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Fixed point theorems for $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings on α-η-complete partial metric spaces

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Abstract

In this paper, the notion of strictly $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings is introduced where the continuity of ξ is relaxed. The existence of fixed point theorems for such mappings in the setting of α-η-complete partial metric spaces are provided. The results of the paper can be viewed as the extension of the recent results obtained in the literature. Furthermore, we assure the fixed point theorems in partial complete metric spaces endowed with an arbitrary binary relation and with a graph using our obtained results.

Keywords: α-η-complete partial metric spaces, α-η-continuity, $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings, α-admissible multi-valued mappings with respect to η.
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1. Introduction and Preliminaries

The metric fixed point theory is one of the most important tools for proving the existence and uniqueness of the solution to various mathematical models. There are many authors who have generalized the metric spaces in many directions. In 1994, Matthews [12] introduced the partial metric spaces and proved the Banach contraction principle in such spaces. Later on, the researchers have studied the fixed point theorems

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for mappings in complete partial metric spaces, see for examples [3, 4, 6, 7, 8] and references contained therein. On the other hand, Nadler [14] proved the multi-valued version of Banach contraction principle. Since then the metric fixed point theory of single-valued mappings has been extended to multi-valued mappings, see for examples [11, 17]. Recently, Kutbi and Sintunavarat [11] proved the existence of fixed point theorems for strictly (α, ψ, ξ)-contractive multi-valued mappings satisfying some certain contractive conditions in the setting of α-complete metric spaces.

In this paper, we relax the continuity of ξ to be the upper semicontinuity from the right at 0 and introduce the notion of strictly $(\alpha, \eta, \psi, \xi)$-contractive mappings. We also prove the existence of fixed point theorems for such mappings in the setting of α-η-complete partial metric spaces. Our results extend the results proved by Kutbi and Sintunavarat [11]. Furthermore, we assure the fixed point theorems in partial complete metric spaces endowed with an arbitrary binary relation and with a graph using our obtained results.

We now recall some definitions and lemmas that will be used in the sequel.

Definition 1.1 ([12])**.** A partial metric on a nonempty set X is a mapping $p : X \times X \to [0, +\infty)$ such that for all $x, y, z \in X$, the following conditions are satisfied:

(P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A set X equipped with a partial metric p is called a partial metric space and denoted by a pair (X, p).

Lemma 1.2 ([1])**.** *Let (X, p) be a partial metric space. If $p(x, y) = 0$, then $x = y$.*

For each partial metric p on X, the function $p^s : X \times X \to [0, +\infty)$ defined by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X.

Definition 1.3 ([12])**.** Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in a partial metric space (X, p) is convergent to a point $x \in X$ if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$.
(ii) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and is finite).
(iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ that is,
$$\lim_{n,m\to\infty} p(x_n, x_m) = p(x, x).$$

Lemma 1.4 ([12])**.** *Let (X, p) be a partial metric space. Then*

(i) a sequence $\{x_n\}$ in a partial metric space (X, p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, p^s);
(ii) a partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Moreover, $\lim_{n\to\infty} p^s(x, x_n) = 0$ if and only if
$$\lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = p(x, x);$$
(iii) a subset E of a partial metric space (X, p) is closed if whenever $\{x_n\}$ is a sequence in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.

Aydi et al. [8] defined a partial Hausdorff metric as follows. Let (X, p) be a partial metric space. Let$CB^p(X)$ be the family of all nonempty closed bounded subsets of a partial metric space (X, p). For any

$A, B \in CB^p(X)$ and $x \in X$, define

$$\delta_p(A,B) = \sup\{p(a,B) : a \in A\} \text{ and } \delta_p(B,A) = \sup\{p(b,A) : b \in B\},$$

where

$$p(x,A) = \inf\{p(x,a), a \in A\}.$$

The mapping $H_p : CB^p(X) \times CB^p(X) \to [0,+\infty)$ defined by

$$H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}$$

is called a partial Hausdorff metric induced by p.

Remark 1.5 ([3]). Let (X,p) be a partial metric space. If A is a nonempty set in (X,p), then

$$a \in \bar{A} \text{ if and only if } p(a,A) = p(a,a),$$

where $\overline{A}$ is the closure of A with respect to the partial metric p.

Lemma 1.6 ([8])**.** *Let (X,p) be a partial metric space and $T : X \to CB^p(X)$ be a multi-valued mapping. If $\{x_n\}$ is a sequence in X such that $x_n \to z$ and $p(z,z) = 0$, then*

$$\lim_{n\to\infty} p(x_n, Tz) = p(z, Tz).$$

In this paper, we denote by Ψ the class of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ψ_1) ψ is a nondecreasing function;
(ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iteration of ψ.

A function $\psi \in \Psi$ is known in the literature as Bianchini-Grandolfi gauge functions (see e.g. [9] and [15]).

Remark 1.7 ([11]). For each $\psi \in \Psi$, the following statements are satisfied,

(i) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$;
(ii) $\psi(t) < t$ for each $t > 0$;
(iii) $\psi(0) = 0$.

Recently, Ali et al. [2] introduced the family Ξ of functions $\xi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ξ_1) ξ is continuous;
(ξ_2) ξ is nondecreasing on $[0,\infty)$;
(ξ_3) $\xi(t) = 0$ if and only if $t = 0$;
(ξ_4) ξ is subadditive.

