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Shadowing orbits of stochastic differential equations

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Abstract

This paper is devoted to the existence of a true solution near a numerical approximate solution of stochastic differential equations. We prove a general shadowing theorem for finite time of stochastic differential equations under some suitable conditions and provide an estimate of shadowing distance by computable quantities. The practical use of this theorem is demonstrated in the numerical simulations of chaotic orbits of the stochastic Lorenz system.

Keywords: Stochastic differential equations, random dynamical system, shadowing, multiplicative ergodic theorem, stochastic Lorenz system.
2010 MSC: 65C20, 65P20, 37H10, 37C50.

1. Introduction

Nowadays shadowing property has an important position in theory and application of random dynamical systems (RDS), especially in the numerical simulations of chaotic systems of stochastic differential equations (SDEs). Due to the sensitivity of the initial value and random noise pumped into the systems constantly, it is difficult to expect that a particular solution of chaotic systems of SDE can be well approximated by a numerical solution for any given length of time. Numerical computations play a significant role in the investigations of the dynamical behavior of SDEs whose applications describe many natural phenomena in meteorology, biology and so on, [1, 11, 14]. In fact, many nice discoveries are derived from numerical experiments. The reliability and feasibility of numerical computations are paid more and more attentions. Therefore, we are mainly concerned that whether a numerical approximative solution implies the dynamics of chaotic systems of SDE.

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There are two main motivations for this work. It follows from the classical shadowing lemma that many studies about the dynamics of deterministic chaotic systems have been performed by B. A. Coomes and K. J. Palmer et al., see [11] and references therein. There is few studies, however, in the random case. The shadowing lemma of random hyperbolic set of RDS φ generated by random diffeomorphisms is proved in [4]. Hong, Li and Wang had completed many nice works on the numerical analysis of RDS [6, 9, 13]. These numerical techniques are applied to problems that are hyperbolic, i.e., for problems where there is a splitting into exponential stable and unstable components. To the best of our knowledge, no investigations of the shadowing theorem for finite time of SDE exist in the literatures. Shadowing is still an interesting method for studying their dynamic behavior of SDE.

As we know, it is very hard to verify the hyperbolicity assumption in specific systems. We overcome this shortcoming by the following method. We only need to construct some conditions such that chaotic systems of SDE possess pseudo hyperbolicity. That is, it only needs to check whether an operator along a sequence of points on chaotic systems is invertible under these conditions. This is the essence of the shadowing which has been investigated from such practical point of view. And this brings great convenience to numerical analysis, so it can be an available method of estimating shadowing distance, i.e. the maximum distance between an (ω, δ)-pseudo orbit and its corresponding nearest true orbit in mean square sense. Therefore, the main difference between the existed work and my study is that there is no hyperbolicity assumption of original systems.

Utilizing generalized Brouwer's fixed point Theorem and the existence of the modified Newton equation's solution, we propose the shadowing theorem for finite time of SDE. The result shows that under some appropriate conditions the numerical approximative orbits of SDE are close to the true orbits of the original systems and shadowing distance can be estimated.

The rest of this paper is organized as follows. Section 2 deals with some preliminaries addressed to clarify the presentation of concepts and norms used later. Section 3 is devoted to the theoretical results of the finite time shadowing. Section 4 presents the details of the numerical implementations. Illustrative numerical experiments for the main theorem are included in Section 5. Section 6 is addressed to summarize the conclusions of the paper.

2. Preliminaries

We consider a class of Stratonovich SDEs of the form

$$dx_t = f(x_t)dt + \sigma x_t \circ dW_t, \qquad x(0) = \xi_0(\omega) \in R^d, \tag{2.1}$$

where $W(t), t \in R^+ = [0, +\infty)$ is a standard one-dimensional Brownian motion defined on a canonical Wiener space $(\Omega, \mathcal{F}, P)$, with $\{\mathcal{F}_t, t \in R^+\}$ being its natural normal filtration, $\Omega = \{\omega \in C(R^+, R) : \omega(0) = 0\}$ which means that the elements of Ω can be identified with paths of a Wiener process $\omega(t) = W_t(\omega)$, the random variable $\xi_0(\omega)$ is independent of $\mathcal{F}_0$ and satisfies the inequality $E|\xi_0(\omega)|^2 < \infty$ and σ is nonzero real number.

2.1. Basic assumptions and notations

It follows from Theorem 2 in [12], i.e., Doss-Sussmann Theorem, that SDE (2.1) can be changed to a random differential equation (RDE) by the Doss-Sussmann transformation as follows.

We define

$$\theta : R^+ \times \Omega \to \Omega, \theta^t\omega(s) = \omega(t + s) - \omega(t)$$

and $0 \le s \le t, s \in R^+, t \in R^+$. Let $O_t(\omega)$ be a one-dimension random stable Ornstein-Uhlenbeck process which satisfies the following linear SDE

$$dO_t = -O_t dt + dW_t.$$

And let

$$z(t, \omega) := \exp(-\sigma O_t(\omega))x_t(\omega) \in R^d,$$

then SDE (2.1) can be changed to a RDE in the form of

$$\frac{dz}{dt} = \exp(-\sigma O_t(\omega))f(\exp(\sigma O_t(\omega))z) + \sigma O_t z = f_1(\theta^t\omega, z). \tag{2.2}$$

It follows from Doss-Sussmann Theorem that the solution of RDE (2.2) is the solution of SDE (2.1).

In this paper, we make the following assumptions:

• $f_1 : \Omega \times R^d \to R^d$ is a measurable function which is locally bounded, locally Lipschitz continuous with respect to the first variable and is a C^1 vector field on R^d.

It follows from Theorem 2.2.2 in [1] that RDE (2.2) generates a unique RDS $\varphi : R^+ \times R^+ \times \Omega \times R^d \to R^d$, which is usually written as $\varphi(s, t, \omega)z := \varphi(s, t, \omega, z) \in R^d$ on the metric dynamical systems $(\Omega, \mathcal{F}, P, \theta^t)$ and is C^1 with respect to z. The RDS φ is given by

$$\varphi(s, t, \omega)z = z + \int_s^t f_1(\theta^\tau\omega, \varphi(s, \tau, \omega)z)d\tau \in R^d. \tag{2.3}$$

We also make use of the following notations.

