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Cyclic hybrid methods for finding common fixed points of a finite family of nonexpansive mappings

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Abstract

In this paper, we propose a cyclic hybrid method for computing a common fixed point of a finite family of nonexpansive mappings. The strong convergence of the method is established. Numerical examples illustrate that the proposed method has an advantage in computing.

Keywords: Common fixed point, hybrid method, cyclic computation, nonexpansive mapping.
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1. Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and C a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

holds for all $x, y \in C$. We denote by $Fix(T)$ the set of fixed points of T, i.e., $Fix(T) = \{x \in C : Tx = x\}$.

Construction of common fixed points for a finite family of nonexpansive mappings have received vast investigation, see [3, 6, 13, 16], since various problems of science and engineering, such as split feasibility problems and multiple-sets split feasibility problems whit applications in intensity-modulated radiation therapy (IMRT) in the field of medical care, see [4, 5], can be reduced to a problem of finding a common fixed point of a family of nonexpansive mappings.

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In 2003, Nakajo and Takahashi [12] firstly introduced a hybrid algorithm for a nonexpansive mapping, thereafter, several researchers generalized the hybrid methods for computing common fixed points of a family of nonlinear mappings, see [7, 8, 14, 17, 18, 21]. For a finite family of relatively nonexpansive mappings $\{T_i\}_{i=1}^N$, Anh and Chung [1] recently proposed a parallel hybrid algorithm as following:

Algorithm AC

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ z_k := P_C(x_k), \\ y_k^i := \alpha_k z_k + (1-\alpha_k)T_i(z_k), \quad i = 1, 2, \cdots, N, \\ i_k := \underset{i=1,2,\cdots,N}{argmax}\{\|y_k^i - x_k\|\}, \\ C_k := \{v \in H : \|v - y_k^{i_k}\| \leq \|v - x_k\|\}, \\ Q_k := \{u \in H : \langle x_0 - x_k, x_k - u\rangle \geq 0\}, \\ x_{k+1} := P_{C_k \bigcap Q_k}(x_0). \end{cases} \tag{1.1}$$

Algorithm AC is inherently parallel and Anh and Chung showed their advantage in parallel computation in numerical examples.

Motivating by Anh and Chung's work, we proposed a cyclic hybrid method which can be regarded as a counterpart of the parallel one. Our ideas consists of determining successively y_k^i for each operator T_i, $i = 1, 2, \ldots, N$ and constructing of y_k^i by using the value of y_k^{i-1}. Subsequent steps are the same with Algorithm AC. The benefit of our approach is using the newly-obtained y_k^{i-1} ($y_k^0 = x_k$).

The remainder of this article is organized as follows. In the next section, some useful facts and tools are given. Convergence analysis of the cyclic algorithm is given in Section 3, while in Section 4 the numerical experiment is considered.

2. Preliminaries

We will use the notation:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1 ([2])**.** *There holds the identity in a real Hilbert space H:*

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle, \quad u, v \in H.$$

Lemma 2.2 ([10])**.** *Let C be a closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $Fix(T) \neq \emptyset$. If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z = Tz$.*

Lemma 2.3 ([2])**.** *Let K be a closed convex subset of real Hilbert space H and let P_K be the (metric or nearest point) projection from H onto K (i.e., for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}$). Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation:*

$$\langle x - z, y - z\rangle \leq 0, \quad \text{for all } y \in K.$$

Lemma 2.4 ([11])**.** *Let K be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_K u$. If $\{x_n\}$ is such that $\omega_w\{x_n\} \subset K$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \quad \text{for all } n,$$

then $x_n \to q$.

3. A cyclic hybrid algorithm and its convergence

Let $\{T_i\}_{i=1}^N$ be a family of nonexpansive mappings from C into itself and assume that the set $F := \bigcap_{i=1}^N F(T_i)$ is not empty.

We consider the following algorithm.