They [2] also introduced the concept of (α,ψ,ξ)-contractive multi-valued mappings as follows.

Definition 1.8 ([2])**.** Let (X,d) be a metric space. A multi-valued mapping $T : X \to CB(X)$ is called an (α,ψ,ξ)-contractive mapping if there exist three functions $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0,\infty)$ such that for all $x, y \in X$,

$$\alpha(x,y) \geq 1 \text{ implies } \xi(H(Tx,Ty)) \leq \psi(\xi(M(x,y))),$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}.$$

In the case when $\psi \in \Psi$ is strictly increasing, the (α,ψ,ξ)-contractive mapping is called a strictly (α,ψ,ξ)-contractive mapping.

On the other hand, Mohamadi et al. [13] introduced the concept of α-admissible multi-valued mappings as follows.

Definition 1.9 ([13])**.** Let X be a nonempty set, $T : X \to N(X)$ where $N(X)$ is a set of nonempty subsets of X and $\alpha : X \times X \to [0, \infty)$. T is α-admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$.

Hussain et al. [10] introduced the concept of an α-completeness of a metric space which is weaker than the concept of a completeness.

Definition 1.10 ([10])**.** Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a mapping. The metric space X is said to be α-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X.

Recently, Kutbi and Sintunavarat [11] introduced the concept of an α-continuities for multi-valued mappings in metric spaces and proved the fixed point theorems for strictly (α, ψ, ξ)-contractive mappings in α-complete metric spaces.

Definition 1.11 ([11])**.** Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to CB(X)$ be two given mappings. T is an α-continuous multi-valued mapping if for all sequence $\{x_n\}$ in X with $x_n \xrightarrow{d} x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $Tx_n \xrightarrow{H} Tx \in X$ as $n \to \infty$, that is

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ and } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N} \text{ imply } \lim_{n\to\infty} H(Tx_n, Tx) = 0.$$

Theorem 1.12 ([11])**.** *Let (X, d) be an α-complete metric space and $T : X \to CB(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping;
(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

2. Main results

In this paper, we relax the continuity of $\xi \in \Xi$ to be the upper semicontinuity from the right at 0. Let Ξ' denote the family of functions $\xi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(ξ'_1) ξ is upper semicontinuous from the right at 0;
(ξ'_2) ξ is nondecreasing on $[0, \infty)$;
(ξ'_3) $\xi(t) = 0$ if and only if $t = 0$;
(ξ'_4) ξ is subadditive.

Example 2.1. The floor function $\xi(x) = \lfloor x \rfloor$ is upper semicontinuous function from the right at 0 and nondecreasing but is not continuous.

The following example illustrates that (ξ'_1) is independent from the conditions (ξ'_2) − (ξ'_4). Roughly, we cannot obtain (ξ'_1) by using (ξ'_2) − (ξ'_4).

Example 2.2. Let $\xi : [0, \infty) \to [0, \infty)$ be defined by

$$\xi(t) = \begin{cases} 0, & \text{if } x = 0 \text{ ;} \\ 2t + 3, & \text{if otherwise.} \end{cases}$$

We see that ξ is nondecreasing, subadditive, $\xi(t) = 0$ if and only if $t = 0$. Moreover, ξ is not upper semicontinuous from the right at 0 since

$$\limsup_{n\to\infty} f(\frac{1}{n}) = \limsup_{n\to\infty} (\frac{2}{n} + 3) = 3 > \xi(0).$$

The following example shows that (ξ'_2) is independent from the conditions $(\xi'_1), (\xi'_3)$ and (ξ'_4).

Example 2.3. Let $\xi : [0, \infty) \to [0, \infty)$ be defined by

$$\xi(t) = \begin{cases} \frac{1}{n}, & \text{if } t = \frac{1}{n} \ ; \\ 0, & \text{if otherwise.} \end{cases}$$

Therefore ξ is upper semicontinuous from the right at 0, subadditive, $\xi(t) = 0$ if and only if $t = 0$, but not nondecreasing.

We now introduce the concepts of α-η-complete partial metric spaces and α-η-continuous multi-valued mappings in partial metric spaces.

Definition 2.4. Let (X, p) be a partial metric space and $\alpha, \eta : X \times X \to [0, \infty)$. The partial metric space X is said to be α-η-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, converges in (X, p).

Definition 2.5. Let (X, p) be a partial metric space, $\alpha, \eta : X \times X \to [0, \infty)$ and $T : X \to CB^p(X)$. T is an α-η-continuous multi-valued mapping if, for all sequence $\{x_n\}$ with $\lim_{n\to\infty} p(x_n, x) = p(x, x)$ and $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} H_p(Tx_n, Tx) = H_p(Tx, Tx).$$

We now prove the key lemma that will be used in proving our main results.

