \max\Big\{C_{i1}, C_{i2}, C_{i3}, C_{i4}\Big\}, i=1,2.
\end{aligned}$$

In the following subsection, we obtain our main results for FDE (1.1) by applying a variety of fixed point theorems. The first results is based on the Schauder fixed point theorem.

Theorem 3.1. *Assume that there exist nonnegative functions $a_{ik}(t)\in L^1([0,1],\mathbb{R}_+)(i=1,2,k=1,2,3,4,5)$ such that the following condition is satisfied*

(H_1) $|f_i(t,x_1,x_2,x_3,x_4)| \le \sum\limits_{k=1}^{4} a_{ik}(t)|x_k|^{\tau_{ik}} + a_{i5}(t), 0<\tau_{ik}<1.$

Then FDE (1.1) has at least one solution.

Proof. First, we can define a ball in Banach space $X\times Y$ as

$$B_R = \{(u,v)|(u,v)\in X\times Y, \|(u,v)\|_{X\times Y}\le R\}, \tag{3.1}$$

where

$$R \ge \max\Big\{(10C_{i1})^{\frac{1}{1-\tau_{i1}}}, (10C_{i2})^{\frac{1}{1-\tau_{i2}}}, (10C_{i3})^{\frac{1}{1-\tau_{i3}}}, (10C_{i4})^{\frac{1}{1-\tau_{i4}}}, 10C_{i5}, i=1,2\Big\}.$$

Consequently, we show that $T : B_R\to B_R$. For any $(u,v)\in B_R$, using Lemma 2.7, the condition (H_1) and

(2.6), we can obtain

$$
\begin{aligned}
|T_1(u,v)(t)| &= |\int_0^1 G_1(t,s) f_1(s,u(s),v(s),D^{\rho_1}u(s),D^{\rho_2}v(s))ds| \\
&\le I_{0^+}^{\alpha_1}|f_1(t,u(t),v(t),D^{\rho_1}u(t),D^{\rho_2}v(t))| \\
&\quad + |\Lambda_1|\gamma_1 I_{0^+}^{\alpha_1+\beta_1}|f_1(\eta_1,u(\eta_1),v(\eta_1),D^{\rho_1}u(\eta_1),D^{\rho_2}v(\eta_1))| \\
&\quad + |\Lambda_1| I_{0^+}^{\alpha_1}|f_1(1,u(1),v(1),D^{\rho_1}u(1),D^{\rho_2}v(1))ds| \\
&\le I_{0^+}^{\alpha_1}\left(\sum_{k=1}^{4} a_{1k}(t)R^{\tau_{1k}} + a_{15}(t)\right) \\
&\quad + |\Lambda_1|\gamma_1 I_{0^+}^{\alpha_1+\beta_1}\left(\sum_{k=1}^{4} a_{1k}(\eta_1)R^{\tau_{1k}} + a_{15}(\eta_1)\right) \\
&\quad + |\Lambda_1| I_{0^+}^{\alpha_1}\left(\sum_{k=1}^{4} a_{1k}(1)R^{\tau_{1k}} + a_{15}(1)\right) \\
&\le \sum_{k=1}^{4}\left[(1+|\Lambda_1|)||I_{0^+}^{\alpha_1-1}a_{1k}||_{L^1} + |\Lambda_1|\gamma_1||I_{0^+}^{\alpha_1+\beta_1-1}a_{1k}||_{L^1}\right]R^{\tau_{1k}} \\
&\quad + \left[(1+|\Lambda_1|)||I_{0^+}^{\alpha_1-1}a_{15}||_{L^1} + |\Lambda_1|\gamma_1||I_{0^+}^{\alpha_1+\beta_1-1}a_{15}||_{L^1}\right] \\
&= \sum_{k=1}^{4} A_{1k}R^{\tau_{1k}} + A_{15}.
\end{aligned}
\tag{3.2}
$$