Algorithm 3.1. Let $x_0 \in C$ be an arbitrarily chosen element and $\{\alpha_k\} \subset (0, \alpha]$ where $\alpha < 1$. For $k \geq 0$, assuming x_k is known, we

•Calculate

$$\begin{cases} y_k^1 : = \alpha_k x_k + (1-\alpha_k)T_1(x_k), \\ y_k^{i+1} : = \alpha_k y_k^i + (1-\alpha_k)T_{i+1}(y_k^i), \quad i = 0, 1, \cdots, N-1. \end{cases} \tag{3.1}$$

•Find

$$i_k := \underset{i=1,2,\cdots,N}{argmax}\{\|y_k^i - x_k\|\}. \tag{3.2}$$

•If $\|y_k^{i_k} - x_k\| = 0$ then stop. Else:

•Define

$$\begin{cases} C_k := \{u \in C : \|u - y_k^{i_k}\| \leq \|u - x_k\|\}, \\ Q_k := \{v \in C : \langle x_0 - x_k, x_k - v\rangle \geq 0\}. \end{cases} \tag{3.3}$$

•Compute

$$x_{k+1} := P_{C_k \bigcap Q_k}(x_0). \tag{3.4}$$

•If $x_{k+1} = x_k$ then stop. Else, set k:=k+1 and repeat.

Lemma 3.2. *If* Algorithm 3.1 *finishes at a step* $k < \infty$*, then* x_k *is a common fixed point of* $T_i, i = 1, 2, \ldots, N$*, i.e.,* $x_k \in Fix(T_i)$.

Proof. Using stopping rule $x_k = x_{k+1}$, we have $x_k \in C_k$. From the definition of C_k, it follows

$$\|x_k - y_k^{i_k}\| \leq \|x_k - x_k\| = 0.$$

Applying the definition of i_k, we get $y_k^i = x_k$ for $i = 1, 2, \ldots, N$. Taking into account (3.1), we have

$$x_k = \alpha_k x_k + (1-\alpha_k)T_i(x_k), \quad i = 1, 2, \cdots, N.$$

Since $\alpha_k < 1$ we see $x_k = T_i(x_k)$ for $i = 1, 2, \cdots, N$, i.e., $x_k \in Fix(T_i)$.

Theorem 3.3. *Let* $\{x_k\}$ *be the (infinite) sequence generated by* Algorithm 3.1*,* T_i *be nonexpansive for* $i = 1, 2, \ldots, N$*. Then* $x_k \to x^\dagger := P_{Fix(T_i)}(x_0)$ *as* $k \to \infty$.

Proof. For each $k \geq 0$, it is easy to see that Q_k is a halfspace or $Q_k = H$. Further, the relation $\|u - y_k^{i_k}\| \leq \|u - x_k\|$ is equivalent to $\langle u, x_k - y_k^{i_k}\rangle \leq \frac{1}{2}\{\|x_k\|^2 - \|y_k^{i_k}\|^2\}$ or $\langle u - \frac{1}{2}(x_k + y_k^{i_k}), x_k - y_k^{i_k}\rangle \leq 0$. Hence, for all $k \geq 0$, C_k is a halfspace in H or $C_k = H$. An explicit formula for $P_{C_k \bigcap Q_k}(x_0)$ can be obtained similarly as in [15]. Therefore, if $C_k \bigcap Q_k \neq \varnothing$ then x_{k+1} is easily computed by (3.4).

Next we show that $Fix(T_i) \subset C_k \bigcap Q_k$. Firstly we show that $Fix(T_i) \subset C_k$ for all $k \geq 0$. To observe this, arbitrarily take $p \in Fix(T_i)$, we have

$$\|p - y_k^{i_k}\| = \|p - \{\alpha_k y_k^{i-1} + (1-\alpha_k)T_i(y_k^{i-1})\}\|$$

$$\begin{aligned}&\le \alpha_k\|p-y_k^{i-1}\|+(1-\alpha_k)\|p-T_i(y_k^{i-1})\|\\&\le \alpha_k\|p-y_k^{i-1}\|+(1-\alpha_k)\|p-y_k^{i-1}\|\\&=\|p-y_k^{i-1}\|\\&\le\cdots\le\|p-y_k^1\|\le\|p-x_k\|.\end{aligned}$$

Therefore, $p\in C_k$ and, hence, $Fix(T_i)\subset C_k$ for all $k\ge 0$.

Next we show $Fix(T_i)\subset Q_k$ for all $k\ge 0$, by induction. For $k=0$, we have $Fix(T_i)\subset C=Q_0$. Assume $Fix(T_i)\subset Q_k$. Since x_{k+1} is the projection of x_0 onto $C_k\bigcap Q_k$, by Lemma 2.3, we have $\langle x_0-x_k,x_k-u\rangle\ge 0$, for $u\in C_k\bigcap Q_k$. As $Fix(T_i)\subset C_k\cap Q_k$, by the induction assumption, the last inequality holds, in particular, for all $u\in Fix(T_i)$. This together with the definition of Q_{k+1} implies that $Fix(T_i)\subset Q_{k+1}$. Therefore we have $Fix(T_i)\subset Q_k$ for all $k\ge 0$. Hence $Fix(T_i)\subset C_k\bigcap Q_k$.