Lemma 2.6. *Let (X, p) be a partial metric space, A and B be nonempty closed bounded subsets of X, $\xi \in \Xi'$ and $h > 1$. Then for all $a \in A$ such that $\xi(p(a, B)) > 0$, there exists $b \in B$ such that $\xi(p(a, b)) < h(\xi(p(a, B)))$.*

Proof. Let $a \in A$ be such that $\xi(p(a, B)) > 0$. By (ξ'_3), we have $p(a, B) > 0$. We can construct a sequence $\{b_n\}$ in B such that $\lim_{n\to\infty} p(a, b_n) = p(a, B)$. Using (ξ'_4), we have

$$\xi(p(a, b_n)) \leq \xi(p(a, b_n) - p(a, B)) + \xi(p(a, B)).$$

This implies that

$$\xi(p(a, b_n)) - \xi(p(a, B)) \leq \xi(p(a, b_n) - p(a, B)).$$

Since ξ is upper semicontinuous from the right at 0 and $\lim_{n\to\infty}(p(a, b_n) - p(a, B)) = 0$, we obtain that

$$\limsup_{n\to\infty}(\xi(p(a, b_n)) - \xi(p(a, B)) \leq \limsup_{n\to\infty} \xi(p(a, b_n) - p(a, B)) \leq \xi(0) = 0.$$

This yields

$$\limsup_{n\to\infty} \xi(p(a, b_n)) \leq \xi(p(a, B)) < h\xi(p(a, B)).$$

It follows that there exists $N \in \mathbb{N}$ such that $\xi(p(a, b_N)) < h\xi(p(a, B))$. This completes the proof. □

Next, we introduce the concepts of α-admissibility with respect to η and $(\alpha, \eta, \psi, \xi)$-contractive multi-valued mappings on α-η-partial metric spaces.

Definition 2.7. Let X be a nonempty set, $T : X \to N(X)$ where $N(X)$ is a set of nonempty subsets of X and $\alpha, \eta : X \times X \to [0, \infty)$. T is α-admissible with respect to η whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq \eta(x, y)$, we have $\alpha(y, z) \geq \eta(y, z)$ for all $z \in Ty$.

Definition 2.8. Let (X, p) be a partial metric space. A multi-valued mapping $T : X \to CB^p(X)$ is called an $(\alpha, \eta, \psi, \xi)$-contractive mapping if there exist $\psi \in \Psi$, $\xi \in \Xi'$ and $\alpha, \eta : X \times X \to [0, \infty)$ such that for all $x, y \in X$,

$$\alpha(x, y) \geq \eta(x, y) \Rightarrow \xi(H_p(Tx, Ty)) \leq \psi(\xi(M(x, y))),$$

where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\}.$$

In the case when $\psi \in \Psi$ is strictly increasing, the $(\alpha, \eta, \psi, \xi)$-contractive mapping is called a strictly $(\alpha, \eta, \psi, \xi)$-contractive mapping.

We now prove the existence of fixed point theorems for strictly $(\alpha, \eta, \psi, \xi)$-contractive mappings in α-η-complete partial metric spaces.

Theorem 2.9. *Let (X, p) be an α-η-complete partial metric space and $T : X \to CB^p(X)$ be a strictly $(\alpha, \eta, \psi, \xi)$-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping with respect to η;
(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$;
(iii) T is an α-η-continuous mapping on (X, p);
(iv) if $\{x_n\}$ is a sequence in X converging to a point x in (X, p) such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $\alpha(x, x) \geq \eta(x, x)$.

Then T has a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$ be such that $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$. If $x_0 = x_1$, then x_0 is a fixed point of T. Assume that $x_0 \neq x_1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Assume that $x_1 \notin Tx_1$. Since T is a strictly $(\alpha, \eta, \psi, \xi)$-contractive mapping, we obtain that

$$\begin{aligned}
\xi(H_p(Tx_0, Tx_1)) &\leq \psi(\xi(M(x_0, x_1))) \\
&= \psi(\xi(\max\{p(x_0, x_1), p(x_0, Tx_0), p(x_1, Tx_1), \frac{p(x_0, Tx_1) + p(x_1, Tx_0)}{2}\})) \\
&\leq \psi(\xi(\max\{p(x_0, x_1), p(x_0, x_1), p(x_1, Tx_1), \frac{p(x_0, Tx_1) + p(x_1, x_1)}{2}\})) \\
&\leq \psi(\xi(\max\{p(x_0, x_1), p(x_1, Tx_1), \\
&\qquad \frac{1}{2}[p(x_0, x_1) + p(x_1, Tx_1) - p(x_1, x_1) + p(x_1, x_1)]\})) \\
&\leq \psi(\xi(\max\{p(x_0, x_1), p(x_1, Tx_1), \frac{p(x_0, x_1) + p(x_1, Tx_1)}{2}\})) \\
&= \psi(\xi(\max\{p(x_0, x_1), p(x_1, Tx_1)\})). \qquad (2.1)
\end{aligned}$$

If $\max\{p(x_0, x_1), p(x_1, Tx_1)\} = p(x_1, Tx_1)$, then we have

$$\begin{aligned}
0 < \xi(p(x_1, Tx_1)) \leq \xi(H_p(Tx_0, Tx_1)) &\leq \psi(\xi(\max\{p(x_0, x_1), p(x_1, Tx_1)\})) \\
&\leq \psi(\xi(p(x_1, Tx_1))) \\
&< \xi(p(x_1, Tx_1)),
\end{aligned}$$

which is a contradiction. Therefore, $\max\{p(x_0, x_1), p(x_1, Tx_1)\} = p(x_0, x_1)$. By (2.1), we have

$$0 < \xi(p(x_1, Tx_1)) \leq \xi(H_p(Tx_0, Tx_1)) \leq \psi(\xi(p(x_0, x_1))). \qquad (2.2)$$

