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Fixed-point theorem for Caputo–Fabrizio fractional Nagumo equation with nonlinear diffusion and convection

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Abstract

We make use of fractional derivative, recently proposed by Caputo and Fabrizio, to modify the nonlinear Nagumo diffusion and convection equation. The proposed fractional derivative has no singular kernel considered as a filter. We examine the existence of the exact solution of the modified equation using the method of fixed-point theorem. We prove the uniqueness of the exact solution and present some numerical simulations.

Keywords: Nonlinear Nagumo equation, Caputo–Fabrizio derivative, fixed-point theorem, uniqueness.
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1. Introduction

The nonlinear diffusion and convection Nagumo equation has been used in population dynamics, environmental studies, neurophysiology, biochemical reactions and flame promulgation. More examples can be found in [8, 9, 10, 16, 17]. Significant attention was devoted to the situations where partial differential equations describe the decreasing nonlinear diffusion; [1, 13, 15, 17]. Another example is when a propagating wave front solution of sharp type endures a constant wave speed; such wave fronts characterize cooperative gesticulation of populations, especially collective spreading, incursion in bionetworks and concentration in biochemical feedbacks [5, 7, 11, 12, 18, 19]. In many situations these physical phenomena can well be described by using the concept of fractional derivative.

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Recently a new fractional derivative with no singular kernel was proposed in [6] and further employed in [2, 3, 4, 14]. We apply the new fractional derivative to the nonlinear Nagumo equation. The main contribution of this study is identifying the new fractional derivative to the nonlinear Nagumo equation and proving in detail the exactness and the uniqueness of solution of the modified equation using a fixed-point theorem. For the readers unfamiliar with this new derivative, we summarize some useful results from the fractional derivative theory in Section 2.

2. On Caputo–Fabrizio derivative

Recently Caputo and Fabrizio have proposed a fractional derivative with no singular kernel. For more information about this derivative see below.

Definition 2.1. Let $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1]$. The Caputo fractional derivative is defined by

$$D_t^\alpha(f(t)) = \frac{M(\alpha)}{1-\alpha}\int_a^t f^{'}(x)\exp[-\alpha\frac{t-x}{1-\alpha}]dx, \tag{2.1}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$, [2, 3, 4, 6, 14]. If $f \notin H^1(a,b)$, then the derivative can be defined by

$$D_t^\alpha(f(t)) = \frac{\alpha M(\alpha)}{1-\alpha}\int_a^t (f(t) - f(x))\exp[-\alpha\frac{t-x}{1-\alpha}]dx. \tag{2.2}$$

Remark 2.2. The authors commented that, if $\sigma = \frac{1-\alpha}{\alpha} \in [0,\infty]$, $\alpha = \frac{1}{1+\sigma} \in [0,1]$, then Equation (2.2) becomes

$$D_t^\sigma(f(t)) = \frac{N(\sigma)}{\sigma}\int_a^t f^{'}(x)\exp[-\frac{t-x}{\sigma}]dx, \quad N(0) = N(\infty) = 1. \tag{2.3}$$

Furthermore,

$$\lim_{\sigma\to 0}\frac{1}{\sigma}\exp\left[-\frac{t-x}{\sigma}\right] = \delta(x-t). \tag{2.4}$$

The corresponding anti-derivative turned out to be important. An integral connected to the Caputo derivative with fractional order, was suggested by Nieto and Losada [2, 3, 4, 6, 14], see the definition below.

Definition 2.3 ([14])**.** Let $0 < \alpha < 1$. The fractional integral of order α of a function f is defined by

$$I_\alpha^t(f(t)) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t f(s)ds, \quad t \geq 0. \tag{2.5}$$

Remark 2.4 ([14]). The remainder occurring in the above definition of the fractional integral of Caputo type of the function of order $0 < \alpha < 1$ is a mean between the function f and its integral of order one. This consequently enforces,

$$\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2\alpha}{(2-\alpha)M(\alpha)} = 1, \tag{2.6}$$

where

$$M(\alpha) = \frac{2}{2-\alpha}, \quad 0 \leq \alpha \leq 1,$$

so that Nieto and Losada noticed that the definition of the Caputo derivative of order $0 < \alpha < 1$ can be reformulated by

$$D_t^\alpha(f(t)) = \frac{1}{1-\alpha}\int_a^t f^{'}(x)\exp[-\alpha\frac{t-x}{1-\alpha}]dx. \tag{2.7}$$