Similarly, Lemma 2.4 enables us to obtain

$$
\begin{aligned}
|D_{0^+}^{\rho_1}T_1(u,v)(t)| &= |I_{0^+}^{\alpha_1-\rho_1} f_1(t,u(t),v(t),D^{\rho_1}u(t),D^{\rho_2}v(t)) \\
&\quad + \frac{\Lambda_1\Gamma(\alpha_1)}{\Gamma(\alpha_1-\rho_1)} t^{\alpha_1-\rho_1-1}[\gamma_1 I_{0^+}^{\alpha_1+\beta_1} f_1(\eta_1,u(\eta_1),v(\eta_1),D^{\rho_1}u(\eta_1),D^{\rho_2}v(\eta_1)) \\
&\quad - I_{0^+}^{\alpha} f_1(1,u(1),v(1),D^{\rho_1}u(1),D^{\rho_2}v(1)]| \\
&\le I_{0^+}^{\alpha_1-\rho_1}|f_1(t,u(t),v(t),D^{\rho_1}u(t),D^{\rho_2}v(t))| \\
&\quad + \frac{|\Lambda_1|\gamma_1\Gamma(\alpha_1)}{\Gamma(\alpha_1-\rho_1)} I_{0^+}^{\alpha_1+\beta_1}|f_1(\eta_1,u(\eta_1),v(\eta_1),D^{\rho_1}u(\eta_1),D^{\rho_2}v(\eta_1))| \\
&\quad + \frac{|\Lambda_1|\Gamma(\alpha_1)}{\Gamma(\alpha_1-\rho_1)} I_{0^+}^{\alpha_1}|f_1(1,u(1),v(1),D^{\rho_1}u(1),D^{\rho_2}v(1)| \\
&\le \sum_{k=1}^{4}\Bigg[||I_{0^+}^{\alpha_1-\rho_1-1}a_{1k}||_{L^1} + \frac{|\Lambda_1|\gamma_1\Gamma(\alpha_1)}{\Gamma(\alpha_1-\rho_1)}||I_{0^+}^{\alpha_1+\beta_1-1}a_{1k}||_{L^1} \\
&\quad + \frac{|\Lambda_1|\Gamma(\alpha_1)}{\Gamma(\alpha_1-\rho_1)}||I_{0^+}^{\alpha_1-1}a_{1k}||_{L^1}\Bigg]R^{\tau_{1k}} + \Bigg[||I_{0^+}^{\alpha_1-\rho_1-1}a_{15}||_{L^1} \\
&\quad + \frac{|\Lambda_1|\gamma_1\Gamma(\alpha_1)}{\Gamma(\alpha_1-\rho_1)}||I_{0^+}^{\alpha_1+\beta_1-1}a_{15}||_{L^1} + \frac{|\Lambda_1|\Gamma(\alpha_1)}{\Gamma(\alpha_1-\rho_1)}||I_{0^+}^{\alpha_1-1}a_{15}||_{L^1}\Bigg] \\
&= \sum_{k=1}^{4} B_{1k}R^{\tau_{1k}} + B_{15}.
\end{aligned}
\tag{3.3}
$$

From (3.2) and (3.3), we have

$$
||T_1(u,v)||_X \le \sum_{k=1}^{4} C_{1k}R^{\tau_{1k}} + C_{15} \le \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R = \frac{R}{2}.
$$

Similarly, we have

$$||T_2(u,v)||_Y \le \sum_{k=1}^{4} C_{2k}R^{\tau_{2k}} + C_{25} \le \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R + \frac{1}{10}R = \frac{R}{2}.$$

That is

$$||T(u,v)||_{X\times Y} = ||T_1(u,v)||_X + ||T_2(u,v)||_Y \le R.$$

Therefore, we obtain $T : B_R \to B_R$.

Notice that $T_1(u,v)(t), T_2(u,v)(t), D^{\rho_1}T_1(u,v)(t), D^{\rho_2}T_2(u,v)(t)$ are continuous on $[0,1]$. Obviously, operator T is also continuous.

Secondly, we prove that operator T is equicontinuous. Let

$$M_i = \max_{t\in[0,1]} \big\{|f_i(t,u(t),v(t),D^{\rho_1}_{0^+}u(t)),D^{\rho_2}_{0^+}v(t))|\big\}, \forall(u,v)\in B_R, i=1,2.$$

For $t_1,t_2\in[0,1](t_1<t_2)$, we have

$$\begin{aligned}
&|T_1(u,v)(t_2) - T_1(u,v)(t_1)|\\
&\quad= |\int_0^1 \big(G_1(t_2,s) - G_1(t_1,s)\big) f_1(s,u(s),v(s),D^{\rho_1}u(s),D^{\rho_2}v(s))ds|\\
&\quad\le |I^{\alpha_1}_{0^+}\big[f_1(t_2,u(t_2),v(t_2),D^{\rho_1}u(t_2),D^{\rho_2}v(t_2)) - f(t_1,u(t_1),v(t_1),D^{\rho_1}u(t_1),D^{\rho_2}v(t_1))\big]|\\
&\qquad+ |\Lambda_1|\gamma_1 I^{\alpha_1+\beta_1}_{0^+}|f_1(\eta_1,u(\eta_1),v(\eta_1),D^{\rho_1}u(\eta_1),D^{\rho_2}v(\eta_1))|\\
&\qquad+ |\Lambda_1|I^{\alpha_1}_{0^+}|f_1(1,u(1),v(1),D^{\rho_1}u(1),D^{\rho_2}v(1))|(t_2^{\alpha_1-1} - t_1^{\alpha_1-1})\\
&\quad\le M_1\Big[\int_0^{t_1} \frac{(t_2-s)^{\alpha_1-1} - (t_1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}ds\Big]\\
&\qquad+ M_1|\Lambda_1|\Big[\int_0^{\eta_1} \frac{\gamma_1(\eta_1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}ds + \int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}ds\Big](t_2^{\alpha_1-1} - t_1^{\alpha_1-1})\\
&\quad\le \frac{M_1}{\Gamma(\alpha_1+1)}(t_2^{\alpha_1} - t_1^{\alpha_1}) + M_1|\Lambda_1|\Big[\frac{\gamma_1\eta_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{1}{\Gamma(\alpha_1+1)}\Big](t_2^{\alpha_1-1} - t_1^{\alpha_1-1}).
\end{aligned} \tag{3.4}$$