Fix $h > 1$ and by using Lemma 2.6, there exists $x_2 \in Tx_1$ such that

$$0 < \xi(p(x_1, x_2)) < h(\xi(p(x_1, Tx_1))). \qquad (2.3)$$

By (2.2) and (2.3), we have

$$0 < \xi(p(x_1, x_2)) < h\psi(\xi(p(x_0, x_1))). \tag{2.4}$$

Since ψ is a strictly increasing mapping, we have

$$0 < \psi(\xi(p(x_1, x_2))) < \psi(h\psi(\xi(p(x_0, x_1)))). \tag{2.5}$$

By setting

$$h_1 = \frac{\psi(h\psi(\xi(p(x_0, x_1))))}{\psi(\xi(p(x_1, x_2)))}, \text{ we obtain that } \quad h_1 > 1.$$

If $x_1 = x_2$ or $x_2 \in Tx_2$, then T has a fixed point. Assume that $x_1 \neq x_2$ and $x_2 \notin Tx_2$. Since $x_1 \in Tx_0$, $x_2 \in Tx_1$, $\alpha(x_0, x_1) \geq \eta(x_0, x_1)$ and T is an α-admissible mapping with respect to η, we have $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$. Since T is a strictly $(\alpha, \eta, \psi, \xi)$-contractive mapping, we obtain that

$$\begin{aligned}
\xi(H_p(Tx_1, Tx_2)) &\leq \psi(\xi(M(x_1, x_2))) \\
&= \psi(\xi(\max\{p(x_1, x_2), p(x_1, Tx_1), p(x_2, Tx_2), \frac{p(x_1, Tx_2) + p(x_2, Tx_1)}{2}\})) \\
&\leq \psi(\xi(\max\{p(x_1, x_2), p(x_1, x_2), p(x_2, Tx_2), \frac{p(x_1, Tx_2) + p(x_2, x_2)}{2}\})) \\
&\leq \psi(\xi(\max\{p(x_1, x_2), p(x_2, Tx_2), \\
&\qquad \frac{1}{2}[p(x_1, x_2) + p(x_2, Tx_2) - p(x_2, x_2) + p(x_2, x_2)]\})) \\
&\leq \psi(\xi(\max\{p(x_1, x_2), p(x_2, Tx_2), \frac{p(x_1, x_2) + p(x_2, Tx_2)}{2}\})) \\
&= \psi(\xi(\max\{p(x_1, x_2), p(x_2, Tx_2)\})).
\end{aligned} \tag{2.6}$$

Assume that $\max\{p(x_1, x_2), p(x_2, Tx_2)\} = p(x_2, Tx_2)$. By (2.6), we have

$$\begin{aligned}
0 < \xi(p(x_2, Tx_2)) \leq \xi(H_p(Tx_1, Tx_2)) &\leq \psi(\xi(\max\{p(x_1, x_2), p(x_2, Tx_2)\})) \\
&\leq \psi(\xi(p(x_2, Tx_2))) \\
&< \xi(p(x_2, Tx_2)),
\end{aligned}$$

which is a contradiction. Then $\max\{p(x_1, x_2), p(x_2, Tx_2)\} = p(x_1, x_2)$. Using (2.6), we obtain that

$$0 < \xi(p(x_2, Tx_2)) \leq \xi(H_p(Tx_1, Tx_2)) \leq \psi(\xi(p(x_1, x_2))). \tag{2.7}$$

By using Lemma 2.6 with $h_1 > 1$, there exists $x_3 \in Tx_2$ such that

$$0 < \xi(p(x_2, x_3)) < h_1(\xi(p(x_2, Tx_2))). \tag{2.8}$$

By (2.7) and (2.8), we have

$$\begin{aligned}
0 < \xi(p(x_2, x_3)) < h_1\psi(\xi(p(x_1, x_2))) &= \frac{\psi(h\psi(\xi(p(x_0, x_1))))}{\psi(\xi(p(x_1, x_2)))}\psi(\xi(p(x_1, x_2))) \\
&= \psi(h\psi(\xi(p(x_0, x_1)))).
\end{aligned}$$

Since ψ is a strictly increasing mapping, we have

$$0 < \psi(\xi(p(x_2, x_3))) < \psi^2(h\psi(\xi(p(x_0, x_1)))). \tag{2.9}$$

Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $x_n \neq x_{n+1} \in Tx_n$,

