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A Lyapunov-type inequality for a fractional q-difference boundary value problem

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Abstract

In this paper, we establish a Lyapunov-type inequality for a fractional q-difference equation subject to Dirichlet-type boundary conditions. The obtained inequality generalizes several existing results from the literature including the standard Lyapunov inequality. We use that result to provide an interval, where a certain Mittag-Leffler function has no real zeros. We present also another application of the obtained inequality, where we prove that existence implies uniqueness for a certain class of fractional q-difference boundary value problems.

Keywords: Lyapunov's inequality, q-fractional derivative, Green's function, Mittag-Leffler function.
2010 MSC: 26D10, 39A13, 26A33, 33E12.

1. Introduction

In [22], Lyapunov established the following result:

Theorem 1.1. *If the boundary value problem*

$$u''(t) + \varphi(t)u(t) = 0, \quad a < t < b,$$
$$u(a) = u(b) = 0,$$

has a nontrivial solution, where $\varphi : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |\varphi(t)| > \frac{4}{b-a}. \tag{1.1}$$

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This result has found several applications in various problems related with differential equations and, since then, there have been several results to improve and generalize Theorem 1.1 in many directions (see [2, 5, 6, 7, 8, 13, 14, 15, 20, 22, 24, 25, 26, 30] and the references therein).

To the best of our knowledge, none of the mentioned works considered a differential equation that depends on a q-fractional derivative operator. This is one of the issues we are going to address in this paper.

The q-difference calculus was firstly developed by Jackson [16, 17]. This concept is rich in history and has several applications as the reader can confirm in [10]. Details, basic definitions and properties of q-difference calculus can be found in the book [18]. The fractional q-difference calculus has been introduced by Al-Salam [3] and Agarwal [1]. Due to the intensive works in the field of fractional calculus, several developments in this theory of fractional q-difference were made (see [4, 9, 11, 12, 21, 23, 27, 28, 29] and references therein).

In this work, we consider the following fractional q-difference boundary value problem

$$({}_aD_q^\alpha u)(t) + \varphi(t)u(t) = 0, \quad a < t < b,\ q \in [0,1),\ 1 < \alpha \le 2, \tag{1.2}$$

$$u(a) = u(b) = 0, \tag{1.3}$$

where ${}_aD_q^\alpha$ denotes the fractional q-derivative of Riemann-Liouville type and $\varphi : [a,b] \to \mathbb{R}$ is a continuous function, and derive a Lyapunov-type inequality for such problem. The obtained inequality generalizes several existing results in the literature including the standard Lyapunov inequality (1.1), Hartman and Wintner inequality [14], a Lyapunov-type inequality due to Ferreira [13], etc. We then use that inequality to obtain an interval, where a certain Mittag-Leffler function has no real zeros. We show that the obtained interval is better than that obtained in [13]. We also present another application of the obtained Lyapunov-type inequality, where we prove that existence implies uniqueness for a certain class of fractional q-difference boundary value problems.

2. Preliminaries and auxiliary results

In this section, we recall briefly some concepts of fractional quantum calculus introduced recently in [29] (see also [28]). Some additional properties are also given.

Let $q \in [0,1)$ and $a \in \mathbb{R}$. For $r \in \mathbb{R}$, we define

$$[r]_q = \frac{1-q^r}{1-q}\cdot$$

The q-analogue of the power function $(x-y)^n$ with $n \in \mathbb{N}_0 := \{0,1,2,\cdots\}$ is

$$(x-y)_a^{(0)} = 1,\ (x-y)_a^{(k)} = \prod_{i=0}^{k-1}((x-a)-(y-a)q^i),\ k \in \mathbb{N},\ (x,y) \in \mathbb{R}^2.$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(x-y)_a^{(\gamma)} = (x-a)^\gamma \prod_{i=0}^{\infty} \frac{(x-a)-q^i(y-a)}{(x-a)-q^{\gamma+i}(y-a)}\cdot$$

The following property will be useful later.