• Let $L^2(\Omega, P)$ be the space of all square-integrable random variables $x : \Omega \to R^d$.

• For any random vector $x = (x_1, x_2, ..., x_d) \in L^2(\Omega, P)$, we define the norm of x in the form of

$$\|x\|_2 = \Big[\int_\Omega [|x_1(\omega)|^2 + |x_2(\omega)|^2 +, ..., +|x_d(\omega)|^2]dP\Big]^{\frac{1}{2}} < \infty.$$

• For a stochastic process $x(t, \omega)$ with $x_t(\omega) \in L^2(\Omega, P)$ and $t \in R^+$, the norm of $x(t, \omega)$ is defined as follows:

$$\|x(t, \omega)\|_2 = \sup_{t \in R^+} \|x_t(\omega)\|_2 < \infty.$$

• We define the norm of random matrix in the form of

$$\|A\|_{L^2(\Omega,P)} = \Big[E(|A|^2)\Big]^{\frac{1}{2}},$$

where A is a random matrix and $|\cdot|$ is the operator norm.

• For simplicity in notations, the norm $\|\cdot\|_2$ and $\|\cdot\|_{L^2(\Omega,P)}$ are usually written as $\|\cdot\|$ unless otherwise stated in sequels.

2.2. Some concepts and lemma

Definition 2.1. For a given positive number δ and P-almost surely $\omega \in \Omega$, if there is a sequence of times $\{t_k\}_{k=0}^N, 0 \le t_0 \le t_1 \le, ..., \le t_N$ and a sequence of random variables $\{(u_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$, which means that $u_k(\theta^{t_k}\omega)$ is $\mathcal{F}_{t_k}$-measurable for $k = 0, 1, 2, ..., N$ and $f_1(u_k(\theta^{t_k}\omega))u_k(\theta^{t_k}\omega) \neq 0$ almost surely, such that the following inequalities hold

$$\|u_{k+1}(\theta^{t_{k+1}}\omega) - \varphi(t_k, t_{k+1}, \theta^{t_k}\omega)u_k(\theta^{t_k}\omega)\| \le \delta, \tag{2.4}$$

then the random variables $\{(u_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ is said to be a (ω, δ)-pseudo orbit of SDE (2.1) in the sense of mean-square, where $\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)u_k(\theta^{t_k}\omega)$ denotes the orbit of RDS φ at the time t_{k+1} which starts from the initial time t_k with the initial value $u_k(\theta^{t_k}\omega)$ and the sample $\theta^{t_k}\omega$.

Definition 2.2. For a given positive number ε, P-almost surely $\omega \in \Omega$ and a (ω, δ)-pseudo orbit $\{(u_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ of SDE (2.1) with associated times $\{t_k\}_{k=0}^N$, if there is a sequence of times $\{h_k\}_{k=0}^N, 0 \le h_0 = t_0 \le h_1 \le, ..., \le h_N$, such that the following inequalities hold

$$\|u_k(\theta^{t_k}\omega) - x_k(\theta^{h_k}\omega)\| \le \varepsilon$$

and

$$0 \le t_k - h_k \le \varepsilon,$$

where the random variables $\{(x_k(\theta^{h_k}\omega), \mathcal{F}_{h_k})\}_{k=0}^N$ are on a true orbits of SDE (2.1), that is

$$x_{k+1}(\theta^{h_{k+1}}\omega) = \varphi(h_k, h_{k+1}, \theta^{h_k}\omega)x_k(\theta^{h_k}\omega), \tag{2.5}$$

then the (ω, δ)-pseudo orbit $\{(u_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ is said to be (ω, ε)-shadowed by a true orbit of SDE (2.1) containing points $\{(x_k(\theta^{h_k}\omega), \mathcal{F}_{h_k})\}_{k=0}^N$ in the sense of mean-square, where the true orbit of RDS φ is a stochastic process.

Since the σ-algebra $\mathcal{F}_{t_k}(t_k \ge 0)$ is nondecreasing and $t_k \ge h_k (k = 0, 1, 2, ..., N)$, the random variables $x_k(\theta^{h_k}\omega)(k = 0, 1, 2, ..., N)$ which are on the true orbit must be $\mathcal{F}_{t_k}$-measurable [1].

Definition 2.3. The RDS $\varphi : R^+ \times R^+ \times \Omega \times R^d \to R^d$ is said to be pseudo hyperbolic in mean square if the constants $\kappa_1, \kappa_2 \ge 1$, $\nu_1, \nu_2 \ge 0$ exist, such that the following inequalities hold with $R^d = E^s(\omega) \oplus E^u(\omega)$,

$$E\|\varphi(s, t_1, \omega)x\|^2 \le \kappa_1 e^{-\nu_1(t_1 - t_2)} E\|\varphi(s, t_2, \omega)x\|^2, \forall t_1 \ge t_2 \ge s \ge 0, x \in E^s(\omega), \quad \text{or}$$
$$E\|\varphi(s, t_2, \omega)x\|^2 \le \kappa_2 e^{-\nu_2(t_1 - t_2)} E\|\varphi(s, t_1, \omega)x\|^2, \forall t_1 \ge t_2 \ge s \ge 0, x \in E^u(\omega).$$

This means that there is a splitting into exponentially stable and unstable components. The famous multiplicative ergodic theorem provides the stochastic analogue of the deterministic spectral theory of matrices and a method to check the pseudo hyperbolicity.