Theorem 2.5. *For the new Caputo derivative of fractional order, if the function $f(t)$ is such that*

$$f^{(s)}(a) = 0, \quad s = 1, 2, ..., n,$$

then, we have

$$D_t^\alpha(D_t^n(f(t))) = D_t^n(D_t^\alpha(f(t))).$$

Proof. For a proof see [6]. □

3. Fixed-point theorem for Nagumo equation with Caputo–Fabrizio

In this section, we aim to show the existence of an exact solution of the Nagumo equation with nonlinear diffusion and convection with time fractional Caputo–Fabrizio derivative. The nonlinear equation under study here is

$$\begin{gathered} {}_0^{CF}D_t^\alpha u(x,t) + \beta u(x,t)^n \partial_x u(x,t) = \partial_x\left(\alpha u(x,t)^n \partial_x u(x,t)\right) + \gamma u(x,t)(1-u^m)(u^m-\delta), \\ 0<\alpha<1, \quad u(x,0)=f(x), \quad u(0,t)=g(t), \end{gathered} \tag{3.1}$$

where α, β, γ and δ are constant. Integrating (3.1), in the sense of Definition 2.3, we obtain

$$u(x,t) - u(x,0) = I_t^\alpha\left(-\beta u(x,t)^n \partial_x u(x,t) + \partial_x\left(\alpha u(x,t)^n \partial_x u(x,t)\right) + \gamma u(x,t)(1-u^m)(u^m-\delta)\right). \tag{3.2}$$

For simplicity, we let

$$K(x,t,u) = -\beta u(x,t)^n \partial_x u(x,t) + \partial_x\left(\alpha u(x,t)^n \partial_x u(x,t)\right) + \gamma u(x,t)(1-u^m)(u^m-\delta). \tag{3.3}$$

Then Equation (3.2) becomes

$$u(x,t) - u(x,0) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}K(x,t,u) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K(x,y,u)dy, \quad t \geq 0. \tag{3.4}$$

To achieve our proof, we first show that the function K satisfies the Lipchitz condition.

Theorem 3.1. *K satisfies the Lipschitz condition.*

Proof. Let u and v be two bounded functions. We have

$$\begin{aligned} \|K(x,t,u) - K(x,t,v)\| = \|&\beta v(x,t)^n \partial_x v(x,t) - \beta u(x,t)^n \partial_x u(x,t) + \partial_x(\alpha u(x,t)^n \partial_x u(x,t) \\ &- \alpha v(x,t)^n \partial_x v(x,t)) + \gamma u(x,t)(1-u^m)(u^m-\delta) \\ &- \gamma v(x,t)(1-v^m)(v^m-\delta)\|. \end{aligned} \tag{3.5}$$

A direct application of the triangular inequality produces

$$\begin{aligned} \|K(x,t,u) - K(x,t,v)\| \leq \|&\beta v(x,t)^n \partial_x v(x,t) - \beta u(x,t)^n \partial_x u(x,t)\| + \|\partial_x(\alpha u(x,t)^n \partial_x u(x,t) \\ &- \alpha v(x,t)^n \partial_x v(x,t))\| + \|\gamma u(x,t)(1-u^m)(u^m-\delta) \\ &- \gamma v(x,t)(1-v^m)(v^m-\delta)\|. \end{aligned} \tag{3.6}$$

We shall investigate case by case

$$\begin{aligned} \|\beta v(x,t)^n \partial_x v(x,t) - \beta u(x,t)^n \partial_x u(x,t)\| &= \frac{\beta}{n+1}\|\partial_x(v(x,t)^{n+1} - u(x,t)^{n+1})\| \\ &\leq \frac{\beta}{n+1}\rho_1\|v(x,t)^{n+1} - u(x,t)^{n+1}\| \\ &\leq \frac{\beta}{n+1}\rho_1\|v(x,t) - u(x,t)\|\left\|\sum_{j=0}^n v(x,t)^j u(x,t)^{n-j}\right\|. \end{aligned} \tag{3.7}$$

Since the two functions are bounded, there exist two positive numbers M and N such that for all (x,t), $\|u(x,t)\| < M$, $\|v(x,t)\| < N$, so that

$$\begin{aligned}\frac{\beta}{n+1}\rho_1\|v(x,t)-u(x,t)\|\left\|\sum_{j=0}^{n} C_n^j v(x,t)^j u(x,t)^{n-j}\right\| <&\frac{\beta}{n+1}\rho_1\|v-u\|\sum_{j=0}^{n} C_n^j N^j M^{n-j}\\ =&\frac{\beta}{n+1}\rho_1\|v-u\|(N+M)^n.\end{aligned} \tag{3.8}$$