Lemma 2.1. *We have*

$$(x-y)_a^{(\gamma)} = (x-a)^\gamma \left(1-\frac{y-a}{x-a}\right)_0^{(\gamma)},\ x \ne a\cdot$$

Proof. We have

$$\begin{aligned}\left(1-\frac{y-a}{x-a}\right)_0^{(\gamma)} &= \prod_{i=0}^{\infty}\frac{1-q^i\frac{y-a}{x-a}}{1-q^{\gamma+i}\frac{y-a}{x-a}}\\ &= \prod_{i=0}^{\infty}\frac{(x-a)-q^i(y-a)}{(x-a)-q^{\gamma+i}(y-a)} = \frac{(x-y)_a^{(\gamma)}}{(x-a)^\gamma},\end{aligned}$$

which gives the desired equality. □

Lemma 2.2 ([11])**.** *Let $\alpha > 0$ and $x \leq y \leq z$. Then*

$$(z-x)_0^{(\alpha)} \geq (z-y)_0^{(\alpha)}.$$

The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)_0^{(x-1)}}{(1-q)^{x-1}},\ x \in \mathbb{R}\backslash\{0,-1,-2,\cdots\}$$

and satisfies

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x).$$

The q-derivative of a function $f : [a,b] \to \mathbb{R}$ $(a < b)$ is defined by

$$({}_aD_q f)(t) = \frac{f(t) - f(qt + (1-q)a)}{(1-q)(t-a)},\ t \neq a$$

and

$$({}_aD_q f)(a) = \lim_{t \to a} ({}_aD_q f)(t).$$

It is easy to observe that if f is differentiable in (a,b), then

$$\lim_{q \to 1^-} ({}_aD_q f)(t) = f'(t),\ t \in (a,b).$$

The q-derivative of a higher order is given by

$$({}_aD_q^0 f)(t) = f(t) \quad \text{and} \quad ({}_aD_q^n f)(t) = {}_aD_q({}_aD_q^{n-1} f)(t),\ n \in \mathbb{N}.$$

The q-derivative of a product and ratio of functions f and g in $[a,b]$ are

$$({}_aD_q fg)(t) = f(t)({}_aD_q g)(t) + g(qt + (1-q)a)({}_aD_q f)(t)$$

and

$${}_aD_q\left(\frac{f}{g}\right)(t) = \frac{({}_aD_q f)(t)g(t) - f(t)({}_aD_q g)(t)}{g(t)g(qt + (1-q)a)}\cdot$$

Lemma 2.3 ([29])**.** *For any $t, s \in [a,b]$, the following formulas hold:*

$${}_t({}_aD_q(t-s)_a^{(\gamma)}) = [\gamma]_q (t-s)_a^{(\gamma-1)}$$

and

$${}_s({}_aD_q(t-s)_a^{(\gamma)}) = -[\gamma]_q (t - (qs + (1-q)a))_a^{(\gamma-1)},$$

where ${}_i({}_aD_q)$ denotes the q-derivative with respect to the variable i.

Lemma 2.4. *Let $f : [a,b] \to \mathbb{R}$ be a given function. Suppose that for every $q \in [0,1)$, we have*

$$({}_aD_q f)(t) \leq 0,\ a < t \leq b.$$

Then f is a decreasing function.

Proof. Let $(x,y) \in [a,b] \times [a,b]$ such that $x < y$. Let

$$q = \frac{x-a}{y-a}\cdot$$

Then $q \in [0,1)$. Moreover, we have

$$({}_aD_q f)(y) = \frac{f(y) - f(x)}{y-x} \leq 0,$$

which yields $f(y) \leq f(x)$. The lemma is proved. □

Similarly, we have,

Lemma 2.5. *Let $f:[a,b]\to\mathbb{R}$ be a given function. Suppose that for every $q\in[0,1)$, we have*

$$({}_aD_qf)(t)\geq 0,\ a<t\leq b.$$

Then f is an increasing function.

The q-integral of a function f defined in the interval $[a,b]$ is given by

$$({}_aI_qf)(t)=\int_a^t f(s)\ {}_ad_qs=(1-q)(t-a)\sum_{i=0}^{\infty}q^if(q^it+(1-q)a),\ t\in[a,b].$$

Clearly, if f is a continuous function, then the above series is convergent. If $a<c<b$, we define

$$\int_c^t f(s)\ {}_ad_qs=\int_a^t f(s)\ {}_ad_qs-\int_a^c f(s)\ {}_ad_qs,\ t\in[c,b].$$

Similarly as done for q-derivatives, it can be defined an operator ${}_aI_q^n$, namely,

$$({}_aI_q^0f)(t)=f(t)\quad\text{and}\quad({}_aI_q^nf)(t)={}_aI_q({}_aI_q^{n-1}f)(t),\ n\in\mathbb{N}.$$

The fundamental theorem of calculus applies to these operators ${}_aD_q$ and ${}_aI_q$, i.e.,

$$({}_aD_{q\,a}I_qf)(t)=f(t),$$

and if f is continuous at $x=a$, then

$$({}_aI_{q\,a}D_qf)(t)=f(t)-f(a).$$

The formula for q-integration by parts in an interval $[a,b]$ is

$$\int_a^b f(s)({}_aD_qg)(s)\ {}_ad_qs=[f(t)g(t)]_{t=a}^{t=b}-\int_a^b g(qs+(1-q)a)({}_aD_qf)(s)\ {}_ad_qs.$$

The following properties follow immediately from the definition of the q-integral.

