Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2019–2030
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Stabilization of a nonlinear control system on the Lie group $SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3$

Camelia Petrişor

Department of Mathematics, Politehnica University of Timişoara, Piaţa Victoriei, Nr. 2, 300006-Timişoara, România.

Communicated by C. Zaharia

Abstract

The stabilization of some equilibrium points of a dynamical system via linear controls is studied. Numerical integration using Lie-Trotter integrator and its properties are also presented.

Keywords: Optimal control problem, Hamilton-Poisson system, nonlinear stability, numerical integration.

2010 MSC: 34H05, 53D17.

1. Introduction

Stability problem is one of the most important issues when a dynamical system is studied. For a Hamilton-Poisson system, like the considered system (1.1), the energy-methods are used in order to establish stability results (see [2] or [4] for instance). New challenges appear when the energy-methods are inconclusive. In this cases, a specific control can be found in order to stabilize a given equilibrium point.

The method was successfully applied in a lot of examples: for Maxwell-Bloch equations (see [6]), for the rigid body (see [1]), for the Chua's system (see [5]), for the Toda lattice (see [7]), and so on.

The goal of this paper is to find appropriate control functions that stabilize some equilibrium points of a dynamical system arisen from a specific case of a drift-free left invariant control system on the Lie group $SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3$.

Email address: camelia.petrisor@upt.ro (Camelia Petrişor)

Received 2015-09-18

Following [3], the system can be written in the form below:

$$\begin{cases} \dot{x}_1 = -x_5x_6 \\ \dot{x}_2 = x_7x_9 \\ \dot{x}_3 = x_4x_5 - x_7x_8 \\ \dot{x}_4 = -x_2x_6 + x_3x_5 \\ \dot{x}_5 = x_1x_6 - x_3x_4 \\ \dot{x}_6 = -x_1x_5 + x_2x_4 \\ \dot{x}_7 = -x_2x_9 + x_3x_8 \\ \dot{x}_8 = x_1x_9 - x_3x_7 \\ \dot{x}_9 = -x_1x_8 + x_2x_7. \end{cases} \tag{1.1}$$

It is easy to see that

$$\begin{aligned} e_1^{MNPQ} &= (0,0,0,M,0,N,0,P,Q), \; M,N,P,Q \in \mathbb{R}, \\ e_2^{MNP} &= (0,0,M,0,0,N,0,0,P), \; M,N,P \in \mathbb{R}, \\ e_3^{MPQ} &= (0,0,0,0,M,0,0,P,Q), \; M,P,Q \in \mathbb{R}, \\ e_4^{MNP} &= (0,M,0,0,N,0,0,P,0), \; M,N,P \in \mathbb{R}, \\ e_5^{MNP} &= (M,N,P,0,0,0,0,0,0), \; M,N,P \in \mathbb{R}, \\ e_6^{MNP} &= (M,0,0,N,0,0,P,0,0), \; M,N,P \in \mathbb{R}, \\ e_7^{MNP} &= (0,0,0,M,0,N,P,0,0), \; M,N,P \in \mathbb{R}, \\ e_8^{MNP} &= (M,0,N,P,0,\frac{NP}{M},0,0,0), \; M,N,P \in \mathbb{R}, \\ e_9^{MNP} &= (0,M,N,0,0,0,0,P,\frac{NP}{M}), \; M,N,P \in \mathbb{R}, \\ e_{10}^{MNP} &= (0,0,0,\frac{NP}{M},M,0,N,P,0), \; M,N,P \in \mathbb{R}, \\ e_{11}^{MNP} &= (M,N,0,\frac{NP}{M},P,0,-\frac{NP}{M},-P,0), \; M,N,P \in \mathbb{R}, \\ e_{12}^{MNP} &= (M,N,0,\frac{NP}{M},P,0,-\frac{NP}{M},P,0), \; M,N,P \in \mathbb{R} \end{aligned}$$

are the equilibrium points of our dynamics (1.1).

The results regarding nonlinear stability of e_1^{MNPQ}, e_3^{MNP} and e_5^{MNP} have been proved in [3]. The goal of our paper is to stabilize some other equilibrium points via linear controls.