Since $Fix(T_i)$ is a nonempty closed convex subset of C, there exists a unique element $x^\dagger\in Fix(T_i)$ such that $x^\dagger=P_{Fix(T_i)}x_0$. From $x_k=P_{Q_k}x_0$ (by the definition of Q_k) and $Fix(T_i)\subset Q_k$, we have $\|x_k-x_0\|\le\|p-x_0\|$ for all $p\in Fix(T_i)$. Due to $x^\dagger\in Fix(T_i)$, then we get

$$\|x_k-x_0\|\le\|x^\dagger-x_0\|, \tag{3.5}$$

which implies that $\{x_k\}$ is bounded.

The fact that $x_{k+1}\in Q_k$ implies that $\langle x_{k+1}-x_k,x_k-x_0\rangle\ge 0$. This together with Lemma 2.1 imply

$$\|x_{k+1}-x_k\|^2\le\|x_{k+1}-x_0\|^2-\|x_k-x_0\|^2. \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$\|x_k-x_{k+1}\|\to 0\quad\text{as }\ k\to\infty. \tag{3.7}$$

Using the definition of C_k and the inclusion $x_{k+1}\in C_k$, we also have

$$\|x_{k+1}-y_k^{i_k}\|\le\|x_{k+1}-x_k\|,$$

which with (3.7) yields

$$\begin{aligned}\|x_k-y_k^{i_k}\|&\le\|x_{k+1}-x_k\|+\|x_{k+1}-y_k^{i_k}\|\\&\le 2\|x_{k+1}-x_k\|\to 0\quad\text{as }\ k\to\infty.\end{aligned}$$

From the definition of i_k in (3.2), it follows that

$$\|x_k-y_k^i\|\to 0\quad\text{as }\ k\to\infty\quad\text{for }\ i=1,2,\cdots,N, \tag{3.8}$$

which implies

$$\|y_k^{i+1}-y_k^i\|\le\|y_k^{i+1}-x_k\|+\|y_k^i-x_k\|\to 0,\qquad\text{for }\ i=1,2,\cdots,N-1.$$

From (3.1) it follows

$$\|x_k-T_1(x_k)\|=\frac{1}{1-\alpha_k}\|y_k^1-x_k\|\to 0, \tag{3.9}$$

and

$$\|y_k^i-T_{i+1}(y_k^i)\|=\frac{1}{1-\alpha_k}\|y_k^i-y_k^{i+1}\|\to 0,\quad i=1,2,\ldots,N-1. \tag{3.10}$$

Using (3.8), (3.10) and nonexpansivity of $\{T_i\}_{i=2}^N$, we get

$$\begin{aligned}\|x_k-T_{i+1}(x_k)\|&\le\|x_k-y_k^i\|+\|y_k^i-T_{i+1}(y_k^i)\|+\|T_{i+1}(x_k)-T_{i+1}(y_k^i)\|\\&\le 2\|x_k-y_k^i\|+\|y_k^i-T_{i+1}(y_k^i)\|\to 0,\quad i=1,2,\ldots,N-1.\end{aligned} \tag{3.11}$$

Equations (3.9), (3.11) and Lemma 2.2 imply that $\omega_w(x_k)\subset Fix(T_i)$, $i=1,2,\ldots,N$, i.e., $\omega_w(x_k)\subset F$. This, together with (3.5) and Lemma 2.4, guarantee strong convergence of x_k to $P_{Fix(T_i)}x_0$. □

4. A numerical example

In this section, we perform Algorithm 3.1 and Algorithm AC for finding a common fixed point of two nonexpansive mappings and compare them through a numerical example.

We take $T_1 : R^2 \to R^2$ by $T_1 : v = (v_1, v_2)^\top \mapsto (\sin \frac{v_1+v_2}{\sqrt{2}}, \cos \frac{v_1+v_2}{\sqrt{2}})^\top$ (see [9]) and $T_2 : R^2 \to R^2$ as $T_2 := P_C$ with $C = \{x \in R^2 | \|x - c\| \leq r\}$ where $c \in [-1, 1]^2$ generated randomly, and $r = 3$. The terminal condition is $\frac{\|x-T(x)\|+\|x-S(x)\|}{\|x\|} \leq \epsilon$. In the numerical results listed in the following table, 'Iter.' and 'Sec.' denote the number of iterations and the cpu time in seconds, respectively.