On the other hand, we have

$$\begin{aligned}
&|D^{\rho_1}_{0^+}T(u,v)(t_2) - D^{\rho_1}_{0^+}T(u,v)(t_1)|\\
&\quad\le |I^{\alpha_1-\rho_1}_{0^+}\big[f_1(t_2,u(t_2),v(t_2),D^{\rho_1}u(t_2),D^{\rho_2}v(t_2)) - f(t_1,u(t_1),v(t_1),D^{\rho_1}u(t_1),D^{\rho_2}v(t_1))\big]|\\
&\qquad+ \frac{|\Lambda_1|\Gamma(\alpha_1)}{\Gamma(\alpha_1-\rho_1)}\Big[\gamma_1|I^{\alpha_1+\beta_1}_{0^+}f_1(\eta_1,u(\eta_1),v(\eta_1),D^{\rho_1}u(\eta_1),D^{\rho_2}v(\eta_1))|\\
&\qquad+ |I^{\alpha_1}_{0^+}f_1(1,u(1),v(1),D^{\rho_1}u(1),D^{\rho_2}v(1))|\Big](t_2^{\alpha_1-\rho_1-1} - t_1^{\alpha_1-\rho_1-1})\\
&\quad\le M_1\Big\{\int_0^{t_1} \frac{(t_2-s)^{\alpha_1-\rho_1-1} - (t_1-s)^{\alpha_1-\rho_1-1}}{\Gamma(\alpha_1-\rho_1)}ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha_1-\rho_1-1}}{\Gamma(\alpha_1-\rho_1)}ds\\
&\qquad+ \frac{|\Lambda_1|\Gamma(\alpha_1)}{\Gamma(\alpha_1-\rho_1)}\Big[\int_0^{\eta_1} \frac{\gamma_1(\eta_1-s)^{\alpha_1+\beta_1-1}}{\Gamma(\alpha_1+\beta_1)}ds + \int_0^1 \frac{(1-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}ds\Big]\\
&\qquad\times (t_2^{\alpha_1-\rho_1-1} - t_1^{\alpha_1-\rho_1-1})\Big\}\\
&\quad\le \frac{M_1}{\Gamma(\alpha_1-\rho_1+1)}(t_2^{\alpha_1-\rho_1} - t_1^{\alpha_1-\rho_1}) + \frac{M_1|\Lambda_1|\Gamma(\alpha_1)}{\Gamma(\alpha_1-\rho_1)}\Big[\frac{\gamma_1\eta_1^{\alpha_1+\beta_1}}{\Gamma(\alpha_1+\beta_1+1)} + \frac{1}{\Gamma(\alpha_1+1)}\Big]\\
&\qquad\times (t_2^{\alpha_1-\rho_1-1} - t_1^{\alpha_1-\rho_1-1}).
\end{aligned} \tag{3.5}$$

Analogously, we can show that

$$|T_2(u,v)(t_2) - T_2(u,v)(t_1)| \leq \frac{M_2}{\Gamma(\alpha_2+1)}(t_2^{\alpha_2} - t_1^{\alpha_2}) + M_2|\Lambda_2|\Big[\frac{\gamma_2\eta_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{1}{\Gamma(\alpha_2+1)}\Big](t_2^{\alpha_2-1} - t_1^{\alpha_2-1}). \tag{3.6}$$

$$\begin{aligned}|D^{\rho_2}T_2(u,v)(t_2) - D^{\rho_2}T_2(u,v)(t_1)| \leq &\frac{M_2}{\Gamma(\alpha_2-\rho_2+1)}(t_2^{\alpha_2-\rho_2} - t_1^{\alpha_2-\rho_2}) + \frac{M_2|\Lambda_2|\Gamma(\alpha_2)}{\Gamma(\alpha_2-\rho_2)}\Big[\frac{\gamma_2\eta_2^{\alpha_2+\beta_2}}{\Gamma(\alpha_2+\beta_2+1)} + \frac{1}{\Gamma(\alpha_2+1)}\Big] \\ &\times (t_2^{\alpha_2-\rho_2-1} - t_1^{\alpha_2-\rho_2-1}).\end{aligned} \tag{3.7}$$