The paper is organized as follows: in the first part, the linear control that stabilizes the equilibrium states e_2^{MNP} of the system (1.1) is found and the spectral and nonlinear stability of this points are established. Numerical integration of the controlled system is analyzed via Lie-Trotter algorithm and some of its properties are sketched. The subject of the second part is the stabilization of the equilibrium states e_4^{MNP} of the system (1.1) followed by the numerical integration of the controlled system via Lie-Trotter algorithm.

2. Stabilization of e_2^{MNP} by one linear control

Let us employ the control $u \in C^\infty(\mathbb{R}^9, \mathbb{R})$,

$$\begin{aligned} u(x_1, x_2, &x_3, x_4, x_5, x_6, x_7, x_8, x_9) \\ &= (-Mx_2, Mx_1, 0, -Mx_5, Mx_4, 0, -Mx_8, Mx_7, 0), \end{aligned} \tag{2.1}$$

for the system (1.1). The controlled system (1.1) – (2.1), explicitly given by

$$\begin{cases} \dot{x}_1 = -x_5x_6 - Mx_2 \\ \dot{x}_2 = x_7x_9 + Mx_1 \\ \dot{x}_3 = x_4x_5 - x_7x_8 \\ \dot{x}_4 = -x_2x_6 + x_3x_5 - Mx_5 \\ \dot{x}_5 = x_1x_6 - x_3x_4 + Mx_4 \\ \dot{x}_6 = -x_1x_5 + x_2x_4 \\ \dot{x}_7 = -x_2x_9 + x_3x_8 - Mx_8 \\ \dot{x}_8 = x_1x_9 - x_3x_7 + Mx_7 \\ \dot{x}_9 = -x_1x_8 + x_2x_7, \end{cases} \tag{2.2}$$

has e_2^{MNP} as an equilibrium state.

Proposition 2.1. *The controlled system* (2.2) *has the Hamilton-Poisson realization*

$$(\mathbb{R}^9, \Pi, H),$$

where

$$\Pi = \begin{bmatrix} 0 & -x_3 & x_2 & 0 & -x_6 & x_5 & 0 & -x_9 & x_8 \\ x_3 & 0 & -x_1 & x_6 & 0 & -x_4 & x_9 & 0 & -x_7 \\ -x_2 & x_1 & 0 & -x_5 & x_4 & 0 & -x_8 & x_7 & 0 \\ 0 & -x_6 & x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_6 & 0 & -x_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_5 & x_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -x_9 & x_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_9 & 0 & -x_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ -x_8 & x_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.3}$$

is the Poisson tensor of the system (1.1), *and the Hamiltonian function is*

$$H(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_7^2) - Mx_3.$$

Proof. Indeed, one obtains immediately that

$$\Pi \cdot \nabla H = [\dot{x}_1\ \dot{x}_2\ \dot{x}_3\ \dot{x}_4\ \dot{x}_5\ \dot{x}_6\ \dot{x}_7\ \dot{x}_8\ \dot{x}_9]^t,$$

and Π is a minus Lie-Poisson structure, see for details [3]. □

Remark 2.2 ([3]). The functions $C_1, C_2, C_3 : \mathbb{R}^9 \to \mathbb{R}$ given by

$$C_1(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = \frac{1}{2}(x_4^2 + x_5^2 + x_6^2),$$

$$C_2(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = \frac{1}{2}(x_7^2 + x_8^2 + x_9^2)$$

and

$$C_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = x_4x_7 + x_5x_8 + x_6x_9$$

are Casimirs of our Poisson configuration.

The goal of this paragraph is to study the spectral and nonlinear stability of the equilibrium state e_2^{MNP} of the controlled system (2.2).

Proposition 2.3. *The controlled system* (2.2) *may be spectral stabilized about the equilibrium states* e_2^{MNP} *for all* $M, N, P \in \mathbb{R}^*$.