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \tag{2.10}$$

and

$$0 < \xi(p(x_{n+1}, x_{n+2})) < \psi^n(h\psi(\xi(p(x_0, x_1)))) \tag{2.11}$$

for all $n \in \mathbb{N} \cup \{0\}$. Let $m > n$. Then by the triangular inequality, we have

$$\begin{aligned}
\xi(p(x_n, x_m)) &\le \xi(p(x_n, x_{n+1}) + p(x_{n+1}, x_m) - p(x_{n+1}, x_{n+1}))\\
&\le \xi(p(x_n, x_{n+1}) + p(x_{n+1}, x_m))\\
&\le \xi(p(x_n, x_{n+1}) + \xi(p(x_{n+1}, x_m))\\
&\le \xi(p(x_n, x_{n+1})) + \xi(p(x_{n+1}, x_{n+2})) + \xi(p(x_{n+2}, x_m))\\
&\le \xi(p(x_n, x_{n+1})) + \xi(p(x_{n+1}, x_{n+2})) + \xi(p(x_{n+2}, x_{n+3})) + \cdots + \xi(p(x_{m-1}, x_m))\\
&= \Sigma_{i=n}^{m-1}\xi(p(x_i, x_{i+1}))\\
&< \Sigma_{i=n}^{m-1}\psi^{i-1}(h\psi(\xi(p(x_0, x_1))))\\
&\le \Sigma_{i=n}^{\infty}\psi^{i-1}(h\psi(\xi(p(x_0, x_1)))).
\end{aligned}$$

Since $\psi \in \Psi$, we have $\lim_{m,n\to\infty} \xi(p(x_n, x_m)) = 0$. If $\lim_{m,n\to\infty} p(x_n, x_m) \neq 0$, then there exist $\varepsilon > 0$ and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k) > n(k) \ge k$ such that

$$p(x_{n(k)}, x_{m(k)}) \ge \varepsilon.$$

Since ξ is nondecreasing, we have $\lim_{k\to\infty} \xi(p(x_{n(k)}, x_{m(k)})) \ge \xi(\varepsilon) > 0$ which is a contradiction. Therefore $\lim_{m,n\to\infty} p(x_n, x_m) = 0$. Then $\{x_n\}$ is a Cauchy sequence in (X, p). By Lemma 1.4, we have $\{x_n\}$ is a Cauchy sequence in metric space (X, p^s). Since (X, p) is α-η-complete, we obtain that (X, p^s) is α-η-complete. Then there exists $z \in X$ such that

$$\lim_{n\to\infty} p^s(x_n, z) = 0. \tag{2.12}$$

Since $\lim_{m,n\to\infty} p(x_n, x_m) = 0$, from Lemma 1.4, we have

$$\lim_{n\to\infty} p(x_n, z) = \lim_{m,n\to\infty} p(x_n, x_m) = p(z, z) = 0. \tag{2.13}$$

This implies that $\{x_n\}$ converges to z in (X, p). Since T is α-η-continuous on (X, p), we have

$$\lim_{n\to\infty} p(x_{n+1}, Tz) \le \lim_{n\to\infty} H_p(Tx_n, Tz) = H_p(Tz, Tz). \tag{2.14}$$

Using the triangular inequality, we have

$$p(z, Tz) \le p(z, x_{n+1}) + p(x_{n+1}, Tz).$$

Letting $n \to \infty$ and using (2.14), we get

$$p(z, Tz) \le \lim_{n\to\infty} p(z, x_{n+1}) + \lim_{n\to\infty} p(x_{n+1}, Tz) \le H_p(Tz, Tz).$$

So we have $p(z, Tz) \le H_p(Tz, Tz)$. We will show that $z \in Tz$. Suppose that $z \notin Tz$. By Remark 1.5, we obtain that $p(z, Tz) \neq 0$. Since $\{x_n\}$ converges to z with $\alpha(x_n, x_{n+1}) \ge \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and by (iv), it follows that $\alpha(z, z) \ge \eta(z, z)$. This implies that

$$\begin{aligned}
\xi(H_p(Tz, Tz)) &\le \psi(\xi(M(z, z)))\\
&\le \psi(\xi(\max\{p(z, z), p(z, Tz), p(z, Tz), \frac{p(z, Tz) + p(Tz, z)}{2}\}))\\
&\le \psi(\xi(\max\{p(z, z), p(z, Tz)\}))\\
&= \psi(\xi(p(z, Tz)))\\
&< \xi(p(z, Tz)\\
&\le \xi(H_p(Tz, Tz)),
\end{aligned}$$

which is a contradiction. Therefore $z \in Tz$ and hence T has a fixed point. □

If we take $\eta(x,y)=1$, we obtain the following results.

Corollary 2.10. *Let (X,p) be an α-complete partial metric space and $T:X\to CB^p(X)$ be a strictly (α,ψ,ξ)-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping;
(ii) there exist $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq 1$;
(iii) T is an α-continuous mapping on (X,p);
(iv) if $\{x_n\}$ be a sequence in X that converges to a point x in (X,p) such that $\alpha(x_n,x_{n+1})\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$, then we have $\alpha(x,x)\geq 1$.

Then T has a fixed point.

We next substitute the α-η-continuity of T by some appropriate conditions.

Theorem 2.11. *Let (X,p) be an α-η-complete partial metric space and $T:X\to CB^p(X)$ be a strictly (α,η,ψ,ξ)-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping with respect to η;
(ii) there exist $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq\eta(x_0,x_1)$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ and $x_n\to x$ as $n\to\infty$, then $\alpha(x_n,x)\geq\eta(x_n,x)$ for all $n\in\mathbb{N}\cup\{0\}$.

Then T has a fixed point.