In view of (3.4)-(3.7), letting $t_1 \to t_2$, then

$$|T_1(u,v)(t_2) - T_1(u,v)(t_1)| \to 0,\ |D^{\rho_1}T_1(u,v)(t_2) - D^{\rho_1}T_1(u,v)(t_1)| \to 0,$$

$$|T_2(u,v)(t_2) - T_2(u,v)(t_1)| \to 0,\ |D^{\rho_2}T_2(u,v)(t_2) - D^{\rho_2}T_2(u,v)(t_1)| \to 0.$$

Therefore

$$||T_1(u,v)(t_2) - T_1(u,v)(t_1)||_X \to 0,\ ||T_2(u,v)(t_2) - T_2(u,v)(t_1)||_Y \to 0.$$

That is, as $t_1 \to t_2$,

$$||T(u,v)(t_2) - T(u,v)(t_1)||_{X\times Y} \to 0.$$

Therefore we prove that $T(B_R)$ is an equicontinuous set. Also, it is uniformly bounded as $T(B_R) \subset B_R$. By applying the Arzelà-Ascoli theorem, we conclude that T is a completely continuous operator. So, it follows from the Schauder fixed theorem that FDE (1.1) has at least one solution (u,v) in B_R. This proof is completed. □

Theorem 3.2. *Assume that there exist nonnegative functions $a_{ik}(t) \in L^1([0,1],\mathbb{R}_+)(i = 1,2, k = 1,2,3,4,5)$ such that the following conditions are satisfied*

(H_2) $|f_i(t,x_1,x_2,x_3,x_4)| \leq \sum\limits_{k=1}^{4} a_{ik}(t)|x_k|^{\tau_{ik}} + a_{i5}(t), \tau_{ik} > 1$,

(H_3) $10\max\Big\{C_{i5}, i = 1,2\Big\} < \min\Big\{(\frac{1}{10C_{ij}})^{\frac{1}{\tau_{ij}-1}}, i = 1,2, j = 1,2,3,4\Big\}$.

Then FDE (1.1) *has at least one solution.*

Proof. Similar to proof of Theorem 3.1, it is easy to obtain the above conclusion. Noticing, the restriction about R in (3.1) should be replaced by the following condition

$$10\max\Big\{C_{i5}, i = 1,2\Big\} < R < \min\Big\{(\frac{1}{10C_{ij}})^{\frac{1}{\tau_{ij}-1}}, , i = 1,2, j = 1,2,3,4\Big\}.$$

□

Remark 3.3. The condition (H_2) can be replaced by the following condition

(H_2') $|f_i(t,x_1,x_2,x_3,x_4)| \leq \sum\limits_{k=1}^{4} a_{ik}(t)|x_k|^{\tau_{ik}}, \tau_{ik} > 1$.

That is $a_{i5}(t) = 0$, and the conclusion of Theorem 3.2 remains true with the condition (H_3) removed. Noticing, R in (3.1) should be replaced by the following restriction

$$R < \min\Big\{(\frac{1}{8C_{ij}})^{\frac{1}{\tau_{ij}-1}}, i = 1,2, j = 1,2,3,4\Big\}.$$

The next result is based on the Leray-Schauder nonlinear alternative.

Theorem 3.4. *Assume that there exist nonnegative functions* $a_{ik}(t) \in L^1([0,1],\mathbb{R}_+)(i=1,2,k=1,2,3,4,5)$ *and* $L>0$ *such that the following conditions are satisfied*

(H_4) $|f_i(t,x_1,x_2,x_3,x_4)| \leq \sum\limits_{k=1}^{4} a_{ik}(t)|x_k| + a_{i5}(t), i=1,2,$

(H_5) $\sum\limits_{k=1}^{4}\big(C_{1k}+C_{2k}\big)L + \big(C_{15}+C_{25}\big) < L.$

Then FDE (1.1) *has at least one solution.*

Proof. First, we prove that T is completely continuous. It is obvious that T is continuous since f_i are continuous. Let $L>0$, $B_L = \{(u,v)|(u,v)\in X\times Y, \|(u,v)\|_{X\times Y} \leq L\}$ be a bounded ball in $X\times Y$. We shall show that $T(B_L)$ is relatively compact.

Similar computation as (3.2) and (3.3), for $\forall (u,v)\in T(B_L)$, it is easy to obtain

$$|T_1(u,v)(t)| \leq \sum_{k=1}^{4} A_{1k}L + A_{15}, |D^{\rho_1}T_1(u,v)(t)| \leq \sum_{k=1}^{4} B_{1k}L + B_{15}. \tag{3.8}$$

Thus

$$||T_1(u,v)||_X \leq \sum_{k=1}^{4} C_{1k}L + C_{15}. \tag{3.9}$$

Similarly, we can obtain

$$||T_2(u,v)||_Y \leq \sum_{k=1}^{4} C_{2k}L + C_{25}. \tag{3.10}$$

From (3.9) and (3.10), we have

$$||T(u,v)||_{X\times Y} = ||T_1(u,v)||_X + ||T_2(u,v)||_Y \leq \sum_{k=1}^{4}\big(C_{1k}+C_{2k}\big)L + \big(C_{15}+C_{25}\big). \tag{3.11}$$

By (3.11), $T(B_L)$ is uniformly bounded. On the other hand, similar computation as (3.4)–(3.7) means that $T(B_L)$ is an equicontinuous set. Applying the Arzelà-Ascoli theorem, we conclude that T is a completely continuous operator.