Proof. Let A be the matrix of linear part of our controlled system (2.2), that is

$$A = \begin{bmatrix} 0 & -M & 0 & 0 & -x_6 & -x_5 & 0 & 0 & 0 \\ M & 0 & 0 & 0 & 0 & 0 & x_9 & 0 & x_7 \\ 0 & 0 & 0 & x_5 & x_4 & 0 & -x_8 & -x_7 & 0 \\ 0 & -x_6 & x_5 & 0 & x_3 - M & -x_2 & 0 & 0 & 0 \\ x_6 & 0 & -x_4 & -x_3 + M & 0 & x_1 & 0 & 0 & 0 \\ -x_5 & x_4 & 0 & x_2 & -x_1 & 0 & 0 & 0 & 0 \\ 0 & -x_9 & x_8 & 0 & 0 & 0 & 0 & x_3 - M & -x_2 \\ x_9 & 0 & -x_7 & 0 & 0 & 0 & -x_3 + M & 0 & x_1 \\ -x_8 & x_7 & 0 & 0 & 0 & 0 & x_2 & -x_1 & 0 \end{bmatrix}.$$

At the equilibrium of interest its characteristic polynomial has the following expression

$$p_{A(e_2^{MNP})}(\lambda) = 4\lambda^5[\lambda^4 + (M^2 + N^2 + P^2)\lambda^2 + N^2P^2].$$

Hence we have five zero eigenvalues and four purely imaginary eigenvalues. So we can conclude that the equilibrium states e_2^{MNP}, $M, N, P \in \mathbb{R}^*$ are spectral stable. □

Moreover we can prove:

Proposition 2.4. *The controlled system* (2.2) *may be nonlinear stabilized about the equilibrium states* e_2^{MNP} *for all* $M, N, P \in \mathbb{R}^*$.

Proof. For the proof we shall use Arnold's technique. Let us consider the following function

$$\begin{aligned} F_{\lambda,\mu,\nu} &= C_2 + \lambda H + \mu C_1 + \nu C_3 \\ &= \frac{1}{2}(x_7^2 + x_8^2 + x_9^2) + \frac{\lambda}{2}(x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_7^2 - 2Mx_3) \\ &\quad + \frac{\mu}{2}(x_4^2 + x_5^2 + x_6^2) + \nu(x_4x_7 + x_5x_8 + x_6x_9). \end{aligned}$$

The following conditions hold:

(*i*) $\nabla F_{\lambda,\mu,\nu}(e_2^{MNP}) = 0$ iff $\mu = \dfrac{P^2}{N^2}$, $\nu = -\dfrac{P}{N}$;

(*ii*) Considering now

$$W = \ker[dH(e_2^{MNP})] \cap \ker[dC_1(e_2^{MNP})] \cap \ker[dC_3(e_2^{MNP})$$

$$= \text{Span}\left\{ \begin{bmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\1\\0 \end{bmatrix} \right\},$$

then, for all $v \in W$, i.e. $v = (a, b, c, d, e, 0, f, g, 0)$, $a, b, c, d, e, f, g \in \mathbb{R}$ we have

$$v \cdot \nabla^2 F_{\lambda, \frac{P^2}{N^2}, -\frac{P}{N}}(e_2^{MNP}) \cdot v^t = \lambda a^2 + \lambda b^2 + \lambda c^2 + \frac{P^2}{N^2} d^2 + \left(\lambda + \frac{P^2}{N^2}\right) e^2 + (\lambda + 1) f^2 + g^2 - 2\frac{P}{N} fd - 2\frac{P}{N} eg$$

positive definite under the restriction $\lambda > 0$, and so

$$\nabla^2 F_{\lambda, \frac{P^2}{N^2}, -\frac{P}{N}}(e_2^{MNP})|_{W \times W}$$

is positive definite.

Therefore, via Arnold's technique, the equilibrium states e_2^{MNP}, $M, N, P \in \mathbb{R}^*$ are nonlinear stable, as required.