Lemma 2.4 ([3])**.** *(Multiplicative ergodic theorem) Let $\phi = \phi(0, t, \omega)x$ be a linear RDS in R^d for $t \in R^+$ on the probability spaces $(\Omega, \mathcal{F}, P)$ and the metric dynamical systems $(\Omega, \mathcal{F}, P, \theta^t)$. Assume that the following integrability conditions are satisfied:*

$$\sup_t \ln^+ \|\phi(0, t, \omega)x\| \in L^1(\Omega), \sup_t \ln^+ \|\phi(-t, 0, \omega)x\| \in L^1(\Omega),$$

where $\ln^+(z) \equiv \max\{\ln(z), 0\}$, denoting the non-negative part of the natural logarithm and $L^1(\Omega) = \{x : E|x| < \infty\}$.

Then there is a θ-invariant set $\tilde{\Omega}$ of full P measure and fixed nonrandom numbers (the Lyapunov exponents of ϕ)

$$\lambda_1 > \lambda_2 >, ..., > \lambda_p$$

with corresponding multiplicities $d_1, d_2, ..., d_p$, where $\sum_{i=1}^p d_i = d$, such that for all $\omega \in \tilde{\Omega}$,

(1) $R^d = E_1(\omega) \oplus ... \oplus E_p(\omega)$, where the $E_i(\omega)$ are measurable random linear subspaces of R^d of dimension d_i which are invariant under ϕ, i.e.,

$$\phi(0, t, \omega)E_i(\omega) = E_i(\theta^t\omega)$$

for $i = 1, 2, ..., p$.

(2) The $E_i(\omega)$ are characterized dynamically by

$$x \in E_i(\omega)\backslash\{0\} \Leftrightarrow \lim_{t\to+\infty} \frac{1}{t} \ln \|\phi(0, t, \omega)x\| = \lambda_i.$$

(3) The Lyapunov exponents of x

$$\lambda(\omega, x) := \lim_{t\to+\infty} \frac{1}{t} \ln \|\phi(0, t, \omega)x\| = \lambda_i$$

exists for each $x \ne 0$ and is a random variable which takes only the values $\lambda_1, ..., \lambda_p$.

This lemma assures the existence of the Lyapunov exponents and provides the foundation to the construction of a local theory of nonlinear RDS including pseudo hyperbolicity in mean square. When all Lyapunov exponents are non-zero, the linear RDS $\phi(0, t, \omega)x$ is pseudo hyperbolic in mean square.

3. Theoretical results of finite time shadowing

3.1. Theoretical foundations

Let $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$ be a (ω, δ)-pseudo orbit of SDE (2.1) obtained by RDE (2.2) and $y_k(\theta^{h_k}\omega) \in L^2(\Omega, P)(k = 0, 1, ..., N)$. Suppose we have a sequence of $d \times d$ random matrices $\{(Y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N-1}$ such that

$$\|Y_k(\theta^{t_k}\omega) - D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega)\| \leq \delta, \quad \forall \quad k = 0, 1, ..., N-1.$$

For $k = 0, 1, ..., N$, we choose $d \times (d-1)$ random matrices $(S_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})$ such that its columns form an approximate orthogonal basis for the subspace orthogonal to $T(x_k)$, where $T(x_k) = f_1(\theta^{t_k}\omega, x_k)$, the approximate orthogonal means that the following inequality holds

$$\|S_k(\theta^{t_k}\omega)S_k^*(\theta^{t_k}\omega) - I\| \leq \delta_1,$$

for some positive number $\delta_1 \in (0, \delta)$, where $*$ denotes the transpose of matrix.

Now we choose $(d-1) \times (d-1)$ random matrices $A_k(\theta^{t_k}\omega)$ satisfying

$$\|A_k(\theta^{t_k}\omega) - S_{k+1}^*(\theta^{t_{k+1}}\omega)Y_k(\theta^{t_k}\omega)S_k(\theta^{t_k}\omega)\| \leq \delta.$$

Next, we define a linear operator L in the following way. If the value of random variables $\xi = \{\xi_k(\theta^{t_k}\omega)\}_{k=0}^{N}$ is in $(R^{d-1})^{N+1}$, then we let $L\xi = \{[L\xi]_k\}_{k=0}^{N-1}$ to be

$$[L\xi]_k = \xi_{k+1}(\theta^{t_{k+1}}\omega) - A_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega), \quad \forall \quad k = 0, 1, ..., N-1.$$

It follows from Subsection 4.2 that the operator L has right inverses and we choose one such right inverse L^{-1}.

At last, we define various constants. Let U be a convex subset of R^d containing the value of the (ω, δ)-pseudo orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$. Therefore, we define

$$\Delta h_{min} = \inf_{0 \leq k \leq N-1} \Delta h_{k+1}.$$

Next, we choose a positive number $0 < \varepsilon_0 \leq \Delta h_{min}$ such that $\|x - y_k(\theta^{t_k}\omega)\| \leq \varepsilon_0$, then the solution $\varphi(s, t, \omega)x(0 \leq s \leq t)$ is defined and remains in U for $0 < t \leq h_k + \varepsilon_0$ P-almost surely.

Finally, we define

$$M_0 = \sup_{x \in U} \|f_1(\theta^t\omega, x(t))\|, M_1 = \sup_{x \in U} \|Df_1(\theta^t\omega, x(t))\|, M_2 = \sup_{x \in U} \|D^2 f_1(\theta^t\omega, x(t))\|$$

and

$$\Theta = \sup_{0 \leq k \leq N-1} \| Y_k(\theta^{t_k}\omega) \|,$$

where

$$Df_1 = \Big[\frac{\partial f_1(\theta^t\omega, x(t))}{\partial x_i}\Big].$$

We first prove the following lemma which will be applied to the main theorem [7].