Now we prove that T has at least one solution in $X\times Y$.

For $\forall (u,v)\in \partial B_L$, suppose that $(u,v)=\lambda T(u,v), 0<\lambda<1$. By (3.8), we have

$$\begin{aligned} |u(t)| &= \lambda |T_1(u,v)(t)| \leq |T_1(u,v)(t)| \leq \sum_{k=1}^{4} A_{1k}L + A_{15}, \\ |D^{\rho_1}u(t)| &= \lambda |D^{\rho_1}T_1(u,v)(t)| \leq |D^{\rho_1}T_1(u,v)(t)| \leq \sum_{k=1}^{4} B_{1k}L + B_{15}. \end{aligned}$$

Thus

$$||u||_X \leq \sum_{k=1}^{4} C_{1k}L + C_{15}. \tag{3.12}$$

Similarly, we can obtain

$$||v||_Y \leq \sum_{k=1}^{4} C_{2k}L + C_{25}. \tag{3.13}$$

From (3.12), (3.13) and the condition (H_5), we have

$$\|(u,v)\|_{X\times Y} = ||u||_X + ||v||_Y \leq \sum_{k=1}^{4}\big(C_{1k}+C_{2k}\big)L + \big(C_{15}+C_{25}\big) < L, \tag{3.14}$$

this contradicts the fact $(u,v)\in\partial B_L$. By Lemma 2.8 we obtain that T has a fixed point $(u,v)\in \overline{B}_L$, and so FDE (1.1) has at least one solution in $X\times Y$. $\square$

Remark 3.5. We established the existence of solutions for FDE (1.1) by Theorem 3.1 $\sim$ 3.4. We provided some growth conditions

$$|f_i(t, x_1, x_2, x_3, x_4)| \leq \sum_{k=1}^{4} a_{ik}(t)|x_k|^{\tau_{ik}} + a_{i5}(t)$$

through three cases: In $(H_1), 0 < \tau_{ik} < 1$; In $(H_2), \tau_{ik} > 1$, for the sake of simplicity, here $a_{i5}(t)$ might be zero; In $(H_4), \tau_{ik} = 1$, and some additional restrictions $(H_3), (H_5)$ are given.

The uniqueness of solutions is based on the Banach contraction principle.

Theorem 3.6. *Assume that there exist nonnegative functions $a_{ik}(t) \in L^1([0,1], \mathbb{R}_+)(i = 1,2, k = 1,2,3,4)$ such that the following conditions are satisfied, for all $t \in [0,1]$ and $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \in \mathbb{R}$,*

(H_6) $|f_i(t, x_1, x_2, x_3, x_4) - f_i(t, y_1, y_2, y_3, y_4)| \leq \sum\limits_{k=1}^{4} a_{ik}(t)|x_k - y_k|, i = 1, 2,$

(H_7) $2D_1 + 2D_2 < 1.$

Then FDE (1.1) has a unique solution.

Proof. Let $\sup\limits_{t\in[0,1]} f_i(t, 0, 0, 0, 0) = E_i < \infty, i = 1, 2$ and take

$$r \geq \frac{E_1' + E_2'}{1 - 2D_1 - 2D_2},$$

where $E_i' = \Big[\frac{1}{\Gamma(\alpha_i+1)} + \frac{1}{\Gamma(\alpha_i-\rho_i+1)} + (1 + \frac{1}{(\Gamma(\alpha_i-\rho_i)})(\frac{|\Lambda_i|\gamma_i}{\Gamma(\alpha_i+\beta_i+1)} + \frac{|\Lambda_i|}{\Gamma(\alpha_i+1)})\Big]E_i, i = 1, 2.$