Proof. As in Theorem 2.9, we can construct a sequence $\{x_n\}$ such that $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$, $x_n\neq x_{n+1}\in Tx_n$ for all $n\in\mathbb{N}\cup\{0\}$ and there exists $z\in X$ such that $x_n\to z$ as $n\to\infty$ and $p(z,z)=0$. From condition (iii), we have

$$\alpha(x_n,z)\geq\eta(x_n,z) \tag{2.15}$$

for all $n\in\mathbb{N}\cup\{0\}$. Suppose that $z\notin Tz$. By Remark 1.5, we have $p(z,Tz)>0$. Since T is a strictly (α,η,ψ,ξ)-contractive mapping and (2.15), we obtain that

$$\begin{aligned}\xi(H_p(Tx_n,Tz)) &\leq \psi(\xi(M(x_n,z)))\\ &=\psi(\xi(\max\{p(x_n,z),p(x_n,Tx_n),p(z,Tz),\frac{p(x_n,Tz)+p(z,Tx_n)}{2}\}))\end{aligned} \tag{2.16}$$

for all $n\in\mathbb{N}$. Let $\varepsilon=\frac{p(z,Tz)}{2}$. Since $\{x_n\}$ converges to z in (X,p), There exists $N_1\in\mathbb{N}$ such that

$$p(x_n,z)=|p(x_n,z)-p(z,z)|<\frac{p(z,Tz)}{2}\ \text{ for all } n\geq N_1. \tag{2.17}$$

Furthermore, we obtain that

$$p(Tx_n,z)\leq p(x_{n+1},z)<\frac{p(z,Tz)}{2}\ \text{ for all } n\geq N_1. \tag{2.18}$$

Since $\{x_n\}$ is a Cauchy sequence in (X,p), there exists $N_2\in\mathbb{N}$ such that

$$p(x_n,Tx_n)\leq p(x_n,x_{n+1})<\frac{p(z,Tz)}{2}\ \text{ for all } n\geq N_2. \tag{2.19}$$

It follows from $x_n\to z$ as $n\to\infty$ and $p(z,z)=0$ via Lemma 1.6, we have $p(x_n,Tz)\to p(z,Tz)$ as $n\to\infty$. This implies that there exists $N_3\in\mathbb{N}$ such that

$$p(x_n,Tz)<\frac{3p(z,Tz)}{2} \tag{2.20}$$

for all $n \geq N_3$. Let $N = \max\{N_1, N_2, N_3\}$. Using (2.17)-(2.20), we have

$$\max\{p(x_n, z), p(x_n, Tx_n), p(z, Tz), \frac{p(x_n, Tz) + p(z, Tx_n)}{2}\} = p(z, Tz), \tag{2.21}$$

for all $n \geq N$. By (2.16) and the triangular inequality, we have

$$\begin{aligned}
\xi(p(z, Tz)) &\leq \xi(p(z, x_{n+1}) + p(x_{n+1}, Tz)) \\
&\leq \xi(p(z, x_{n+1})) + \xi(p(x_{n+1}, Tz)) \\
&\leq \xi(p(z, x_{n+1})) + \xi(H_p(Tx_n, Tz)) \\
&\leq \xi(p(z, x_{n+1})) + \psi(\xi(M(x_n, z))) \\
&\leq \xi(p(z, x_{n+1})) + \psi(\xi(p(z, Tz))).
\end{aligned}$$

Since ξ is upper semicontinuous from the right at 0 and by taking the limit superior in the above inequality, we have

$$\begin{aligned}
\xi(p(z, Tz)) &\leq \limsup_{n\to\infty} \xi(p(z, x_{n+1})) + \psi(\xi(p(z, Tz))) \\
&\leq \xi(0) + \psi(\xi(p(z, Tz))) \\
&= \psi(\xi(p(z, Tz))) \\
&< \xi(p(z, Tz)),
\end{aligned}$$

which is a contradiction. Then $z \in Tz$ and hence T has a fixed point. □

If we take $\eta(x, y) = 1$, we have the following result.

Corollary 2.12. *Let (X, p) be an α-complete partial metric space and $T : X \to CB^p(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping;
(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

Using Corollary 2.12, we can extend the result proved by Kutbi and Sintunavarat (Theorem 2.6, [11]).

Corollary 2.13 ([11])**.** *Let (X, d) be an α-complete metric space and $T : X \to CB(X)$ be a strictly (α, ψ, ξ)-contractive mapping. Assume that the following conditions hold:*

(i) T is an α-admissible mapping;
(ii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to^d x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a fixed point.

We give an example for supporting Theorem 2.11.

Example 2.14. Let $X = (-1, 5]$ and a partial metric $p : X \times X \to \mathbb{R}$ defined by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Define $T : X \to CB^p(X)$ by

$$Tx = \begin{cases} \{2x\}, & \text{if } x \in (-1, 0) \ ; \\ \{\frac{x}{16}\}, & \text{if } x \in [0, 5]. \end{cases}$$

Also, we define mappings $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2, & \text{if } x, y \in [0, 5] \text{ ;} \\ \frac{1}{6}, & \text{if otherwise,} \end{cases}$$

$$\eta(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 5] \text{ ;} \\ \frac{1}{4}, & \text{if otherwise .} \end{cases}$$

Define $\psi, \xi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{4}$ and $\xi(t) = \sqrt{t}$. We see that $\psi \in \Psi$ and $\xi \in \Xi'$.