Lemma 2.6. *Let $f,g:[a,b]\to\mathbb{R}$. Then*

(i) $f\leq g\Longrightarrow\displaystyle\int_a^b f(s)\ {}_ad_qs\leq\int_a^b g(s)\ {}_ad_qs$,

(ii) $\displaystyle\left|\int_a^b f(s)\ {}_ad_qs\right|\leq\int_a^b|f(s)|\ {}_ad_qs$.

Definition 2.7. Let $f:[a,b]\to\mathbb{R}$. The fractional q-integral of Riemann-Liouville type is given by

$$({}_aI_q^0f)(t)=f(t),\ t\in[a,b]$$

and

$$({}_aI_q^\alpha f)(t)=\frac{1}{\Gamma_q(\alpha)}\int_a^t(t-(qs+(1-q)a))_a^{(\alpha-1)}f(s)\ {}_ad_qs,\ \alpha>0,\ t\in[a,b].$$

The fractional q-derivative of Riemann-Liouville type is defined by

$$({}_aD_q^0f)(t)=f(t),\ t\in[a,b]$$

and

$$({}_aD_q^\alpha f)(t)=({}_aD_{q\,a}^\ell I_q^{\ell-\alpha}f)(t),\ \alpha>0,\ t\in[a,b],$$

where ℓ is the smallest integer greater than or equal to α.

Lemma 2.8 ([29]). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then*

(i) ${}_aD_q^\alpha({}_aI_q^\alpha f)(t) = f(t),\ \alpha > 0,\ t \in [a,b]$,

(ii) ${}_aI_q^\alpha\, {}_aI_q^\beta f(t) = {}_aI_q^{\alpha+\beta} f(t),\ \alpha,\beta > 0,\ t \in [a,b]$.

Lemma 2.9 ([29]). *Let $f : [a,b] \to \mathbb{R}$, p be a positive integer, and $\alpha > p-1$. Then*

$$ {}_aI_q^\alpha\, {}_aD_q^p f(t) = {}_aD_q^p\, {}_aI_q^\alpha f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)}\, {}_aD_q^k f(a),\ t \in [a,b]. $$

3. Main result

At first, we need an integral representation of the solution to problem (1.2)-(1.3). We note that such integral representation was obtained in [11] for the case $a = 0$.

Lemma 3.1. *If $u \in C[a,b]$ is a solution to the fractional q-difference boundary value problem* (1.2)-(1.3), *then u satisfies the integral equation*

$$ u(t) = \int_a^t G_1(t, qs + (1-q)a))\varphi(s)u(s)\ {}_ad_qs + \int_t^b G_2(t,s)\varphi(s)u(s)\ {}_ad_qs, \quad a \le t \le b, $$

where

$$ G_1(t,s) = \frac{1}{\Gamma_q(\alpha)} \left(\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} (b-s)_a^{(\alpha-1)} - (t-s)_a^{(\alpha-1)} \right), \quad a \le s \le t \le b $$

and

$$ G_2(t,s) = \frac{(t-a)^{\alpha-1}}{\Gamma_q(\alpha)(b-a)^{\alpha-1}} (b - (qs + (1-q)a))_a^{(\alpha-1)}, \quad a \le t \le s \le b. $$

Proof. By the definition of the fractional q-derivative, we have

$$ {}_aD_q^2\, {}_aI_q^{2-\alpha} u(t) = -\varphi(t)u(t),\ a < t < b. $$

Then

$$ {}_aI_q^\alpha\, {}_aD_q^2\, {}_aI_q^{2-\alpha} u(t) = -{}_aI_q^\alpha(\varphi(t)u(t)),\ a < t < b. $$