Therefore Equation (3.7) becomes

$$\|\beta v(x,t)^n\partial_x v(x,t)-\beta u(x,t)^n\partial_x u(x,t)\| <\frac{\beta}{n+1}\rho_1\|v-u\|(N+M)^n = \lambda_1\|u-v\|. \tag{3.9}$$

We shall evaluate the following

$$\|\partial_x(\alpha u(x,t)^n\partial_x u(x,t)-\alpha v(x,t)^n\partial_x v(x,t))\| < \rho_2\alpha\|u(x,t)^n\partial_x u(x,t)-v(x,t)^n\partial_x v(x,t)\|. \tag{3.10}$$

Now following the demonstration presented earlier, we obtain

$$\|\partial_x(\alpha u(x,t)^n\partial_x u(x,t)-\alpha v(x,t)^n\partial_x v(x,t))\| <\frac{\beta}{n+1}\rho_2\alpha\|v-u\|(N+M)^n = \lambda_2\|u-v\|. \tag{3.11}$$

We evaluate the following

$$\begin{aligned}\|\gamma u(x,t)(1-u^m)(u^m-\delta)-\gamma v(x,t)(1-v^m)(v^m-\delta)\| \leq&\gamma[\|u^{m+1}-v^{m+1}\|+\delta\|v-u\|\\ &+\|v^{2m+1}-u^{2m+1}\|+\delta\|u^{m+1}-v^{m+1}\|],\end{aligned} \tag{3.12}$$

where

$$\|u^{m+1}-v^{m+1}\| \leq \|u-v\|\left\|\sum_{j=0}^{m-1} C_{m-1}^j v^{m-j-1}u^j\right\| < \|u-v\|(N+M)^m. \tag{3.13}$$

Thus,

$$\|u^{2m+1}-v^{2m+1}\| \leq \|u-v\|\left\|\sum_{j=0}^{2m-1} C_{m-1}^j v^{2m-j-1}u^j\right\| < \|u-v\|(N+M)^{2m}. \tag{3.14}$$

Replacing (3.13) and (3.14) into (3.12), we obtain

$$\begin{aligned}\|\gamma u(x,t)(1-u^m)(u^m-\delta)-\gamma v(x,t)(1-v^m)(v^m-\delta)\| <&\gamma[\|u-v\|(M+N)^m+\delta\|v-u\|\\ &+\|u-v\|(M+N)^{2m}+\delta\|u-v\|(M+N)^m]\\ =&\lambda_3\|u-v\|.\end{aligned} \tag{3.15}$$

Thus replacing (3.15), (3.11) and (3.9) into (3.5), we obtain

$$\|K(x,t,u)-K(x,t,v)\| < L\|u-v\|, \qquad L=\lambda_1+\lambda_2+\lambda_3. \tag{3.16}$$

This completes the proof. □

Now taking into account that the function K, which is considered here as a nonlinear kernel, Equation (3.4) can be converted to the following recursive formula

$$\begin{aligned}u_n(x,t) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}(K(x,t,u_{n-1})-K(x,t,u_{n-2}))\\&\quad+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t(K(x,y,u_{n-1})-K(x,y,u_{n-2}))dy,\\u_0(x,t) &= u(x,0).\end{aligned}\tag{3.17}$$

We consider the variance between the two consecutive terms

$$\begin{aligned}V_n(x,t)=u_n(x,t)-u_{n-1}(x,t) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}(K(x,t,u_{n-1})-K(x,t,u_{n-2}))\\&\quad+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t(K(x,y,u_{n-1})-K(x,y,u_{n-2}))dy.\end{aligned}\tag{3.18}$$

We notice that

$$u_n(x,t)=\sum_{j=0}^{n}V_j(x,t).\tag{3.19}$$

Therefore

$$\begin{aligned}\|V_n(x,t)\|=\|u_n(x,t)-u_{n-1}(x,t)\| &= \|\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}(K(x,t,u_{n-1})-K(x,t,u_{n-2}))\\&\quad+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t(K(x,y,u_{n-1})-K(x,y,u_{n-2}))dy\|\\&\le\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|(K(x,t,u_{n-1})-K(x,t,u_{n-2}))\|\\&\quad+\frac{2\alpha}{(2-\alpha)M(\alpha)}\|\int_0^t(K(x,y,u_{n-1})-K(x,y,u_{n-2}))dy\|\\&\le\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|(K(x,t,u_{n-1})-K(x,t,u_{n-2}))\|\\&\quad+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t\|(K(x,y,u_{n-1})-K(x,y,u_{n-2}))\|dy.\end{aligned}\tag{3.20}$$