Lemma 3.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be convex sets in finite-dimensional random vector spaces and B be an open subset of $\mathcal{X}$. Let v_0 be a given element of B and $\bar{\varepsilon}$ be a given positive number. Assume that $G : B \to \mathcal{Y}$ be a C^2 function satisfying the following properties:*

(i) the derivative $DG(v_0)$ at $v_0 \in B$ has a right inverse $\mathcal{K}$;

(ii) the closed ball about v_0 with radius $\bar{\varepsilon}$ is contained in B, where $\bar{\varepsilon} = 2\|\mathcal{K}\|\|G(v_0)\|$;

(iii) the inequality $2M\|\mathcal{K}\|^2\|G(v_0)\| \leq 1$ *holds, where*

$$M = \sup\Big\{\|D^2G(v)\| : v \in B, \|v - v_0\| \leq \bar{\varepsilon}\Big\}.$$

Then there is a solution $\bar{v}$ *of the equation* $G(\bar{v}) = 0$ *satisfying* $\| \bar{v} - v_0 \| \leq \bar{\varepsilon}$.

Proof. We apply generalized Brouwer's fixed point Theorem to this case. Let the operator $F : B \to \mathcal{X}$ be defined in the form

$$F(v) = v_0 - \mathcal{K}[G(v) - DG(v)(v - v_0)].$$

We conclude that if $F(v) = v$, then the equality $G(v) = 0$ holds. In fact, if $\|v - v_0\| \leq \bar{\varepsilon}$, we have

$$\begin{aligned}\|F(v) - v_0\| &= \|\mathcal{K}[G(v) - G(v_0) - DG(v)(v - v_0) + G(v_0)]\| \\ &\leq \|\mathcal{K}\|\|[G(v) - G(v_0) - DG(v)(v - v_0) + G(v_0)]\| \\ &\leq \|\mathcal{K}\|\|\frac{1}{2}D^2G(v)(v - v_0)^2\| + \|\mathcal{K}\|\|G(v_0)\| \leq \frac{1}{2}M\|\mathcal{K}\|\bar{\varepsilon}^2 + \frac{1}{2}\bar{\varepsilon}.\end{aligned}$$

It follows from the hypothesis (ii) that

$$\|F(v) - v_0\| \leq M\|\mathcal{K}\|^2\|G(v_0)\|\bar{\varepsilon} + \frac{1}{2}\bar{\varepsilon} \leq \frac{1}{2}\bar{\varepsilon} + \frac{1}{2}\bar{\varepsilon} = \bar{\varepsilon},$$

where the last inequality follows from the hypothesis (iii).

Therefore, the conclusion of Lemma 3.1 follows from generalized Brouwer's fixed point Theorem. This completes the proof. □

3.2. Main results

Now we are in the position of the statement and proof of the main theorem in this paper.

Theorem 3.2. *Let* $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ *be a bounded* (ω, δ)*-pseudo orbit of SDE* (2.1) *obtained by RDE* (2.2) *and let*

$$C = \max\{M_0^{-1}(1 + \Theta\|L^{-1}\|), \|L^{-1}\|\}. \tag{3.1}$$

If the parameters δ, ε_0 *and these quantities shown in Subsection* 3.1 *satisfy the following inequalities*

(i) $C_1 = C\delta < \frac{1}{3}$;

(ii) $C_2 = 3C\delta < \min(\varepsilon_0, \Delta h_{min})$;

(iii) $C_3 = \frac{9}{2}C^2\delta(M_0M_1 + 2M_1\exp(M_1\Delta h) + M_2\Delta h \cdot \exp(2M_1\Delta h)) \leq 1$.

Then there exists a sequence of times $\{h_k\}_{k=0}^N (0 \leq h_0 \leq h_1 \leq, ..., \leq h_N)$ such that the (ω, δ)-pseudo orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ is (ω, ε)-shadowed by a true orbit of SDE (2.1) containing points $\{(x_k(\theta^{h_k}\omega), \mathcal{F}_{h_k})\}_{k=0}^N$ in mean-square. Moreover, shadowing distance satisfies $\varepsilon \leq 3C\delta$.

Proof. Given a (ω, δ)-pseudo orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ of SDE (2.1) obtained by RDE (2.2), we wish to show that $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ is shadowed by a true orbit containing $\{(x_k(\theta^{h_k}\omega), \mathcal{F}_{h_k})\}_{k=0}^N$, where $x_k(\theta^{h_k}\omega)$ lies in the random hyperplane $\mathcal{H}_k(\theta^{t_k}\omega)$ through $y_k(\theta^{t_k}\omega)$.

And we assume the random hyperplane $\mathcal{H}_k(\theta^{t_k}\omega)$ is normal to $T(y_k) = f_1(\theta^{t_k}\omega, y_k)$ at the point $y_k(\theta^{t_k}\omega)$. In fact, we will find a sequence of times $\{h_k\}_{k=0}^N = \{t_k\}_{k=0}^N, 0 \leq h_0 \leq h_1 \leq, ..., \leq h_N$ and a sequence of points $\{(x_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^N$ with $x_k(\theta^{t_k}\omega) \in \mathcal{H}_k(\theta^{t_k}\omega)$ being contained in the ε-neighborhood of $y_k(\theta^{t_k}\omega)$ such that

$$x_{k+1}(\theta^{t_{k+1}}\omega) = \varphi(t_k, t_{k+1}, \theta^{t_k}\omega)x_k(\theta^{t_k}\omega).$$

The random hyperplane $\mathcal{H}_k(\theta^{t_k}\omega)$ can be viewed as a subspace of the tangent space at $y_k(\theta^{t_k}\omega)$. It follows from the assumption that $S_k(\theta^{t_k}\omega)$ is a $d\times(d-1)$ random matrix whose columns form an approximate orthogonal basis for $\mathcal{H}_k(\theta^{t_k}\omega)$. Thus we may identify $\mathcal{H}_k(\theta^{t_k}\omega)$ via the map $\mathbf{z}\longmapsto y_k(\theta^{t_k}\omega)+S_k(\theta^{t_k}\omega)\mathbf{z}$.