Using Lemma 2.8 and Lemma 2.9 with $p = 2$, we get

$$ u(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} - \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - (qs + (1-q)a))_a^{(\alpha-1)} \varphi(s)u(s)\ {}_ad_qs, $$

for some constants $c_1, c_2 \in \mathbb{R}$. The boundary conditions given by (1.3) yield

$$ c_1 = \frac{1}{(b-a)^{\alpha-1}\Gamma_q(\alpha)} \int_a^b (b - (qs + (1-q)a))_a^{(\alpha-1)} \varphi(s)u(s)\ {}_ad_qs \quad \text{and} \quad c_2 = 0. $$

Thus we have

$$ \begin{aligned} u(t) =& \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}\Gamma_q(\alpha)} \int_a^b (b - (qs + (1-q)a))_a^{(\alpha-1)} \varphi(s)u(s)\ {}_ad_qs \\ & - \frac{1}{\Gamma_q(\alpha)} \int_a^t (t - (qs + (1-q)a))_a^{(\alpha-1)} \varphi(s)u(s)\ {}_ad_qs \\ =& \frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}\Gamma_q(\alpha)} \int_a^t (b - (qs + (1-q)a))_a^{(\alpha-1)} \varphi(s)u(s)\ {}_ad_qs \end{aligned} $$

$$+\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}\Gamma_q(\alpha)}\int_t^b (b-(qs+(1-q)a))_a^{(\alpha-1)}\varphi(s)u(s)\ {}_ad_qs$$
$$-\frac{1}{\Gamma_q(\alpha)}\int_a^t (t-(qs+(1-q)a))_a^{(\alpha-1)}\varphi(s)u(s)\ {}_ad_qs,$$

which yields the desired result. □

Lemma 3.2. *We have*

$$0 \le G_1(t,qs+(1-q)a) \le G_2(s,s), \quad a<s\le t\le b.$$

Proof. By Lemma 2.1, we have

$$(t-(qs+(1-q)a))_a^{(\alpha-1)} = (t-a)^{\alpha-1}\left(1-\frac{q(s-a)}{t-a}\right)_0^{(\alpha-1)}, \quad a<s\le t\le b$$

and

$$(b-(qs+(1-q)a))_a^{(\alpha-1)} = (b-a)^{\alpha-1}\left(1-\frac{q(s-a)}{b-a}\right)_0^{(\alpha-1)}, \quad a\le s\le b.$$

Then

$$G_1(t,qs+(1-q)a) = \frac{(t-a)^{\alpha-1}}{\Gamma_q(\alpha)}\left[\left(1-\frac{q(s-a)}{b-a}\right)_0^{(\alpha-1)} - \left(1-\frac{q(s-a)}{t-a}\right)_0^{(\alpha-1)}\right].$$

On the other hand, since

$$\frac{q(s-a)}{b-a} \le \frac{q(s-a)}{(t-a)} \le 1, \quad a<s\le t\le b,$$

using Lemma 2.2, we obtain

$$G_1(t,qs+(1-q)a)\ge 0, \quad a<s\le t\le b.$$

Moreover, for $s\in(a,b]$, using Lemma 2.3, we have

$$\begin{aligned}{}_t({}_aD_qG_1(t,s)) =& \frac{1}{\Gamma_q(\alpha)}\left(\frac{(b-s)_a^{(\alpha-1)}}{(b-a)^{\alpha-1}}{}_t({}_aD_q(t-a)^{\alpha-1}) - {}_t({}_aD_q(t-s)_a^{(\alpha-1)})\right)\\ =& \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\left(\frac{(b-s)_a^{(\alpha-1)}(t-a)^{\alpha-2}}{(b-a)^{\alpha-1}} - (t-s)_a^{(\alpha-2)}\right).\end{aligned}$$

On the other hand, using Lemma 2.1, we have

$$(b-s)_a^{(\alpha-1)} = (b-a)^{\alpha-1}\left(1-\frac{s-a}{b-a}\right)_0^{(\alpha-1)}$$

and

$$(t-s)_a^{(\alpha-2)} = (t-a)^{\alpha-2}\left(1-\frac{s-a}{t-a}\right)_0^{(\alpha-2)}.$$