Using the Lipschitz condition of the function K, we obtain

$$\begin{aligned}&\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|(K(x,t,u_{n-1})-K(x,t,u_{n-2}))\|\\&\quad+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t\|(K(x,y,u_{n-1})-K(x,y,u_{n-2}))\|dy\\&\le\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L\|(u_{n-1}-u_{n-2})\|+\frac{2\alpha}{(2-\alpha)M(\alpha)}L\int_0^t\|(u_{n-1}-u_{n-2})\|dy\\&\le\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L\|V_{n-1}\|+\frac{2\alpha}{(2-\alpha)M(\alpha)}L\int_0^t\|V_{n-1}\|dy.\end{aligned}\tag{3.21}$$

Then

$$\|V_n(x,t)\|\le\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L\|V_{n-1}\|+\frac{2\alpha}{(2-\alpha)M(\alpha)}L\int_0^t\|V_{n-1}\|dy.\tag{3.22}$$

With the above relation in hand, we shall state the following theorem

Theorem 3.2. *Providing that the physical problem under investigation in this paper takes place in a confined medium,* Equation (3.4) *has an exact solution.*

Proof. Under the conditions in the theorem, together with the Lipchitz condition of the kernel K, we have

$$\|V_n(x,t)\| \le u(x,0)\left[\left(\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L\right)^n + \left(\frac{2\alpha}{(2-\alpha)M(\alpha)}LT\right)^n\right]. \tag{3.23}$$

However to demonstrate that the above is solution of Equation (3.4), we assume that the exact solution is given by

$$\begin{aligned} u(x,t) = u_n(x,t) - P_n(x,t) =& \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}K(x,t,u-P_n(x,t)) \\ &+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K(x,t,u-P_n(x,t))dy. \end{aligned} \tag{3.24}$$

In this case, the function $P_n(x,t)$ tends to zero for large n

$$\begin{aligned} &u(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}K(x,t,u) - u(x,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K(x,y,u)dy \\ &= P_n(x,t) + \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}K(x,t,u) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t (K(x,y,u-P_n(x,y)) - K(x,y,u))dy. \end{aligned} \tag{3.25}$$

Then,

$$\begin{aligned} &\|u(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}K(x,t,u) - u(x,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K(x,y,u)dy\| \\ &\qquad \le \|P_n(x,t)\| + \left(\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L + \frac{2\alpha}{(2-\alpha)M(\alpha)}LT\right)\|P_n(x,t)\|. \end{aligned} \tag{3.26}$$

Taking the limit when n tends to infinity, we obtain

$$\left\|u(x,t) - \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}K(x,t,u) - u(x,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K(x,y,u)dy\right\| = 0. \tag{3.27}$$

This implies

$$u(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}K(x,t,u) + u(x,0) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K(x,y,u)dy. \tag{3.28}$$

This completes the proof. □

4. Uniqueness of the exact solution

Now we prove the uniqueness of the exact solution of Equation (3.4). So assume that there exists another solution of Equation (3.4), say $v(x,t)$. We would have

$$u(x,t) - v(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}(K(x,t,u) - K(x,t,v)) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t (K(x,y,u) - K(x,y,v))dy. \tag{4.1}$$

Thus

$$\begin{aligned} \|u(x,t) - v(x,t)\| \le& \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|(K(x,t,u) - K(x,t,v))\| \\ &+ \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \|(K(x,y,u) - K(x,y,v))\|dy. \end{aligned} \tag{4.2}$$

Making use of the Lipchitz condition of the kernel K, we obtain

$$\|u-v\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L\|u-v\| + \frac{2\alpha}{(2-\alpha)M(\alpha)}LT\|u-v\|. \tag{4.3}$$

This leads to

$$\|u-v\|\left(1-\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L-\frac{2\alpha}{(2-\alpha)M(\alpha)}L\right) \leq 0. \tag{4.4}$$

Theorem 4.1. *Equation* (3.4) *has a unique exact solution if the following condition holds*

$$\left(1-\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L-\frac{2\alpha}{(2-\alpha)M(\alpha)}L\right) \neq 0. \tag{4.5}$$

