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Existence results for three-point boundary value problems for nonlinear fractional differential equations

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Abstract

In this paper, we study a new class of nonlinear fractional differential equations with three-point boundary conditions. Existence of solutions are obtained by using Krasnoselskii's fixed point theorem and Leray-Schauder nonlinear alternative. An illustrative example is presented at the end of the paper to illustrate the validity of our results.

Keywords: Fractional differential equation, boundary value problem, existence, fixed point theorem, three-point.
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1. Introduction and Preliminaries

Differential equations and inclusions of fractional order have central role in the modeling of many natural phenomena and physical processes. The increasing interest of fractional differential equations and inclusions

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are motivated by their applications in various fields of science such as physics, chemistry, engineering, biology, economics, fluid mechanics, control theory, etc. For this reason, in the recent years, many papers have been published about fractional differential equations and inclusions by mathematicians and other researchers (for example, see [1, 2, 4, 5, 6, 7, 8, 9, 12, 16, 17, 18, 19, 22] and the references therein).

In [3], Bashir Ahmad investigated the existence of solutions for the following nonlinear fractional differential equation with anti-periodic type fractional boundary conditions

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t), {}^cD^\beta x(t)), & t \in [0,T],\ T>0,\ 1<\alpha\le 2, \\ x(0)+\mu_1 x(T)=\sigma_1, & {}^cD^\gamma x(0)+\mu_2\, {}^cD^\gamma x(T)=\sigma_2, \end{cases} \tag{1.1}$$

where ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α, $0<\gamma,\beta<1$, $\mu_1\neq 1$, $\mu_1\neq 0$ and σ_1 and σ_2 are real constants.

In [11], Xi Fu discussed the existence of solutions for the following fractional differential equation with three-point boundary conditions

$$\begin{cases} {}^cD^\alpha x(t) = f(t, x(t)), & t \in [0,T],\ T>0,\ 1<\alpha\le 2, \\ a_1 x(0)+b_1 x(T)=c_1, & a_2({}^cD^\gamma x(\eta))+b_2({}^cD^\gamma x(T))=c_2, \end{cases} \tag{1.2}$$

where ${}^cD^\alpha$denotes the Caputo fractional derivative of order α, $0<\gamma<1$, $0<\eta<T$ and $a_i, b_i, c_i \in \mathbb{R}$, $i=1,2$.

Motivated by the above papers, in this paper, we study the existence of solutions for the nonlinear fractional differential equation

$$ {}^cD^\alpha u(t) = f(t, u(t), u'(t)) \tag{1.3}$$

subject to three-point boundary conditions

$$\begin{cases} \beta u(0)+\gamma u(1)=u(\eta), \\ \beta u'(0)+\gamma u'(1)=u'(\eta), \\ \beta\, {}^cD^p u(0)+\gamma\, {}^cD^p u(1)={}^cD^p u(\eta), \end{cases} \tag{1.4}$$

where $t\in[0,1]$, $2<\alpha\le 3$, $1<p\le 2$, $0<\eta<1$ and $\beta,\gamma\in\mathbb{R}^+$. Also, f is a continuous function from $[0,1]\times\mathbb{R}^2$ into $\mathbb{R}$.

Here, we bring some important definitions and lemmas which are needed in the sequel. For more details See [10, 14, 15] and [20].

Definition 1.1. Let $\alpha>0$, $n-1<\alpha<n$, $n=[\alpha]+1$ and $u\in C([a,b],\mathbb{R})$. The Caputo derivative of fractional order α for the function u is defined by

$$ {}^cD^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} u^{(n)}(\tau)d\tau. \tag{1.5}$$

Definition 1.2. The Riemann-Liouville fractional order integral of the function u is defined by

$$ I^\alpha u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}d\tau, \qquad (t>0), \tag{1.6}$$

whenever the integral exists.

Lemma 1.3 ([15]). *Let $n-1<\alpha<n$ and the function $g:[0,T]\to\mathbb{R}$ be continuous for each $T>0$. Then, the general solution of the fractional differential equation ${}^cD^\alpha g(t)=0$ is given by $g(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}$, where $c_0,\cdots,c_{n-1}$ are real constants and $n=[\alpha]+1$.*

Also, in [15], authors have been proved that for each $T>0$ and $u\in C([0,T])$ we have

$$ I^\alpha\ {}^cD^\alpha u(t) = u(t)+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}, \tag{1.7}$$

where $c_0,\cdots,c_{n-1}$ are real constants and $n=[\alpha]+1$.

Now we state the following known fixed point theorems which are needed in proving our results.

Theorem 1.4 ([21], Krasnoselskii fixed point theorem)**.** *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Consider the operators A and B such that:*

(i) $Ax + By \in M$ whenever $x, y \in M$;

(ii) A is compact and continuous;

(iii) B is a contraction mapping.

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 1.5 ([13], Leray-Schauder nonlinear alternative)**.** *Let E be a Banach space, C a closed and convex subset of E and U an open subset of C with $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous and compact map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

2. Main results

Now, we are ready to prove our main results. Let $X = \{u : u, u' \in C([0,1], \mathbb{R})\}$ endowed with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)| + \sup_{t\in[0,1]} |u'(t)|$. Then, $(X, \|\cdot\|)$ is a Banach space.