Thus by Lemma 2.2, we get

$$\begin{aligned}{}_t({}_aD_qG_1(t,s)) =& \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}\left(\left(1-\frac{s-a}{b-a}\right)_0^{(\alpha-1)}(t-a)^{\alpha-2} - (t-a)^{\alpha-2}\left(1-\frac{s-a}{t-a}\right)_0^{(\alpha-2)}\right)\\ =& \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}(t-a)^{\alpha-2}\left(\left(1-\frac{s-a}{b-a}\right)_0^{(\alpha-1)} - \left(1-\frac{s-a}{t-a}\right)_0^{(\alpha-2)}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{[\alpha-1]_q}{\Gamma_q(\alpha)}(t-a)^{\alpha-2}\left(\left(1-\frac{s-a}{b-a}\right)_0^{(\alpha-1)}-\left(1-\frac{s-a}{b-a}\right)_0^{(\alpha-2)}\right)\\
&\leq 0.
\end{aligned}
$$

By Lemma 2.4, we deduce that the function $G_1(\cdot,s)$ is decreasing for every $s\in(a,b]$. Since $a<s\leq t\leq b$, we obtain

$$G_1(t,qs+(1-q)a)\leq G_1(qs+(1-q)a,qs+(1-q)a),\quad a<s\leq t\leq b.$$

On the other hand, for $a\leq s\leq b$, we have

$$
\begin{aligned}
G_1(qs+(1-q)a,qs+(1-q)a) &= \frac{1}{\Gamma_q(\alpha)}\frac{(q(s-a))^{\alpha-1}}{(b-a)^{\alpha-1}}(b-(qs+(1-q)a))_a^{(\alpha-1)}\\
&\leq \frac{1}{\Gamma_q(\alpha)}\frac{(s-a)^{\alpha-1}}{(b-a)^{\alpha-1}}(b-(qs+(1-q)a))_a^{(\alpha-1)}\\
&= G_2(s,s).
\end{aligned}
$$

This finishes the proof. □

Lemma 3.3. *We have*

$$G_2(a,s)=0\leq G_2(t,s)\leq G_2(s,s),\quad a\leq t\leq s\leq b.$$

Proof. It is sufficient to observe that for every $s\in(a,b]$, the function $G_2(\cdot,s)$ is increasing. □

Let $h:[a,b]\to\mathbb{R}$ be the function defined by

$$h(s)=G_2(s,s)=\frac{(s-a)^{\alpha-1}}{\Gamma_q(\alpha)(b-a)^{\alpha-1}}(b-(qs+(1-q)a))_a^{(\alpha-1)}.$$

Now, we are ready to state and prove our main result in this paper.

Theorem 3.4. *If a nontrivial continuous solution to the fractional q-difference boundary value problem,*

$$
\begin{aligned}
&({}_aD_q^{\alpha}u)(t)+\varphi(t)u(t)=0,\quad a<t<b,\ q\in[0,1),\ 1<\alpha\leq 2,\\
&u(a)=u(b)=0,
\end{aligned}
$$

exists, where $\varphi:[a,b]\to\mathbb{R}$ is a continuous function, then

$$\int_a^b (s-a)^{\alpha-1}(b-(qs+(1-q)a))_a^{(\alpha-1)}|\varphi(s)|\ {}_ad_qs\geq\Gamma_q(\alpha)(b-a)^{\alpha-1}.$$

Proof. Let u be a nontrivial continuous solution to the considered fractional q-difference boundary value problem. Let

$$\|u\|_\infty=\max_{a\leq t\leq b}|u(t)|.$$

By Lemma 3.1, for all $a\leq t\leq b$, we have

$$u(t)=\int_a^t G_1(t,qs+(1-q)a))\varphi(s)u(s)\ {}_ad_qs+\int_t^b G_2(t,s)\varphi(s)u(s)\ {}_ad_qs.$$

By Lemma 2.6, Lemma 3.2 and Lemma 3.3, for all $a\leq t\leq b$, we get

$$
\begin{aligned}
|u(t)| &\leq \int_a^b h(s)|\varphi(s)||u(s)|\ {}_ad_qs\\
&\leq \|u\|_\infty\int_a^b h(s)|\varphi(s)|\ {}_ad_qs.
\end{aligned}
$$

Since u is nontrivial, we get

$$1\leq\int_a^b h(s)|\varphi(s)|\ {}_ad_qs,$$

which yields the desired inequality. □

4. Particular cases

In this section, we deduce several Lyapunov-type inequalities from our main Theorem 3.4.

4.1. The limite case $q \to 1^-$

We have the following result.