First, we prove that $T(B_r) \subset B_r$, where $B_r = \{(u,v)|(u,v) \in X \times Y : ||(u,v)||_{X\times Y} \leq r\}$. For any $(u, v) \in B_r$, we can obtain

$$\begin{aligned}
|T_1(u,v)(t)| \leq & I_{0^+}^{\alpha_1}\Big[|f_1(t, u(t), v(t), D^{\rho_1}u(t), D^{\rho_2}u(t)) - f_1(t, 0, 0, 0, 0)| + |f_1(t, 0, 0, 0, 0)|\Big] \\
& + |\Lambda_1|\gamma_1 I_{0^+}^{\alpha_1+\beta_1}\Big[|f_1(\eta_1, u(\eta_1), v(\eta_1), D^{\rho_1}u(\eta_1)), D^{\rho_2}v(\eta_1)) - f_1(\eta_1, 0, 0, 0, 0)| \\
& + |f_1(\eta_1, 0, 0, 0, 0)|\Big] + |\Lambda_1| I_{0^+}^{\alpha_1}\Big[|f_1(1, u(1), v(1), D^{\rho_1}u(1), D^{\rho_2}v(1)) \\
& - f_1(1, 0, 0, 0, 0)| + |f_1(1, 0, 0, 0, 0)|\Big] \\
\leq & I_{0^+}^{\alpha_1}\Big[(a_{11}(t) + a_{13}(t))||u||_X + (a_{12}(t) + a_{14}(t))||v||_Y + E_1\Big] \\
& + |\Lambda_1|\gamma_1 I_{0^+}^{\alpha_1+\beta_1}\Big[(a_{11}(t) + a_{13}(t))||u||_X + (a_{12}(t) + a_{14}(t))||v||_Y + E_1\Big] \\
& + |\Lambda_1| I_{0^+}^{\alpha_1}\Big[(a_{11}(t) + a_{13}(t))||u||_X + (a_{12}(t) + a_{14}(t))||v||_Y + E_1\Big] \\
\leq & (A_{11} + A_{13})||u||_X + (A_{12} + A_{14})||v||_Y + \frac{(1 + |\Lambda_1|)E_1}{\Gamma(\alpha_1 + 1)} + \frac{|\Lambda_1|\gamma_1 E_1}{\Gamma(\alpha_1 + \beta_1 + 1)}.
\end{aligned} \tag{3.15}$$

On the other hand, using (3.3), we can obtain

$$\begin{aligned}
|D^{\rho_1}T_1(u,v)(t)| \leq & (B_{11} + B_{13})||u||_X + (B_{12} + B_{14})||v||_Y + \frac{E_1}{\Gamma(\alpha_1-\rho_1+1)} \\
& + \frac{|\Lambda_1|\gamma_1 E_1}{\Gamma(\alpha_1-\rho_1)\Gamma(\alpha_1+\beta_1+1)} + \frac{|\Lambda_1|E_1}{\Gamma(\alpha_1-\rho_1)\Gamma(\alpha_1+1)}.
\end{aligned} \tag{3.16}$$

By (3.15) and (3.16), we have

$$||T_1(u,v)||_X \leq (C_{11} + C_{13})||u||_X + (C_{12} + C_{14})||v||_Y + E_1' \leq 2D_1 r + E_1'.$$

Similarly, one has

$$||T_2(u,v)||_Y \leq (C_{21} + C_{23})||u||_X + (C_{22} + C_{24})||v||_Y + E_2' \leq 2D_2 r + E_2'.$$

Thus

$$||T(u,v)||_{X\times Y} = ||T_1(u,v)||_X + ||T_2(u,v)||_Y \leq 2(D_1+D_2)r + E_1' + E_2' \leq r.$$

Second, for any $(u_2,v_2),(u_1,v_1) \in B_r$, we have

$$\begin{aligned} |T_1(u_2,v_2)(t) - T_1(u_1,v_1)(t)| &= |\int_0^1 G_1(t,s)\big(f_1(s,u_2(s),v_2(s),D^{\rho_1}u_2(s),D^{\rho_2}v_2(s)) \\ &\quad - f_1(s,u_1(s),v_1(s),D^{\rho_1}u_1(s),D^{\rho_2}v_1(s))\big)ds| \\ &\leq (A_{11}+A_{13})||u_2-u_1||_X + (A_{12}+A_{14})||v_2-v_1||_Y, \end{aligned} \tag{3.17}$$

and

$$|D^{\rho_2}T_1(u_2,v_2)(t) - D^{\rho_2}T_1(u_1,v_1)(t)| \leq (B_{11}+B_{13})||u_2-u_1||_X + (B_{12}+B_{14})||v_2-v_1||_Y. \tag{3.18}$$

From (3.17) and (3.18), we have

$$\begin{aligned} ||T_1(u_2,v_2) - T_1(u_1,v_1)||_X &\leq (C_{11}+C_{13})||u_2-u_1||_X + (C_{12}+C_{14})||v_2-v_1||_Y \\ &\leq 2D_1||u_2-u_1||_X + 2D_1||v_2-v_1||_Y. \end{aligned}$$

Similarly, we can obtain

$$||T_2(u_2,v_2) - T_2(u_1,v_1)||_Y \leq 2D_2||u_2-u_1||_X + 2D_2||v_2-v_1||_Y.$$

Thus

$$||T(u_2,v_2) - T(u_1,v_1)||_{X\times Y} \leq (2D_1+2D_2)\Big[||u_2-u_1||_X + ||v_2-v_1||_Y\Big].$$

Since $2D_1+2D_2<1$, T is a contraction operator. Applying the Banach contraction principle, the operator T has a unique fixed point which is the unique solution of FDE (1.1). □

Next,we present the following three examples to illustrate our results.