□

We shall discuss now the numerical integrator of the dynamics (2.2) via the Lie-Trotter integrator, see for details [8]. For the beginning, let us observe that the Hamiltonian vector field X_H splits as follows

$$X_H = X_{H_1} + X_{H_2} + X_{H_3} + X_{H_4} + X_{H_5} + X_{H_6},$$

where

$$H_1 = \frac{x_1^2}{2}, \quad H_2 = \frac{x_2^2}{2}, \quad H_3 = \frac{x_3^2}{2}, \quad H_4 = \frac{x_5^2}{2}, \quad H_5 = \frac{x_7^2}{2}, \quad H_6 = -Mx_3.$$

Their corresponding integral curves are, respectively, given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \\ x_7(t) \\ x_8(t) \\ x_9(t) \end{bmatrix} = A_i(t) \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \\ x_5(0) \\ x_6(0) \\ x_7(0) \\ x_8(0) \\ x_9(0) \end{bmatrix} \quad i = \overline{1, 6},$$

where

$$A_1(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos at & \sin at & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin at & \cos at & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos at & \sin at & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin at & \cos at & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos at & \sin at \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sin at & \cos at \end{bmatrix} \tag{2.4}$$

$a = x_1(0)$,

$$A_2(t) = \begin{bmatrix} \cos bt & 0 & -\sin bt & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin bt & 0 & \cos bt & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos bt & 0 & -\sin bt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin bt & 0 & \cos bt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos bt & 0 & -\sin bt \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin bt & 0 & \cos bt \end{bmatrix} \tag{2.5}$$

$b = x_2(0)$,

$$A_3(t) = \begin{bmatrix} \cos ct & \sin ct & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin ct & \cos ct & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos ct & \sin ct & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin ct & \cos ct & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos ct & \sin ct & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sin ct & \cos ct & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.6}$$

$c = x_3(0)$,

$$A_4(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -dt & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & dt & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.7}$$

$d = x_5(0)$,

$$A_5(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & et \\ 0 & 0 & 1 & 0 & 0 & 0 & -et & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.8}$$

$e = x_7(0)$,

$$A_6(t) = \begin{bmatrix} \cos Mt & -\sin Mt & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin Mt & \cos Mt & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos Mt & -\sin Mt & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin Mt & \cos Mt & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos Mt & -\sin Mt & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin Mt & \cos Mt & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$M \in \mathbb{R}^*$.

Then the Lie-Trotter integrator is given by

$$\begin{bmatrix} x_1^{n+1} \\ x_2^{n+1} \\ x_3^{n+1} \\ x_4^{n+1} \\ x_5^{n+1} \\ x_6^{n+1} \\ x_7^{n+1} \\ x_8^{n+1} \\ x_9^{n+1} \end{bmatrix} = A_1(t)A_2(t)A_3(t)A_4(t)A_5(t)A_6(t) \begin{bmatrix} x_1^{n} \\ x_2^{n} \\ x_3^{n} \\ x_4^{n} \\ x_5^{n} \\ x_6^{n} \\ x_7^{n} \\ x_8^{n} \\ x_9^{n} \end{bmatrix} \tag{2.9}$$

that is

$$\begin{aligned} x_1^{n+1} =&(\cos bt \cos ct \cos Mt + \cos bt \sin ct \sin Mt)x_1^n \\ &+ (\cos bt \sin ct \cos Mt - \cos bt \cos ct \sin Mt)x_2^n \\ &- \sin bt x_3^n - dt \sin bt \cos Mt x_4^n + dt \sin bt \sin Mt x_5^n - dt \cos bt \cos ct x_6^n \\ &+ et \sin bt \cos Mt x_7^n - et \sin bt \sin Mt x_8^n + et \cos bt \sin ct x_9^n, \end{aligned}$$

$$\begin{aligned} x_2^{n+1} = &[(\sin at \sin bt \cos ct - \cos at \sin ct) \cos Mt \\ &+(\cos at \cos ct + \sin at \sin bt \sin ct) \sin Mt]x_1^n \\ &+[(\cos at \cos ct + \sin at \sin bt \sin ct) \cos Mt \\ &-(\sin at \sin bt \cos ct - \cos at \sin ct) \sin Mt]x_2^n \\ &+ \sin at \cos bt x_3^n + dt \sin at \cos bt \cos Mt x_4^n - dt \sin at \cos bt \sin Mt x_5^n \\ &+dt(- \sin at \sin bt \cos ct + \sin at \cos ct)x_6^n \\ &-et \sin at \cos bt \cos Mt x_7^n + et \sin at \cos bt \sin Mt x_8^n \\ &+(\cos at \cos ct + \sin at \sin bt \sin ct)x_9^n, \end{aligned}$$