The problem of finding appropriate sequences of t_k and x_k becomes that of finding a sequence of times $\{t_k\}_{k=0}^{N-1}$ and a sequence of points $\{(\mathbf{z}_k(\theta^{t_k}\omega),\mathcal{F}_{t_k})\}_{k=0}^{N}$ such that

$$y_{k+1}(\theta^{t_{k+1}}\omega)+S_{k+1}(\theta^{t_{k+1}}\omega)\mathbf{z}_{k+1}(\theta^{t_{k+1}}\omega)=\varphi(t_k,t_{k+1},\theta^{t_k}\omega)(y_k(\theta^{t_k}\omega)+S_k(\theta^{t_k}\omega)\mathbf{z}_k(\theta^{t_k}\omega)).$$

Next, we introduce the set $\mathcal{X}=(R^+)^N\times(R^{d-1})^{N+1}$ with norm

$$\|(\{s_k\}_{k=0}^{N-1},\{\zeta_k\}_{k=0}^{N})\|=\max\Big\{\sup_{0\le k\le N-1}|s_k|,\ \sup_{0\le k\le N}\|\zeta_k\|\Big\}$$

and the space $\mathcal{Y}=(R^d)^N$ with norm

$$\|\{\mathbf{g}_k\}_{k=0}^{N-1}\|=\max_{0\le k\le N-1}\|\mathbf{g}_k\|,$$

where $s_k\in R^+$, $\zeta_k\in R^{d-1}$ and $\mathbf{g}_k\in R^d$.

Now we let B be a properly chosen ε-open neighborhood of $v_0=(\{h_k\}_{k=0}^{N-1},0)$ in $\mathcal{X}$ which contain the point $v=(\{s_k\}_{k=0}^{N-1},\{\zeta_k\}_{k=0}^{N})$ and we introduce the function $G:B\to\mathcal{Y}$ given by

$$[G(v)]_k=y_{k+1}(\theta^{s_{k+1}}\omega)+S_{k+1}(\theta^{s_{k+1}}\omega)\zeta_{k+1}(\theta^{s_{k+1}}\omega)-\varphi(s_k,s_{k+1},\theta^{s_k}\omega)(y_k(\theta^{s_k}\omega)+S_k(\theta^{s_k}\omega)\zeta_k(\theta^{s_k}\omega)).\quad(3.2)$$

We find that Theorem 3.2 will be proved if we find a solution $\bar{v}=(\{t_k\}_{k=0}^{N-1},\{\mathbf{z}_k(\theta^{t_k}\omega)\}_{k=0}^{N})$ of the equation

$$G(\bar{v})=0,\quad a.s.$$

in the closed ball of radius ε about $v_0=(\{h_k\}_{k=0}^{N-1},0)$.

Therefore, we now only need to verify that the map G as (3.2) does indeed satisfy the hypotheses $(i)-(iii)$ of Lemma 3.1.

Verification of hypothesis (i) of Lemma 3.1:

First note that $\|G(v_0)\|\le\delta$. Secondly note that the Gateaux derivative of G at v_0 is given for $u=(\{\tau_k\}_{k=0}^{N-1},\{\xi_k(\theta^{t_k}\omega)\}_{k=0}^{N})\in\mathcal{X}$ by

$$\begin{aligned}[DG(v_0)u]_k&=\lim_{\varepsilon\to0}\frac{[G(v_0+\varepsilon u)-G(v_0)]_k}{\varepsilon}\\&=-\tau_kT(y_{k+1})+S_{k+1}(\theta^{t_{k+1}}\omega)\cdot\xi_{k+1}(\theta^{t_{k+1}}\omega)\\&\quad-D\varphi(h_k,h_{k+1},\theta^{h_k}\omega)y_k(\theta^{t_k}\omega)\cdot S_k(\theta^{t_k}\omega)\cdot\xi_k(\theta^{t_k}\omega).\end{aligned}\quad(3.3)$$

Let $\mathcal{T}_ku$ be the approximation of $[DG(v_0)u]_k$ and $\mathcal{T}$ be the approximation of $DG(v_0)$ [5], we have

$$\mathcal{T}_ku=-\tau_kT(y_{k+1})+S_{k+1}(\theta^{t_{k+1}}\omega)\cdot\xi_{k+1}(\theta^{t_{k+1}}\omega)-Y_k(\theta^{t_k}\omega)\cdot S_k(\theta^{t_k}\omega)\cdot\xi_k(\theta^{t_k}\omega).\quad(3.4)$$

Now we need to prove that $\mathcal{T}_k$ is invertible. Therefore, we must show that for all $\mathbf{g}=\{\mathbf{g}_k\}_{k=0}^{N-1}\in\mathcal{Y}$, there is a solution of the following equation

$$\mathcal{T}_ku=\mathbf{g}_k,$$

that is,

$$-\tau_kT(y_{k+1})+S_{k+1}(\theta^{t_{k+1}}\omega)\xi_{k+1}(\theta^{t_{k+1}}\omega)-Y_k(\theta^{t_k}\omega)S_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega)=\mathbf{g}_k(\theta^{t_k}\omega).\quad(3.5)$$

As we know, the matrix

$$\Big[\frac{T(y_k)}{\|T(y_k)\|}\Big|S_k(\theta^{t_k}\omega)\Big]$$

is orthogonal for each k. Then this set of equations is equivalent to the following two sets of equations, one set obtained by premultiplying the kth member in (3.5) by $T^*(y_{k+1})$, the other set obtained by premultiplying the kth member in (3.5) by $S^*_{k+1}(\theta^{t_{k+1}}\omega)$. Therefore, we obtain

$$-\tau_k\|T(y_{k+1})\|^2 - T(y_{k+1})^* Y_k(\theta^{t_k}\omega)S_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega) = T(y_{k+1})^*\mathbf{g}_k(\theta^{t_k}\omega), \tag{3.6}$$

$$\xi_{k+1}(\theta^{t_{k+1}}\omega) - A_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega) = S^*_k(\theta^{t_{k+1}}\omega)\mathbf{g}_k(\theta^{t_k}\omega). \tag{3.7}$$