Corollary 4.1. *If a nontrivial continuous solution to the Riemann-Liouville fractional boundary value problem,*

$$\begin{aligned} &({}_aD^\alpha u)(t) + \varphi(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \le 2,\\ &u(a) = u(b) = 0, \end{aligned}$$

exists, where $\varphi : [a,b] \to \mathbb{R}$ is a continuous function and ${}_aD^\alpha$ denotes the Riemann-Liouville fractional derivative of order α, then

$$\int_a^b (s-a)^{\alpha-1}(b-s)^{\alpha-1}|\varphi(s)|\ ds \ge \Gamma(\alpha)(b-a)^{\alpha-1}.$$

Proof. It follows from Theorem 3.4 by letting $q \to 1^-$. □

By the arithmetic-geometric-harmonic mean inequality,

$$(s-a)(b-s) \le \left(\frac{b-a}{2}\right)^2,$$

we deduce from Corollary 4.1 the following inequality established recently by Ferreira [13].

Corollary 4.2. *If a nontrivial continuous solution to the Riemann-Liouville fractional boundary value problem,*

$$\begin{aligned} &({}_aD^\alpha u)(t) + \varphi(t)u(t) = 0, \quad a < t < b,\ 1 < \alpha \le 2,\\ &u(a) = u(b) = 0, \end{aligned}$$

exists, where $\varphi : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |\varphi(s)|\ ds \ge \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}.$$

Taking $\alpha = 2$ in Corollary 4.1, we obtain the following inequality obtained by Hartman and Wintner [14].

Corollary 4.3. *If a nontrivial continuous solution to the boundary value problem,*

$$\begin{aligned} &u''(t) + \varphi(t)u(t) = 0, \quad a < t < b,\\ &u(a) = u(b) = 0, \end{aligned}$$

exists, where $\varphi : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b (s-a)(b-s)|\varphi(s)|\ ds \ge (b-a).$$

Observe that the standard Lyapunov inequality follows immediately from Corollary 4.2 by taking $\alpha = 2$.

4.2. The case $\alpha = 2$

Taking $\alpha = 2$ in Theorem 3.4, we obtain the following Lyapunov-type inequality.

Corollary 4.4. *If a nontrivial continuous solution to the q-difference boundary value problem,*

$$\begin{aligned}&({}_aD_q^2u)(t) + \varphi(t)u(t) = 0, \quad a < t < b,\ q \in (0,1),\\ &u(a) = u(b) = 0,\end{aligned}$$

exists, where $\varphi : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b (s-a)(b-(qs+(1-q)a))|\varphi(s)|\ {}_ad_qs \geq (b-a).$$

Let us consider the function

$$f(s) = (s-a)(b-(qs+(1-q)a)),\ s \in [a,b].$$

Differentiating f with respect to s, we get

$$f'(s) = (b-(qs+(1-q)a)) - q(s-a),\ s \in [a,b].$$

Then $f'(s)$ has a unique zero, attained in the point

$$s^* = \frac{(2q-1)a+b}{2q}.$$

Observe that for $\frac{1}{2} < q < 1$, we have $a < s^* < b$. In this case, it is easily seen that $f'(s) > 0$ on (a, s^*) and $f'(s) < 0$ on (s^*, b). Then we conclude that $\max\limits_{a\leq s\leq b} f(s) = f(s^*)$. Evaluating $f(s^*)$, we get

$$0 \leq f(s) \leq f(s^*) = \frac{(b-a)^2}{4q}, \quad s \in [a,b].$$

However, for $0 < q \leq \frac{1}{2}$, we have $s^* \geq b$. In this case, we conclude that $\max\limits_{a\leq s\leq b} f(s) = f(b)$. Evaluating $f(b)$, we get

$$0 \leq f(s) \leq f(b) = (b-a)^2(1-q), \quad s \in [a,b].$$

From Corollary 4.4 and the above analysis, we get the following results.

Corollary 4.5. *If a nontrivial continuous solution to the q-difference boundary value problem,*

$$\begin{aligned}&({}_aD_q^2u)(t) + \varphi(t)u(t) = 0, \quad a < t < b,\ q \in (1/2,1),\\ &u(a) = u(b) = 0,\end{aligned}$$

exists, where $\varphi : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |\varphi(s)|\ {}_ad_qs \geq \frac{4q}{b-a}.$$

Corollary 4.6. *If a nontrivial continuous solution to the q-difference boundary value problem,*

$$\begin{aligned}&({}_aD_q^2u)(t) + \varphi(t)u(t) = 0, \quad a < t < b,\ q \in (0,1/2],\\ &u(a) = u(b) = 0,\end{aligned}$$

exists, where $\varphi : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |\varphi(s)|\ {}_ad_qs \geq \frac{1}{(b-a)(1-q)}.$$

Observe that by letting $q \to 1^-$ in Corollary 4.5, we obtain the standard Lyapunov inequality.