$$\begin{aligned} x_3^{n+1} =&[(\cos at \sin bt \cos ct + \sin at \sin ct) \cos Mt \\ &- (\sin at \cos ct - \cos at \sin bt \sin ct) \sin Mt]x_1^n \\ &+ [(- \sin at \cos ct + \cos at \sin bt \sin ct) \cos Mt \\ &- (\cos at \sin bt \cos ct + \sin at \sin ct) \sin Mt]x_2^n \\ &+ \cos at \cos bt x_3^n + dt \cos at \cos bt \cos Mt x_4^n - dt \cos at \cos bt \sin Mt x_5^n \\ &- d(\cos at \sin bt \cos ct + \sin at \sin ct)x_6^n \\ &- et \cos at \cos bt \cos Mt x_7^n + et \cos at \cos bt \sin Mt x_8^n \\ &- (\sin at \cos ct + \cos at \sin bt \sin ct)x_9^n, \end{aligned}$$

$$\begin{aligned} x_4^{n+1} =&(\cos bt \cos ct \cos Mt + \cos bt \sin ct \sin Mt)x_4^n \\ &+ (\cos bt \sin ct \cos Mt - \cos bt \cos ct \sin Mt)x_5^n - \sin bt x_6^n, \end{aligned}$$

$$\begin{aligned} x_5^{n+1} =&[(\sin at \sin bt \cos ct - \cos at \sin ct) \cos Mt \\ &+ (\cos at \cos ct + \sin at \sin bt \sin ct) \sin Mt]x_4^n \end{aligned}$$

$$+ [(\cos at \cos ct + \sin at \sin bt \sin ct) \cos Mt$$
$$- (\sin at \sin bt \cos ct - \cos at \sin ct) \sin Mt]x_5^n + \sin at \cos bt x_6^n,$$

$$\begin{aligned} x_6^{n+1} =& [(\cos at \sin bt \cos ct + \sin at \sin ct) \cos Mt \\ &- (\sin at \cos ct - \cos at \sin bt \sin ct) \sin Mt]x_4^n \\ &+ [(-\sin at \cos ct + \cos at \sin bt \sin ct) \cos Mt \\ &- (\cos at \sin bt \cos ct + \sin at \sin ct) \sin Mt]x_5^n + \cos at \cos bt x_6^n, \end{aligned}$$

$$\begin{aligned} x_7^{n+1} =& (\cos bt \cos ct \cos Mt + \cos bt \sin ct \sin Mt)x_7^n \\ &+ (\cos bt \sin ct \cos Mt - \cos bt \cos ct \sin Mt)x_8^n - \sin bt x_9^n, \end{aligned}$$

$$\begin{aligned} x_8^{n+1} =& [(\sin at \sin bt \cos ct - \cos at \sin ct) \cos Mt \\ &+ (\cos at \cos ct + \sin at \sin bt \sin ct) \sin Mt]x_7^n \\ &+ [(\cos at \cos ct + \sin at \sin bt \sin ct) \cos Mt \\ &- (\sin at \sin bt \cos ct - \cos at \sin ct) \sin Mt]x_8^n + \sin at \cos bt x_8^n, \end{aligned}$$

$$\begin{aligned} x_9^{n+1} =& [(\cos at \sin bt \cos ct + \sin at \sin ct) \cos Mt \\ &- (\sin at \cos ct - \cos at \sin bt \sin ct) \sin Mt]x_7^n \\ &+ [(-\sin at \cos ct + \cos at \sin bt \sin ct) \cos Mt \\ &- (\cos at \sin bt \cos ct + \sin at \sin ct) \sin Mt]x_8^n + \cos at \cos bt x_9^n. \end{aligned}$$