If we write $\bar{\mathbf{g}} = \{S^*_{k+1}(\theta^{t_{k+1}}\omega)\mathbf{g}_k(\theta^{t_k}\omega)\}_{k=0}^{N-1}$, it follows from the condition (3.1) that the solution of Eq.(3.7) is

$$\xi_k = (L^{-1}\bar{\mathbf{g}})_k. \tag{3.8}$$

If (3.8) is substituted into Eq.(3.6), we obtain

$$\tau_k = -\frac{T(y_{k+1})^*}{\|T(y_{k+1})\|^2}\cdot\Big[Y_k(\theta^{t_k}\omega)S_k(\theta^{t_k}\omega)L^{-1}S_{k+1}(\theta^{t_{k+1}}\omega)+1\Big]\mathbf{g}_k(\theta^{t_k}\omega). \tag{3.9}$$

Taking into account (3.8) and (3.9), we define the right inverse of $\mathcal{T}_k$ in the form of

$$\mathcal{T}_k^{-1}\mathbf{g} = \Big[\{\tau_k\}_{k=0}^{N-1}, \{\xi_k(\theta^{t_k}\omega)\}_{k=0}^{N}\Big].$$

It follows from (3.1) that $\mathcal{T}$ is invertible and the following inequality holds

$$\|\mathcal{T}^{-1}\| \le C. \tag{3.10}$$

Therefore, we can construct the invertibility of $DG(v_0)$. By the operator theory, we obtain

$$\mathcal{K} = \Big[I + \mathcal{T}^{-1}(DG(v_0)-\mathcal{T})\Big]^{-1}\mathcal{T}^{-1}. \tag{3.11}$$

It follows from (3.3), (3.4) and the assumption (i) of Theorem 3.2 that

$$\begin{aligned}\mathcal{T}^{-1}(DG(v_0)-\mathcal{T}) &\le \|\mathcal{T}^{-1}\|\|DG(v_0)-\mathcal{T}\|\\ &\le \|\mathcal{T}^{-1}\|\cdot\Big[\sup\|(D\varphi(t_k,t_{k+1},\theta^{t_k}\omega)y_k(\theta^{t_k}\omega) - Y_k(\theta^{t_k}\omega)S_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega)\|\Big]\\ &\le C\delta < \frac{1}{3}.\end{aligned}$$

Then the inverse $[I + T^{-1}(DG(v_0) - T)]^{-1}$ exits and $\mathcal{K}$ is a right inverse of $DG(v_0)$. Furthermore,

$$\|[I + \mathcal{T}^{-1}(DG(v_0)-\mathcal{T})]^{-1}\| \le \frac{3}{2}. \tag{3.12}$$

Therefore, this satisfies the assumption (i) of Lemma 3.1.

Verification of hypothesis (ii) of Lemma 3.1:

Taking into account (3.10), (3.11) and (3.12), we obtain

$$\|\mathcal{K}\| \le \frac{3}{2}C.$$

and

$$\|G(v_0)\| = \sup_k \|y_{k+1}(\theta^{t_{k+1}}\omega) - \varphi(t_k,t_{k+1},\theta^{t_k}\omega)y_k(\theta^{t_k}\omega)\| \le \delta.$$

It follows from the assumption (ii) of Theorem 3.2 that

$$\varepsilon = 2\|\mathcal{K}\|\|G(v_0)\| \le 3C\delta < \varepsilon_0.$$

Therefore, this satisfies the assumption (ii) of Lemma 3.1.

Verification of hypothesis (iii) of Lemma 3.1:

We only need to estimate $\|D^2G(v)\|$. Then we choose $\bar{u} = (\{r_k\}_{k=0}^{N-1}, \{\eta_k\}_{k=0}^{N})$ and calculate the second order Gateaux differential of $G(v)$ as follows

$$\begin{aligned}[DG(v)u\bar{u}]_k :=& \lim_{t\to 0}\frac{[DG(v+t\bar{u})u - DG(v)u]_k}{|t|}\\ =& -\tau_k r_k DT[y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega)]\cdot T[y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega)]\\ & -\tau_k DT[y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega)]\cdot\\ & D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)(y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega))\cdot S_k(\theta^{t_k}\omega)\eta_k(\theta^{t_k}\omega)\\ & - r_k DT[y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega)]\cdot\\ & D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)(y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega))\cdot S_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega)\\ & - D^2\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)(y_k(\theta^{t_k}\omega) + S_k(\theta^{t_k}\omega)\zeta_k(\theta^{t_k}\omega))\\ & \cdot[S_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega)]\cdot[S_k(\theta^{t_k}\omega)\eta_k(\theta^{t_k}\omega)].\end{aligned}$$

By the norm property, i.e., sub-additivity, we obtain

$$M = \sup_k \|D^2G(v)\| \le M_0M_1 + 2M_1\exp(M_1\Delta h) + M_2\Delta h\exp(2M_1\Delta h).$$

It follows from the assumption (iii) of Theorem 3.2 and

$$\|G(v_0)\| \le \delta, \|\mathcal{K}\|^2 \le \frac{9}{4}C^2,$$

that

$$2M\|\mathcal{K}\|^2\|G(v_0)\| \le 1.$$

Then this satisfies the assumption (iii) of Lemma 3.1. Therefore, the conclusion follows from Lemma 3.1. The proof is completed. □

4. Numerical implementation methods

In the computation we approximate the local error δ using the local error control mechanism of the numerical scheme. We only pay attention to the magnification of the local error, C, that gives shadowing distance.

4.1. Basic methods

Step 1. Utilizing the one-step numerical scheme (eg. Taylor-like scheme [10]) to simultaneously solve the following equations from t_k to t_{k+1} with the initial values $z(0) = y_k(\theta^{t_k}\omega)$ and $v(0) = I$,

$$\begin{cases} dz = f_1(\theta^t\omega, z)dt \\ dv_t = Df_1(\theta^t\omega, z)v_t dt, \end{cases}$$

then we obtain the approximations of $z_{k+1}(\theta^{t_{k+1}}\omega)$ and $D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega)$ respectively,

$$z_{k+1}(\theta^{t_{k+1}}\omega) \approx \varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega),$$

$$D\varphi(t_k, t_{k+1}, \theta^{t_k}\omega)y_k(\theta^{t_k}\omega) \approx v_{k+1}(\theta^{t_{k+1}}\omega).$$

Step 2. Using the methods shown in Subsection 3.1 and 4.2, we can find C such that (3.1) holds.