Firstly, we will show that T is a strictly $(\alpha, \eta, \psi, \xi)$-contractive mapping. For $x, y \in X$ and $\alpha(x, y) \geq \eta(x, y)$, we have $x, y \in [0, 5]$ and then

$$\begin{aligned} \xi(H_p(Tx, Ty)) &= \sqrt{\max\{\frac{x}{16}, \frac{y}{16}\}} \\ &= \frac{1}{4}\sqrt{\max\{x, y\}} \\ &= \frac{1}{4}\sqrt{p(x, y)} \\ &\leq \frac{1}{4}\sqrt{M(x, y)} = \psi(\xi(M(x, y))). \end{aligned}$$

It is clear that ψ is a strictly increasing function. Therefore, T is strictly $(\alpha, \eta, \psi, \xi)$-contractive mapping. We next show that T is an α-admissible with respect to η. Let $x \in X$, $y \in Tx$ and $z \in Ty$ with $\alpha(x, y) \geq \eta(x, y)$, we have $x, y \in [0, 5]$, it follows that $Ty \in [0, 5]$. Since $z \in Ty$, we have $z \in [0, 5]$. So $\alpha(y, z) \geq \eta(y, z)$. Therefore T is an α-admissible with respect to η. We will prove that (X, p) is an α-η-complete partial metric space. If $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then $\{x_n\} \subseteq [0, 5]$ for all $n \in \mathbb{N}$. Since $x \in \overline{[0, 5]}$ iff $p(x, [0, 5]) = p(x, x)$ iff $\inf_{y \in [0,5]} \max\{x, y\} = \inf_{y \in [0,5]} p(x, y) = p(x, x)$ iff $x \in [0, 5]$, we obtain that $[0, 5]$ is closed in (X, p). Now, since $([0, 5], p)$ is a complete partial metric space, then the sequence $\{x_n\}$ converges in $[0, 5] \subseteq X$. Next, there exist $x_0 = 1 \in X$ and $x_1 = 2 \in Tx_0$ such that

$$\alpha(x_0, x_1) = \alpha(1, 2) = 2 > 1 = \eta(1, 2) = \eta(x_0, x_1).$$

Then the condition (ii) of Theorem 2.11 is satisfied. Finally, for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $x_n \to x$ as $n \to \infty$ for all $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N}$. Thus the condition (iv) of Theorem 2.11 is satisfied. Then all the conditions of Theorem 2.11 are satisfied and so T has a fixed point which is $x = 0$.

3. Consequences

3.1. Fixed point results in partial metric spaces endowed with binary relations

Let (X, p) be a partial metric space and $\mathcal{R}$ be a binary relation over X. Denote $\mathcal{S} := \mathcal{R} \cup \mathcal{R}^{-1}$, that is

$$x, y \in X, \;\; x\mathcal{S}y \text{ if and only if } x\mathcal{R}y \;\text{ or }\; y\mathcal{R}x.$$

Definition 3.1 ([11])**.** Let X be a nonempty set and $\mathcal{R}$ be a binary relation over X. A multi-valued mapping $T : X \to N(X)$ is said to be weakly comparative if for each $x \in X$ and $y \in Tx$ with $x\mathcal{S}y$, we have $y\mathcal{S}z$ for all $z \in Ty$.

We now introduce the notions of $\mathcal{S}$-completeness, $\mathcal{S}$-continuity and $(\mathcal{S}, \psi, \xi)$-contractive mappings on partial metric spaces as follows.

Definition 3.2. Let (X, p) be a partial metric space and $\mathcal{R}$ be a binary relation over X. The partial metric space X is said to be $\mathcal{S}$-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$, converges in (X, p).

Definition 3.3. Let (X, p) be a partial metric space and $\mathcal{R}$ be a binary relation over X. $T : X \to CB^p(X)$ is an $\mathcal{S}$-continuous mapping if for given $x \in X$ and a sequence $\{x_n\}$ with $\lim_{n\to\infty} p(x_n, x) = p(x, x)$ and $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$ imply $\lim_{n\to\infty} H_p(Tx_n, Tx) = H_p(Tx, Tx)$.

Definition 3.4. Let (X, p) be a partial metric space and $\mathcal{R}$ be a binary relation over X. A mapping $T : X \to CB^p(X)$ is called an $(\mathcal{S}, \psi, \xi)$-contractive mapping if there exist $\psi \in \Psi$ and $\xi \in \Xi'$ such that for all $x, y \in X$,

$$x\mathcal{S}y \text{ implies } \xi(H_p(Tx, Ty)) \leq \psi(\xi(M(x, y))),$$

where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\}.$$

In the case when $\psi \in \Psi$ is strictly increasing, the $(\mathcal{S}, \psi, \xi)$-contractive mapping is called a strictly $(\mathcal{S}, \psi, \xi)$-contractive mapping.

We now assure the fixed point theorems for strictly $(\mathcal{S}, \psi, \xi)$-contractive mappings on partial metric spaces with binary relations.