Now, a direct computation or using MATHEMATICA 8.0 leads us to

Proposition 2.5. *Lie-Trotter integrator* (2.9) *has the following properties:*
(i) *It preserves the Poisson structure* Π*;*
(ii) *It preserves the Casimirs* C_1*,* C_2 *and* C_3 *of our Poisson configuration* $(\mathbb{R}^9, \Pi)$*;*
(iii) *It does not preserve the Hamiltonian* H *of our system* (2.2)*;*
(iv) *Its restriction to the coadjoint orbit* $(\mathcal{O}_k, \omega_k)$*, where*

$$\mathcal{O}_k = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \in \mathbb{R}^9 \mid x_4^2 + x_5^2 + x_6^2 = const,$$
$$x_7^2 + x_8^2 + x_9^2 = const,\ x_4x_7 + x_5x_8 + x_6x_9 = const\}$$

and ω_k *is the Kirilov-Konstant-Souriau symplectic structure on* $\mathcal{O}_k$ *gives rise to a symplectic integrator.*

Proof. The items (i), (ii) and (iv) hold because Lie-Trotter is a Poisson integrator.
The item (iii) is essentially due to the fact that

$$\{H_i, H_j\} \neq 0, \quad i \neq j\ .$$

□

3. Stabilization of e_4^{MNP} by one linear control

In order to stabilize the equilibrium states e_4^{MNP} of the system (1.1) we employ the linear control $u \in C^\infty(\mathbb{R}^9, \mathbb{R})$ given by

$$u(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = (Mx_3 + 2Nx_6, 0, -Mx_1 - 2Nx_4, Mx_6, 0, -Mx_4, Mx_9, 0, -Mx_7), \quad (3.1)$$

so the controlled system (1.1) − (3.1) can be explicitly written:

$$\begin{cases} \dot{x}_1 = -x_5x_6 + Mx_3 + 2Nx_6 \\ \dot{x}_2 = x_7x_9 \\ \dot{x}_3 = x_4x_5 - x_7x_8 - Mx_1 - 2Nx_4 \\ \dot{x}_4 = -x_2x_6 + x_3x_5 + Mx_6 \\ \dot{x}_5 = x_1x_6 - x_3x_4 \\ \dot{x}_6 = -x_1x_5 + x_2x_4 - Mx_4 \\ \dot{x}_7 = -x_2x_9 + x_3x_8 + Mx_9 \\ \dot{x}_8 = x_1x_9 - x_3x_7 \\ \dot{x}_9 = -x_1x_8 + x_2x_7 - Mx_7. \end{cases} \tag{3.2}$$

Using the same arguments like in Proposition 2.1 we obtain the following result:

Proposition 3.1. *The controlled system* (3.2) *has the Hamilton-Poisson realization*

$$(\mathbb{R}^9, \Pi, \bar{H}),$$

where Π *is given by* (2.3) *and the Hamiltonian function is*

$$\bar{H}(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_7^2) - Mx_2 - 2Nx_5.$$

Proposition 3.2. *The controlled system* (3.2) *may be spectral stabilized about the equilibrium states* e_4^{MNP} *for all* $M, N, P \in \mathbb{R}^*$.

Proof. Let $\bar{A}$ be the matrix of linear part of our controlled system (3.2), that is

$$\bar{A} = \begin{bmatrix} 0 & 0 & M & 0 & -x_6 & -x_5 + 2N & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_9 & 0 & x_7 \\ -M & 0 & 0 & x_5 - 2N & x_4 & 0 & -x_8 & -x_7 & 0 \\ 0 & -x_6 & x_5 & 0 & x_3 & -x_2 + M & 0 & 0 & 0 \\ x_6 & 0 & -x_4 & -x_3 & 0 & x_1 & 0 & 0 & 0 \\ -x_5 & x_4 & 0 & x_2 - M & -x_1 & 0 & 0 & 0 & 0 \\ 0 & -x_9 & x_8 & 0 & 0 & 0 & 0 & x_3 & -x_2 + M \\ x_9 & 0 & -x_7 & 0 & 0 & 0 & -x_3 & 0 & x_1 \\ -x_8 & x_7 & 0 & 0 & 0 & 0 & x_2 - M & -x_1 & 0 \end{bmatrix}.$$

At the equilibrium of interest its characteristic polynomial has the following expression,

$$p_{\bar{A}(e_4^{MNP})}(\lambda) = 4\lambda^5[\lambda^4 + (M^2 + 2N^2 + P^2)\lambda^2 + N^2(N^2 + P^2)].$$

Hence we have five zero eigenvalues and four purely imaginary eigenvalues. So we can conclude that the equilibrium states e_4^{MNP}, $M, N, P \in \mathbb{R}^*$ are spectral stable. □

Moreover we can prove,

Proposition 3.3. *The controlled system* (3.2) *may be nonlinear stabilized about the equilibrium states* e_4^{MNP} *for all* $M, N, P \in \mathbb{R}^*$.