Step 3. If all inequalities in Section 3 hold and the time h_k can be constructed by $h_k = t_k - \epsilon'$ for $k = 0, 1, ..., N$, where $0 < \epsilon' < \epsilon$, then the shadowing distance is $\epsilon = 3C\delta$.

4.2. Choice of the operator L^{-1}

We are going to verify that the linear operator L along the obtained (ω, δ)-pseudo orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$ is invertible for P-almost surely $\omega \in \Omega$.

Let $g = \{g_k(\theta^{t_k}\omega)\}_{k=0}^{N-1}$ be in $\mathcal{Y}$. To find $\xi = L^{-1}g$, we have to solve the random difference equation

$$\xi_{k+1}(\theta^{t_{k+1}}\omega) = A_k(\theta^{t_k}\omega)\xi_k(\theta^{t_k}\omega) + g_k(\theta^{t_k}\omega).$$

Now as chosen in Section 3, the random matrix $A_k(\theta^{t_k}\omega)$ is upper triangular with positive diagonal entries. Therefore, we expect there to be an integer l such that for most k the first l diagonal entries of $A_k(\theta^{t_k}\omega)$ exceed 1 and the rest are less than 1 in mean square for P-almost surely $\omega \in \Omega$. We can partition the random matrix $A_k(\theta^{t_k}\omega)$ in the form

$$A_k(\theta^{t_k}\omega) = \begin{bmatrix} P_k(\theta^{t_k}\omega) & Q_k(\theta^{t_k}\omega) \\ 0 & R_k(\theta^{t_k}\omega) \end{bmatrix}, k = 0, 1, ..., N-1,$$

where $P_k(\theta^{t_k}\omega)$ is $l \times l$ random matrix, $Q_k(\theta^{t_k}\omega)$ is $l \times (d-l-1)$ random matrix and $R_k(\theta^{t_k}\omega)$ is $(d-l-1) \times (d-l-1)$ random matrix.

It follows from Lemma 2.4 that the Lyapunov exponents of $A_k(\theta^{t_k}\omega)$ are non-zero. Then it suggests that the RDS φ generated by SDE (2.1) along the obtained (ω, δ)-pseudo orbit $\{(y_k(\theta^{t_k}\omega), \mathcal{F}_{t_k})\}_{k=0}^{N}$ is pseudo hyperbolicity in mean square for P-almost surely $\omega \in \Omega$. It can be written as

$$\begin{cases} \xi_{k+1}^{(1)} = P_k(\theta^{t_k}\omega)\xi_k^{(1)} + Q_k(\theta^{t_k}\omega)\xi_k^{(2)} + g_k^{(1)} \\ \xi_{k+1}^{(2)} = R_k(\theta^{t_k}\omega)\xi_k^{(2)} + g_k^{(2)}, \end{cases} \quad k = 0, 1, ..., N-1.$$

In the second equation above, we set $\xi_2^{(2)} = 0$ and solve forwards, then we substitute the resulting solution $\xi_k^{(2)}$ into the first equation above, set $\xi_N^{(2)} = 0$ and solve it backwards, obtaining the solutions $\xi_k^{(1)}$. Therefore, we obtain the right inverse L^{-1} by

$$[L^{-1}g]_k = [\xi_k^{(1)}, \xi_k^{(2)}]^T, k = 0, 1, ..., N.$$

Therefore, the operator L is invertible. And this verify the important assumption of the invertibility of the operator L.

5. Numerical experiments

5.1. Experimental preparation

We consider the Stratonovich stochastic Lorenz systems (SLS)

$$\begin{cases} \dot{x} = \sigma(-x+y) + \lambda x \circ dW_t \\ \dot{y} = -xz + \rho x - y + \lambda y \circ dW_t \\ \dot{z} = xy - \beta z + \lambda z \circ dW_t. \end{cases}$$

Therefore, its Itô SLS is the form of

$$\begin{cases} dx = (\sigma(-x+y) + \frac{\lambda^2}{2}x)dt + \lambda x dW_t \\ dy = (-xz + \rho y + (\frac{\lambda^2}{2} - 1)y)dt + \lambda y dW_t \\ dz = (xy - \beta z + \frac{\lambda^2}{2}z)dt + \lambda z dW_t. \end{cases}$$

Make the following transformation

$$\begin{cases} \bar{x}(t,\omega) = \exp(-\lambda O_t(\omega))x \\ \bar{y}(t,\omega) = \exp(-\lambda O_t(\omega))y \\ \bar{z}(t,\omega) = \exp(-\lambda O_t(\omega))z, \end{cases}$$

where $O_t(\omega)$ is a one-dimension stable Ornstein-Uhlenbeck stochastic process and statisfies

$$dO_t = -O_t dt + dW_t.$$

It follows from the transformation that Itô SLS can be transformed to the RDE in the form of

$$\begin{cases} \frac{d\bar{x}}{dt} = \sigma(-\bar{x}+\bar{y}) + \lambda O_t(\omega)\bar{x} \\ \frac{d\bar{y}}{dt} = -\bar{x}\bar{z} + \rho\bar{x} - \bar{y} + \lambda O_t(\omega)\bar{y} \\ \frac{d\bar{z}}{dt} = \bar{x}\bar{y} - \beta\bar{z} + \lambda O_t(\omega)\bar{z}. \end{cases} \tag{5.1}$$

It follows from Theorem 4.4 and Lemma 6.3 in [8] that although Eq. (5.1) does not satisfy a linear growth condition, the existence and uniqueness of its solution are proved and the solution operator of Eq. (5.1) can generate a RDS.