For randomly chosen initial values, we compare Algorithm 3.1 and Algorithm AC with different terminal condition many times, and results in the Table 1 were the average values. From Table 1 we observe that Algorithm 3.1 is better than Algorithm AC in the sense of the average.

Table 1: Numerical results for a given tolerance

	Algorithm 3.1		**Algorithm AC**	
x_0	Iter.	Sec.	Iter.	Sec.
$\epsilon = 0.01$	86	0.0234	114	0.0468
$\epsilon = 0.001$	265	0.05772	322	0.07176
$\epsilon = 0.0001$	559	0.11388	663	0.12168

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References

[1] P. K. Anh, C. V. Chung, *Parallel hybrid methods for a finite family of relatively nonexpansive mappings*, Numer. Func. Anal. Optim., **35** (2014), 649–664. 1

[2] H. H. Bauschke, P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces,* Springer, Berlin, (2011). 2.1, 2.3

[3] L. C. Ceng, N. C. Wong, J. C. Yao, *Strong and weak convergence theorems for an infinite family of nonexpansive mappings and applications*, Fixed Point Theory Appl., **2012** (2012), 21 pages. 1

[4] Y. Censor, T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms, **8** (1994), 221–239. 1

[5] Y. Censor, T. Elfving, N. Kopf, T. Bortfeld, *The multiple-sets split feasibility problem and its applications for inverse problems*, Inverse Problems, **21** (2005), 2071–2084. 1

[6] J. P. Chancelier, *Iterative schemes for computing fixed points of nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl., **353** (2009), 141–153. 1

[7] Q. L. Dong, Y. Y. Lu, *A new hybrid algorithm for a nonexpansive mapping*, Fixed Point Theory Appl., **2015** (2015), 7 pages. 1

[8] Q. L. Dong, H. B. Yuan, *Accelerated Mann and CQ algorithms for finding a fixed point of a nonexpansive mapping*, Fixed Point Theory Appl., **2015** (2015), 12 pages. 1

[9] S. He, C. Yang, P. Duan, *Realization of the hybrid method for Mann iterations*, Appl. Math. Comput., **217** (2010), 4239–4247. 4

[10] K. Goebel, W. A. Kirk, *Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, (1990). 2.2

[11] C. Martinez-Yanes, H. K. Xu, *Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal., **64** (2006), 2400–2411. 2.4

[12] K. Nakajo, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl., **279** (2003), 372–379. 1

[13] K. Nammanee, R. Wangkeeree, *New iterative approximation methods for a countable family of nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., **2011** (2011), 24 pages. 1

[14] W. Nilsrakoo, S. Saejung, *Weak and strong convergence theorems for countable Lipschitzian mappings and its applications*, Nonlinear Anal., **69** (2008), 2695–2708. 1

[15] M. V. Solodov, B. F. Svaiter, *Forcing strong convergence of proximal point iterations in Hilbert space*, Math. Program., **87** (2000), 189–202. 3

[16] W. Takahashi, *Viscosity approximation methods for countable families of nonexpansive mappings in Banach spaces*, Nonlinear Anal., **70** (2009), 719–734. 1

[17] W. Takahashi, Y. Takeuchi, R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **341** (2008), 276–286. 1

[18] L. Wei, Y. J. Cho, H. Y. Zhou, *A strong convergence theorem for common fixed points of two relatively nonexpansive mappings and its applications*, J. Appl. Math. Comput., **29** (2009), 95–103. 1

[19] Y. Yao, Y. C. Liou, N. C. Wong, *Iterative algorithms based on the implicit midpoint rule for nonexpansive mappings*, J. Nonlinear Convex A., (preprint).

[20] Y. Yao, M. Postolache, Y. C. Liou, Z. Yao, *Construction algorithms for a class of monotone variational inequalities*, Optim. Lett., (in press).

[21] H. Zhou, Y. Su, *Strong convergence theorems for a family of quasi-asymptotic pseudo-contractions in Hilbert spaces*, Nonlinear Anal., **70** (2009), 4047–4052. 1