Example 3.7. Consider the following coupled system of nonlinear FDE

$$\begin{cases} D_{0^+}^{\frac{5}{2}}u(t) &= a_{11}(t)(u(t))^{\tau_{11}} + a_{12}(t)(v(t))^{\tau_{12}} + a_{13}(t)(D_{0^+}^{\frac{1}{4}}u(t))^{\tau_{13}} \\ & \quad +a_{14}(t)(D_{0^+}^{\frac{1}{3}}v(t))^{\tau_{14}} + a_{15}(t), t\in(0,1), \\ D_{0^+}^{\frac{7}{3}}v(t) &= a_{21}(t)(u(t))^{\tau_{21}} + a_{22}(t)(v(t))^{\tau_{22}} + a_{23}(t)(D_{0^+}^{\frac{1}{4}}u(t))^{\tau_{23}} \\ & \quad +a_{24}(t)(D_{0^+}^{\frac{1}{3}}v(t))^{\tau_{24}} + a_{25}(t), t\in(0,1), \\ u(0) &= u'(0) = 0, u(1) = 2I_{0^+}^{\frac{5}{4}}u(\frac{1}{5}), \\ v(0) &= v'(0) = 0, v(1) = \frac{1}{2}I_{0^+}^{\frac{1}{3}}v(\frac{1}{8}), \end{cases} \tag{3.19}$$

where $\alpha_1 = \frac{5}{2}, \alpha_2 = \frac{7}{3}, \rho_1 = \frac{1}{4}, \rho_2 = \frac{1}{3}, \gamma_1 = 2, \gamma_2 = \frac{1}{2}, \beta_1 = \frac{5}{4}, \beta_2 = \frac{1}{3}, \eta_1 = \frac{1}{5}, \eta_1 = \frac{1}{8}, 0 < \tau_{ij} < 1 (i=1,2; j=1,2,3,4)$ and $a_{ik}(t)(i=1,2;k=1,2,3,4,5)$ are nonnegative functions. Obviously, Theorem 3.1 implies that FDE (3.19) has at least one solution.

Example 3.8. Consider the following coupled system of nonlinear FDE

$$\begin{cases} D_{0^+}^{\frac{5}{2}}u(t) = \frac{t}{10}u + \frac{t^2}{10(1+|v|)}v^2 + \frac{t^3}{10}(D_{0^+}^{\frac{1}{2}}u(t)) + \frac{t}{20}(D_{0^+}^{\frac{1}{2}}v(t)) + \frac{(1-t)^2}{100}, t\in(0,1), \\ D_{0^+}^{\frac{7}{3}}v(t) = \frac{(1-t)u^3}{50(1+u^2)} + \frac{t^2}{20}v + \frac{t}{30}(D_{0^+}^{\frac{1}{2}}u(t)) + \frac{t^2}{30}(D_{0^+}^{\frac{1}{2}}v(t)) + \frac{(1+t)^2}{40}, t\in(0,1), \\ u(0) = u'(0) = 0, u(1) = I_{0^+}^{\frac{1}{3}}u(\frac{1}{2}), \\ v(0) = v'(0) = 0, u(1) = 2I_{0^+}^{\frac{5}{4}}u(\frac{1}{3}), \end{cases} \tag{3.20}$$

where $\alpha_1 = \frac{5}{2}, \alpha_2 = \frac{7}{4}, \rho_1 = \rho_2 = \frac{1}{2}, \eta_1 = \frac{1}{2}, \eta_2 = \frac{1}{3}, \beta_1 = \frac{1}{3}, \beta_2 = \frac{5}{4}, \gamma_1 = 1, \gamma_2 = 2$. By (3.20), we have

$$|f_1(t, x_1, x_2, x_3, x_4)| \leq \frac{t}{10}|x_1| + \frac{t^2}{10}|x_2| + \frac{t^3}{10}|x_3| + \frac{t}{20}|x_4| + \frac{(1-t)^2}{100},$$
$$|f_2(t, x_1, x_2, x_3, x_4)| \leq \frac{(1-t)}{50}|x_1| + \frac{t^2}{20}|x_2| + \frac{t}{30}|x_3| + \frac{t^2}{30}|x_4| + \frac{(1+t)^2}{40},$$