Proof. Let us consider the function:

$$\begin{aligned} F_{\lambda,\mu,\nu} &= C_2 + \lambda\bar{H} + \mu C_1 + \nu C_3 \\ &= \frac{1}{2}(x_7^2 + x_8^2 + x_9^2) + \frac{\lambda}{2}(x_1^2 + x_2^2 + x_3^2 + x_5^2 + x_7^2 - 2Mx_2 - 4Nx_5) \\ &\quad + \frac{\mu}{2}(x_4^2 + x_5^2 + x_6^2) + \nu(x_4x_7 + x_5x_8 + x_6x_9). \end{aligned}$$

Then we have successively:

(i) $\nabla F_{\lambda,\mu,\nu}(e_4^{MNP}) = 0$ iff $\mu = \lambda + \dfrac{P^2}{N^2}$, $\nu = -\dfrac{P}{N}$;

(ii) Considering now

$$W = \ker[d\bar{H}(e_4^{MNP})] \cap \ker[dC_1(e_4^{MNP})] \cap \ker[dC_3(e_4^{MNP})] =$$

$$= \text{Span}\left\{ \begin{bmatrix} 1\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1\\0\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1\\0\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\1 \end{bmatrix} \right\},$$

then, for all $v \in W$, i.e. $v = (a, b, c, d, 0, e, f, 0, g)$, $a, b, c, d, e, f, g \in \mathbb{R}$, we have

$$v \cdot \nabla^2 F_{\lambda, \frac{P^2}{N^2}+\lambda, -\frac{P}{N}}(e_4^{MNP}) \cdot v^t = \lambda a^2 + \lambda b^2 + \lambda c^2 + \left(\lambda + \frac{P^2}{N^2}\right) d^2 + \left(\lambda + \frac{P^2}{N^2}\right) e^2 + (\lambda+1) f^2 + g^2 - 2\frac{P}{N} fd - 2\frac{P}{N} eg$$

positive definite under the restriction $\lambda > 0$, and so

$$\nabla^2 F_{\lambda, \frac{P^2}{N^2}+\lambda, -\frac{P}{N}}(e_4^{MNP})|_{W\times W}$$

is positive definite.

Therefore, via Arnold's technique, the equilibrium states e_4^{MNP}, $M, N, P \in \mathbb{R}^*$ are nonlinear stable, as required.

□

We shall discuss now the numerical integrator of the dynamics (3.2) via the Lie-Trotter integrator, see for details [8]. For the beginning, let us observe that the Hamiltonian vector field X_H splits as follows:

$$X_{\bar{H}} = X_{\bar{H}_1} + X_{\bar{H}_2} + X_{\bar{H}_3} + X_{\bar{H}_4} + X_{\bar{H}_5} + X_{\bar{H}_6} + X_{\bar{H}_7},$$

where

$$\bar{H}_1 = \frac{x_1^2}{2}, \quad \bar{H}_2 = \frac{x_2^2}{2}, \quad \bar{H}_3 = \frac{x_3^2}{2}, \quad \bar{H}_4 = \frac{x_5^2}{2},$$
$$\bar{H}_5 = \frac{x_7^2}{2}, \quad \bar{H}_6 = -Mx_2, \quad \bar{H}_7 = -2Nx_5.$$

Their corresponding integral curves are, respectively, given by

$$\begin{bmatrix} x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\\x_5(t)\\x_6(t)\\x_7(t)\\x_8(t)\\x_9(t) \end{bmatrix} = A_i(t) \begin{bmatrix} x_1(0)\\x_2(0)\\x_3(0)\\x_4(0)\\x_5(0)\\x_6(0)\\x_7(0)\\x_8(0)\\x_9(0) \end{bmatrix} \quad i = \overline{1,7},$$

where $A_i(t)$, $i = \overline{1,5}$ are given by the relations (2.4) – (2.8) and

$$A_6(t) = \begin{bmatrix} \cos Mt & 0 & \sin Mt & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sin Mt & 0 & \cos Mt & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos Mt & 0 & \sin Mt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin Mt & 0 & \cos Mt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos Mt & 0 & \sin Mt \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sin Mt & 0 & \cos Mt \end{bmatrix},$$

$M \in \mathbb{R}^*$,

$$A_7(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2Nt & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2Nt & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$N \in \mathbb{R}^*$.