Theorem 3.5. *Let (X, p) be a partial metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to CB^p(X)$ be a strictly $(\mathcal{S}, \psi, \xi)$-contractive mapping. Assume that the following conditions hold:*

(i) (X, p) is an $\mathcal{S}$-complete partial metric space;
(ii) T is a weakly comparative mapping;
(iii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \mathcal{S} x_1$;
(iv) T is an $\mathcal{S}$-continuous multi-valued mapping;
(v) if $\{x_n\}$ is a sequence in X converging to a point x in (X, p), where $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $x\mathcal{S}x$.

Then T has a fixed point.

Proof. Define mappings $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in x\mathcal{S}y; \\ 0, & \text{if otherwise,} \end{cases}$$

$$\eta(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x, y \in x\mathcal{S}y; \\ 2, & \text{if otherwise.} \end{cases}$$

Therefore we can obtain the result by using Theorem 2.9. □

By using Theorem 2.11, we immediately obtain the following result.

Theorem 3.6. *Let (X, p) be a partial metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to CB^p(X)$ be a strictly $(\mathcal{S}, \psi, \xi)$-contractive mapping. Assume that the following conditions hold:*

(i) (X, p) is an $\mathcal{S}$-complete partial metric space;
(ii) T is a weakly comparative mapping;
(iii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x_0 \mathcal{S} x_1$;
(iv) if $\{x_n\}$ is a sequence in X with $x_n \to x \in X$ as $n \to \infty$ and $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, then we have $x_n \mathcal{S} x$.

Then T has a fixed point.

3.2. Fixed point results in partial metric spaces endowed with graph

Let (X, p) be a partial metric space. Let G be a graph such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops.

Definition 3.7 ([11])**.** Let (X, p) be a nonempty set endowed with a graph G and $T : X \to N(X)$ be a multi-valued mapping. We say that T weakly preserves edges if for each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for all $z \in Ty$.

We now introduce the notions of $E(G)$-completeness, $E(G)$-continuity and $(E(G), \psi, \xi)$-contractive mappings on partial metric spaces as follows.

Definition 3.8. Let (X, p) be a partial metric space endowed with a graph G. The partial metric space X is said to be $E(G)$-complete if and only if every Cauchy sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, converges in (X, p).

Definition 3.9. Let (X, p) be a partial metric space endowed with a graph G. $T : X \to CB^p(X)$ is an $E(G)$-continuous mapping if for given $x \in X$ and a sequence $\{x_n\}$ with $\lim_{n\to\infty} p(x_n, x) = p(x, x)$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ imply $\lim_{n\to\infty} H_p(Tx_n, Tx) = H_p(Tx, Tx)$.

Definition 3.10. Let (X, p) be a partial metric space endowed with a graph G. A mapping $T : X \to CB^p(X)$ is called an $(E(G), \psi, \xi)$-contractive mapping if there exist $\psi \in \Psi$ and $\xi \in \Xi'$ such that for all $x, y \in X$,

$$(x, y) \in E(G) \text{ implies } \xi(H_p(Tx, Ty)) \le \psi(\xi(M(x, y))),$$

where

$$M(x, y) = \max\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\}.$$

In the case when $\psi \in \Psi$ is strictly increasing, the $(E(G), \psi, \xi)$-contractive mapping is called a strictly $(E(G), \psi, \xi)$-contractive mapping.

Theorem 3.11. *Let (X, p) be a partial metric space endowed with a graph G and $T : X \to CB^p(X)$ be a strictly $(E(G), \psi, \xi)$-contractive mapping. Assume that the following conditions hold:*

(i) (X, p) is an $E(G)$-complete partial metric space;
(ii) T weakly preserves edges;
(iii) there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;
(iv) T is an $E(G)$-continuous mapping on (X, p);
(v) if $\{x_n\}$ is a sequence in X converging to a point x in (X, p), where $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}\cup\{0\}$, then we have $(x, x) \in E(G)$.

Then T has a fixed point.

Proof. Define mappings $\alpha, \eta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } (x, y) \in E(G); \\ 0, & \text{if otherwise,} \end{cases}$$

$$\eta(x, y) = \begin{cases} \frac{1}{2}, & \text{if } (x, y) \in E(G); \\ 2, & \text{if otherwise .} \end{cases}$$

Therefore we can obtain the result by using Theorem 2.9. □

By using Theorem 2.11, we immediately obtain the the following result.

Theorem 3.12. *Let (X, p) be a partial metric space endowed with a graph G and $T : X \to CB^p(X)$ be a strictly $(E(G), \psi, \xi)$-contractive mapping. Assume that the following conditions hold:*

(i) (X, p) *is an* $E(G)$*-complete partial metric space;*

(ii) T *weakly preserves edges;*

(iii) there exist $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $(x_0, x_1) \in E(G)$*;*

(iv) if $\{x_n\}$ *is a sequence in* X *with* $x_n \to x \in X$ *as* $n \to \infty$ *and* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in \mathbb{N} \cup \{0\}$*, then we have* $(x_n, x) \in E(G)$*.*

Then T *has a fixed point.*

Remark 3.13. Our results extend and improve several results in the literature as the following:

(1) Theorem 2.11 extends Theorem 2.6 [2], Theorem 2.2 [5], Theorem 3.2 [8], Theorem 2.6 [11], Theorem 3.4 [13] and Theorem 2.2 [16].

(2) Theorem 3.6 extends Theorem 3.6 [11].

(3) Theorem 3.12 extends Theorem 3.12 [11].

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