where $a_{11}(t) = \frac{t}{10}, a_{12}(t) = \frac{t^2}{10}, a_{13}(t) = \frac{t^3}{10}, a_{14}(t) = \frac{t}{20}, a_{15}(t) = \frac{(1-t)^2}{100}, a_{21}(t) = \frac{(1-t)}{50}, a_{22}(t) = \frac{t^2}{20}, a_{23}(t) = \frac{t}{30}, a_{24}(t) = \frac{t^2}{30}, a_{25}(t) = \frac{(1+t)^2}{40}$. Let us evaluate $\sum_{k=1}^{4} \big(C_{1k} + C_{2k}\big)L + \big(C_{15} + C_{25}\big) - L$. By direct calculation, we have

$$A_{11} = 0.01962,\ A_{12} = 0.00871,\ A_{13} = 0.00475,\ A_{14} = 0.00981,\ A_{15} = 0.00384,$$
$$A_{21} = 0.00194,\ A_{22} = 0.00066,\ A_{23} = 0.00106,\ A_{24} = 0.00021,\ A_{25} = 0.00401,$$
$$B_{11} = 0.03132,\ B_{12} = 0.01483,\ B_{13} = 0.00854,\ B_{14} = 0.01566,\ B_{15} = 0.00539,$$
$$B_{21} = 0.03879,\ B_{22} = 0.00577,\ B_{23} = 0.00425,\ B_{24} = 0.00235,\ B_{25} = 0.03001,$$
$$C_{11} = 0.05094,\ C_{12} = 0.02354,\ C_{13} = 0.01329,\ C_{14} = 0.02547,\ C_{15} = 0.00923,$$
$$C_{21} = 0.04073,\ C_{22} = 0.00643,\ C_{23} = 0.00531,\ C_{24} = 0.00256,\ C_{25} = 0.03402.$$

Thus $\sum_{k=1}^{4} \big(C_{1k} + C_{2k}\big)L + \big(C_{15} + C_{25}\big) - L = 0.16827 \times 1 + 0.04325 - 1 = -0.7885 < 0$ for $L = 1$. Theorem 3.4 implies that FDE (3.20) has at least one solution.

Example 3.9. Consider the following coupled system of nonlinear FDE

$$\begin{cases} D_{0^+}^{\frac{5}{2}} u(t) = \frac{t}{10}u(t) + \frac{t^2}{10}v(t) + \frac{t^3}{10}D_{0^+}^{\frac{1}{2}}u(t) + \frac{t}{20}D_{0^+}^{\frac{1}{2}}v(t) + \sin t, t \in (0,1), \\ D_{0^+}^{\frac{7}{3}} v(t) = \frac{1-t}{50}u(t) + \frac{t^2}{20}v(t) + \frac{t}{30}D_{0^+}^{\frac{1}{2}}u(t) + \frac{t^2}{30}D_{0^+}^{\frac{1}{2}}v(t) + \frac{e^t}{10}, t \in (0,1), \\ u(0) = u'(0) = 0, u(1) = I_{0^+}^{\frac{1}{3}}u(\frac{1}{2}), \\ v(0) = v'(0) = 0, u(1) = 2I_{0^+}^{\frac{5}{4}}u(\frac{1}{3}), \end{cases} \tag{3.21}$$

where $\alpha_1 = \frac{5}{2}, \alpha_2 = \frac{7}{4}, \rho_1 = \rho_2 = \frac{1}{2}, \eta_1 = \frac{1}{2}, \eta_2 = \frac{1}{3}, \beta_1 = \frac{1}{3}, \beta_2 = \frac{5}{4}, \gamma_1 = 1, \gamma_2 = 2$. By (3.21), we have

$$|f_1(t, x_1, x_2, x_3, x_4)| = \frac{t}{10}x_1 + \frac{t^2}{10}x_2 + \frac{t^3}{10}x_3 + \frac{t}{20}x_4 + sint,$$
$$|f_2(t, x_1, x_2, x_3, x_4)| = \frac{(1-t)}{50}x_1 + \frac{t^2}{20}x_2 + \frac{t}{30}x_3 + \frac{t^2}{30}x_4 + \frac{e^t}{10},$$

and

$$|f_1(t, x_1, x_2, x_3, x_4) - f_1(t, y_1, y_2, y_3, y_4)| \leq \frac{t}{10}|x_1 - y_1| + \frac{t^2}{10}|x_2 - y_2| + \frac{t^3}{10}|x_3 - y_3| + \frac{t}{20}|x_4 - y_4|,$$
$$|f_2(t, x_1, x_2, x_3, x_4) - f_2(t, y_1, y_2, y_3, y_4)| \leq \frac{1-t}{50}|x_1 - y_1| + \frac{t^2}{20}|x_2 - y_2| + \frac{t}{30}|x_3 - y_3| + \frac{t^2}{30}|x_4 - y_4|.$$

From the calculation result of (3.20), we can obtain that $D_1 = 0.05094,\ D_2 = 0.04703$. Thus, $2D_1 + 2D_2 = 0.19594 < 1$. Theorem 3.6 implies that FDE (3.21) has a unique solution.

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