Then, the Lie-Trotter integrator is given by

$$\begin{bmatrix} x_1^{n+1} \\ x_2^{n+1} \\ x_3^{n+1} \\ x_4^{n+1} \\ x_5^{n+1} \\ x_6^{n+1} \\ x_7^{n+1} \\ x_8^{n+1} \\ x_9^{n+1} \end{bmatrix} = A_1(t)A_2(t)A_3(t)A_4(t)A_5(t)A_6(t)A_7(t) \begin{bmatrix} x_1^{n} \\ x_2^{n} \\ x_3^{n} \\ x_4^{n} \\ x_5^{n} \\ x_6^{n} \\ x_7^{n} \\ x_8^{n} \\ x_9^{n} \end{bmatrix}. \tag{3.3}$$

Now, a direct computation or using MATHEMATICA 8.0 leads us to

Proposition 3.4. *Lie-Trotter integrator* (3.3) *has the following properties:*

(i) It preserves the Poisson structure Π*;*

(ii) It preserves the Casimirs C_1*,* C_2 *and* C_3 *of our Poisson configuration* $(\mathbb{R}^9, \Pi)$*;*

(iii) It does not preserve the Hamiltonian $\bar{H}$ *of our system* (3.2)*;*

(iv) Its restriction to the coadjoint orbit $(\mathcal{O}_k, \omega_k)$*, where*

$$\mathcal{O}_k = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \in \mathbb{R}^9 \mid x_4^2 + x_5^2 + x_6^2 = const,$$

$$x_7^2 + x_8^2 + x_9^2 = const,\ x_4x_7 + x_5x_8 + x_6x_9 = const\}$$

and ω_k *is the Kirilov-Konstant-Souriau symplectic structure on* $\mathcal{O}_k$*, gives rise to a symplectic integrator.*

4. Conclusion

The paper presents the stabilization of two equilibrium points of a dynamical system for which the energy-methods fail. In order to do this, for each equilibrium point, a specific linear control is found. Numerical integration using the Lie-Trotter algorithm is analyzed and some properties of the Lie-Trotter integrator are presented.

Acknowledgements

I would like to thank the referees very much for their valuable comments and suggestions.

References

[1] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, G. Sanchez de Alvarez, *Stabilization of Rigid Body Dynamics by Internal and External Torques,* Automatica J., **28** (1992), 745–756. 1
[2] C. Petrişor, *Some New Remarks about the Dynamics of an Automobile with Two Trailers*, J. Appl. Math., **2014** (2014), 6 pages. 1
[3] C. Pop, *A Drift-Free Left Invariant Control System on the Lie Group* $SO(3)\times\mathbb{R}^3\times\mathbb{R}^3$, Math. Probl. Eng., **2015** (2015), 9 pages. 1, 1, 2, 2.2
[4] C. Pop, A. Aron, C. Petrişor, *Geometrical aspects of the ball-plate problem,* Balkan J. Geom. Appl., **16** (2011), 114–121. 1
[5] C. Pop Arieşanu, *Stability Problems for Chua System with One Linear Control*, J. Appl. Math., **2013** (2013), 5 pages. 1
[6] M. Puta, *On the Maxwell-Bloch equations with one control,* C. R. Acad. Sci. Paris, Sér. I Math., **318** (1994), 679–683. 1
[7] M. Puta, *On an Extension of the 3-Dimensional Toda Lattice*, Differential geometry and applications (Brno, **1995**), 455–462, Masaryk Univ., Brno, (1996). 1
[8] H. F. Trotter, *On the Product of Semi-Operators*, Proc. Amer. Math. Soc., **10** (1959), 545–551. 2, 3