In this experiment we take the initial value $(0, 1, 0)$, time step size $7e-3$ and iterative step $4.5e+5$. The pseudo orbits of Eq. (5.1) in Figs. 1 and 2 are generated by the Taylor-like scheme[10, 13].

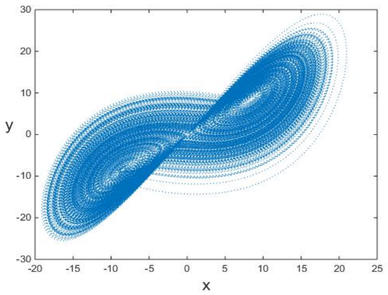

Figure 1: Pseudo orbit of SLS projected on the (x, y) plane

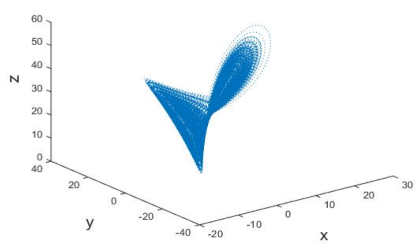

Figure 2: Pseudo orbit of SLS in the (x, y, z) space

It follows from [2] that the forward invariant random compact set $\mathcal{U}$ of RDS φ generated by Eqs. (5.1) is the closed ball with center zero and radius $\mathcal{R}(\omega)$, where

$$\mathcal{R}(\omega) = c_2 \int_{-T}^{0} \exp(c_1 s - 2\sigma W_s(\omega))ds,$$

$$c_1 = \min(1, \beta, \sigma), c_2 > 0, 2\langle Bu, u\rangle < -c_1|u|^2 + c_2, T \in (0, t_N]$$

and

$$B = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}.$$

Then it suggests that the RDS φ generated by Eqs. (5.1) is pseudo hyperbolic in mean square for P-almost surely $\omega \in \Omega$ on the finite interval and lies in the forward invariant random compact set $\mathcal{U}$. It is shown as Figs. 3 and 4.

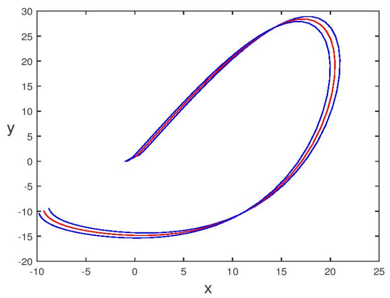

Figure 3: The approximative structure of pseudo hyperbolicity of an orbit of length 100 on SLS projected on the (x, y) plane

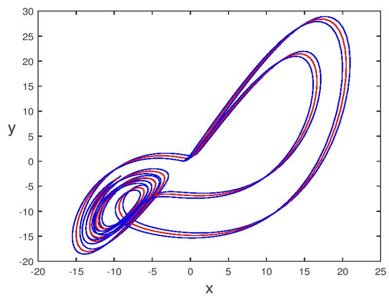

Figure 4: The approximative structure of pseudo hyperbolicity of an orbit of length 700 on SLS projected on the (x, y) plane

Therefore, this verify that the RDS φ along the finite computational points possesses pseudo hyperbolic in mean square for P-almost surely $\omega \in \Omega$.

5.2. Numerical results

It follows from the methods shown in Section 3 and 4, we can determine the parameters of Theorem 3.2. Tables 1 and 2 present the numerical results and show the existence of shadowing orbits.

Table 1: Value of the parameters.

parameters	value	parameters	value
Δh_k	0.03	M_1	≤ 0.1855
(x_0, y_0, z_0)	$(0.0, 1.0, 0.0)$	M_2	0.0014
N	10^6	Θ	$\leq 1.9369e + 03$
ε_0	0.2	δ	$\leq 3.1128e - 03$
M_0	≤ 5.4477	$\parallel L^{-1} \parallel$	$\leq 3.0712e - 03$

Table 2: Comparison of the inequalities.

inequalities	value
C	≤ 1.0681
C_1	$\leq 4.3213e - 13$
C_2	≤ 0.01
C_3	≤ 0.0221
shadowing distance ε	0.01
shadowing time t	$3 * 10^4$

In conclusion, there is explicit dependent relationship between the shadowing distance and the pseudo orbit error and there exists the true orbit in the appropriate neighborhood of the pseudo orbit of SLS. Furthermore, the higher the order of the scheme is, the shorter the shadowing distance will be. The symbolic drawing of such relation between pseudo orbits and true orbits of Eqs. (5.1) is depicted in Fig. 5, that is, a (ω, δ)-pseudo orbit is shown as the red line, there exists a true orbit in the domain between two blue lines.

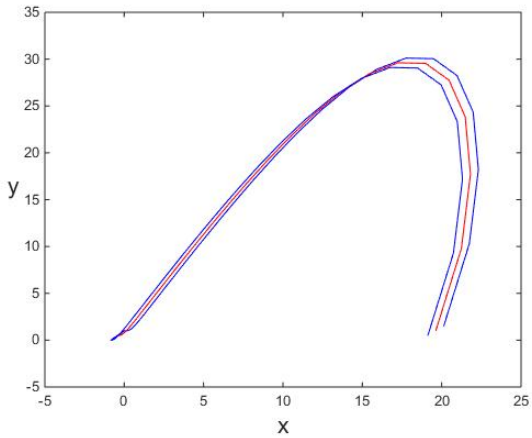

Figure 5: The symbolic drawing of the relation between true orbit and pseudo orbit

6. Conclusion

The main result presented here is the shadowing theorem for finite time of SDE. To conduct the study we have extended the well-known deterministic shadowing lemma to the random scenario by taking advantage of mean square and stochastic calculus. We show that the existence of the shadowing orbits of the SLS so that the numerical experiments are performed and match the results of theoretical analysis. Although some progresses are made, other kinds of shadowing such as random periodic shadowing, random quasi-periodic shadowing and so on are needed in reality, which will be shown in my further work.

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