Proof. If the above condition holds, then

$$\|u-v\|\left(1-\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L-\frac{2\alpha}{(2-\alpha)M(\alpha)}L\right) \leq 0 \tag{4.6}$$

implies

$$\|u-v\| = 0. \tag{4.7}$$

This completes the proof. □

5. Simulations

In this section, we present numerical solutions of Equation (3.4). So, let us consider the following recursive formula

$$\begin{gathered} u_n(x,t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}K(x,t,u_{n-1}(x,t)) + \frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t K(x,y,u_{n-1}(x,y))dy, \\ u_0(x,t) = u(x,0). \end{gathered} \tag{5.1}$$

Then, the solution is given by

$$u(x,t) = \sum_{j=0}^{n} V_j(x,t). \tag{5.2}$$

In Table 1 we present the theoretical parameters used for this simulation.

Table 1: Theoretical parameters used for the simulations

Parameters	Value
a	2
β	3
γ	1
δ	3
m	5
n	6

The simulations are done for different values of α. The numerical solutions are presented in Figures 1, 2 and 3. From the figures, one can see that the fractional order is a controlling parameter.

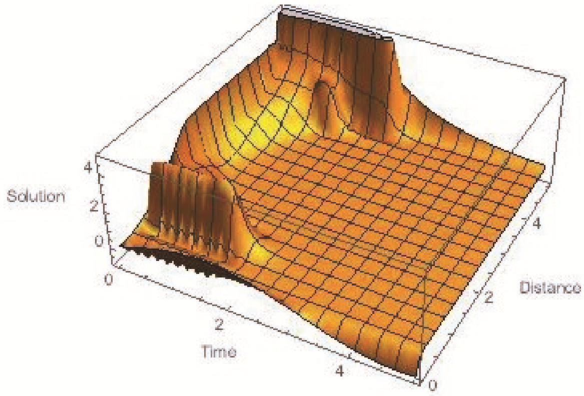

Figure 1: Numerical simulation for $\alpha = 0.85$

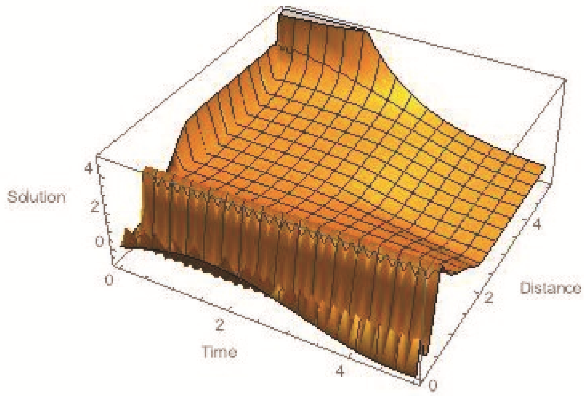

Figure 2: Numerical simulation for $\alpha = 0.45$

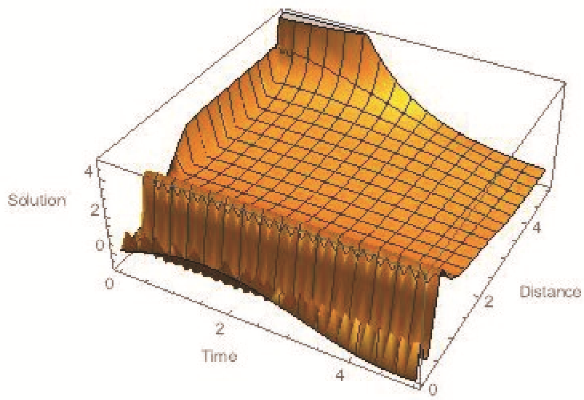

Figure 3: Numerical simulation for $\alpha = 0.25$

6. Conclusion

A kernel with exponential function is more realistic that the one with a power function, due to the fact that the singularity does not occur at the end of the interval within which the fractional derivative of a given function is taken. In addition to this, the exponent function is perhaps a better filter than the power function, therefore a fractional derivative with an exponential kernel is preferable than the one with power function. Using the proposed fractional derivative, the nonlinear Nagumo equation was extended to the scope of fractional calculus. A fixed-point theorem method was used to show the existence of the exact solution of the nonlinear time-fractional equation. The presented proof of the uniqueness of the exact solution is detailed. Some simulations are presented to show the effect of the fractional order.

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