5. On real zeros of the Mittag-Leffler function

In this section, we use the result given by Corollary 4.1 to determine intervals for the real zeros of the Mittag-Leffler function,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\alpha)}, \quad z \in \mathbb{C},\ \Re(\alpha) > 0.$$

Such problem was considered recently by Ferreira in [13]. In that work, he obtained the following result.

Theorem 5.1. *Let $1 < \alpha \le 2$. Then the Mittag-Leffler function $E_\alpha(z)$ has no real zeros for*

$$|z| \le \Gamma(\alpha)4^{\alpha-1}. \tag{5.1}$$

In this section, using the Lyapunov-type inequality given by Corollary 4.1, we obtain a better estimate than that in (5.1).

Let now $(a,b) = (0,1)$ and consider the following fractional eigenvalue problem

$$\begin{aligned} &({}_0D^\alpha u)(t) + \lambda u(t) = 0, \quad 0 < t < 1,\ 1 < \alpha \le 2, \\ &u(0) = u(1) = 0. \end{aligned} \tag{5.2}$$

By [19, Corollary 5.1], we know that the eigenvalues $\lambda \in \mathbb{R}$ of (5.2) are the solutions to

$$E_\alpha(-\lambda) = 0 \tag{5.3}$$

and the corresponding eigenfunctions are given by

$$u(t) = t^{\alpha-1}E_\alpha(-\lambda t^\alpha),\ t \in [0,1].$$

By Corollary 4.1, if $\lambda \in \mathbb{R}$ is an eigenvalue of (5.1), i.e., $-\lambda$ is a zero of equation (5.3), then

$$|\lambda| \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}\, ds \ge \Gamma(\alpha). \tag{5.4}$$

On the other hand,

$$\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}\, ds = B(\alpha,\alpha),$$

where B is the beta function. Using the identity

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

we have

$$B(\alpha,\alpha) = \frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)}.$$

Rewriting inequality (5.4), we obtain

$$|\lambda| \ge \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}.$$

We deduce the following,

Theorem 5.2. *Let $1 < \alpha \le 2$. Then the Mittag-Leffler function $E_\alpha(z)$ has no real zeros for*

$$|z| < \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}. \tag{5.5}$$

Remark 5.3. Note that

$$\Gamma(x)4^{x-1} \leq \frac{\Gamma(2x)}{\Gamma(x)}, \quad 1 < x \leq 2,$$

which proves that the estimate (5.5) is better than that given by (5.1).

Finally, we end the paper with the following observation.

Let us consider the nonhomogeneous fractional q-difference boundary value problem consisting of the equation

$$({}_aD_q^\alpha x)(t) + \varphi(t)x(t) = \psi(t), \quad a < t < b,\ q \in [0,1),\ 1 < \alpha \leq 2, \tag{5.6}$$

where $\varphi, \psi \in C[a,b]$, under the boundary conditions

$$x(a) = k_1,\ x(b) = k_1, \tag{5.7}$$

where $k_1, k_2 \in \mathbb{R}$.

Theorem 5.4. *Assume that*

$$\int_a^b (s-a)^{\alpha-1}(b-(qs+(1-q)a))_a^{(\alpha-1)}|\varphi(s)|\ {}_ad_qs < \Gamma_q(\alpha)(b-a)^{\alpha-1}. \tag{5.8}$$

If $x \in C[a,b]$ is a solution to (5.6)-(5.7)*, then x is the unique solution to* (5.6)-(5.7)*.*

Proof. Assume the contrary, i.e., (5.6)-(5.7) has two different solutions $x, y \in C[a,b]$. Let

$$u(t) = x(t) - y(t), \quad a \leq t \leq b.$$

Then u is a nontrivial continuous solution to the fractional q-difference boundary value problem (1.2)-(1.3). By Theorem 3.4, we get

$$\int_a^b (s-a)^{\alpha-1}(b-(qs+(1-q)a))_a^{(\alpha-1)}|\varphi(s)|\ {}_ad_qs \geq \Gamma_q(\alpha)(b-a)^{\alpha-1},$$

which contradicts assumption (5.8). □

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