Remark 2.1. Throughout the paper, let

$$M = \frac{\Gamma(3-p)}{|\gamma - \eta^{2-p}|} \neq 0, \qquad Q = |\beta + \gamma - 1| \neq 0, \qquad S = |2(\eta-\gamma)^2 + Q(\eta^2 - \gamma)| \neq 0. \tag{2.1}$$

Lemma 2.2. *Let $y \in C([0,1], \mathbb{R})$. Then the integral solution of the linear problem*

$$\begin{cases} {}^cD^\alpha u(t) = y(t), \\ \beta u(0) + \gamma u(1) = u(\eta), \\ \beta u'(0) + \gamma u'(1) = u'(\eta), \\ \beta {}^cD^p u(0) + \gamma {}^cD^p u(1) = {}^c D^p u(\eta), \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} u(t) = &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \frac{1}{Q}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \frac{\gamma}{Q}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\ &+ \frac{A(t)}{Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds - \frac{\gamma A(t)}{Q^2}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds \\ &+ \frac{MB(t)}{2Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)ds - \frac{\gamma MB(t)}{2Q^2}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)ds, \end{aligned} \tag{2.3}$$

where

$$A(t) = \eta - \gamma + tQ, \;\; B(t) = S + 2Qt(\eta-\gamma) + t^2Q^2. \tag{2.4}$$

Proof. It is well known that the solution of equation ${}^cD^\alpha u(t) = y(t)$ can be written as

$$u(t) = I^\alpha y(t) + c_0 + c_1 t + c_2 t^2, \tag{2.5}$$

where $c_0, c_1, c_2 \in \mathbb{R}$ are arbitrary constants. Then, we have

$$u'(t) = I^{\alpha-1} y(t) + c_1 + 2c_2 t$$

and

$$^cD^p u(t) = I^{\alpha-p}y(t) + c_2\frac{2t^{2-p}}{\Gamma(3-p)}, \quad 1 < p \le 2.$$

By using the three-point boundary conditions, we obtain

$$\begin{aligned} c_0 &= \frac{1}{Q}(I^\alpha y(\eta) - \gamma I^\alpha y(1)) + \frac{\eta-\gamma}{Q}\Big[\frac{1}{Q}(I^{\alpha-1}y(\eta) - \gamma I^{\alpha-1}y(1)) \\ &\quad + \frac{M(\eta-\gamma)}{Q}(I^{\alpha-p}y(\eta) - \gamma I^{\alpha-p}y(1))\Big] + \frac{M(\eta^2-\gamma)}{2Q}(I^{\alpha-p}y(\eta) - \gamma I^{\alpha-p}y(1)), \\ c_1 &= \frac{1}{Q}(I^{\alpha-1}y(\eta) - \gamma I^{\alpha-1}y(1)) + \frac{M(\eta-\gamma)}{Q}(I^{\alpha-p}y(\eta) - \gamma I^{\alpha-p}y(1)) \end{aligned}$$

and

$$c_2 = \frac{M}{2}(I^{\alpha-p}y(\eta) - \gamma I^{\alpha-p}y(1)).$$

Substituting the values of constants c_0, c_1 and c_1 in (2.5), we get (2.3); that is

$$\begin{aligned} u(t) &= I^\alpha y(t) + \frac{1}{Q}I^\alpha y(\eta) - \frac{\gamma}{Q}I^\alpha y(1) + \frac{A(t)}{Q^2}I^{\alpha-1}y(\eta) - \frac{\gamma A(t)}{Q^2}I^{\alpha-1}y(1) \\ &\quad + \frac{MB(t)}{2Q^2}I^{\alpha-p}y(\eta) - \frac{\gamma MB(t)}{2Q^2}I^{\alpha-p}y(1). \end{aligned}$$

The proof is completed. □

Remark 2.3. In this paper, the following relations hold:

$$\begin{aligned} |A(t)| &\le |\eta-\gamma+Q| = A_1, \\ |B(t)| &\le |S + 2Q(\eta-\gamma) + Q^2| = B_1, \\ |A'(t)| &\le |Q| = A_1', \\ |B'(t)| &\le 2|Q(\eta-\gamma) + Q^2| = B_1'. \end{aligned} \tag{2.6}$$

For the sake of brevity, we set

$$\Delta_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{\eta^\alpha+\gamma}{Q\Gamma(\alpha+1)} + \frac{A_1(\eta^{\alpha-1}+\gamma)}{Q^2\Gamma(\alpha)} + \frac{MB_1(\eta^{\alpha-p}+\gamma)}{2Q^2\Gamma(\alpha-p+1)}, \tag{2.7}$$

$$\Delta_2 = \frac{1}{\Gamma(\alpha)} + \frac{A_1'(\eta^{\alpha-1}+\gamma)}{Q^2\Gamma(\alpha)} + \frac{MB_1'(\eta^{\alpha-p}+\gamma)}{2Q^2\Gamma(\alpha-p+1)}, \tag{2.8}$$

$$\Lambda_1 = \frac{\eta^\alpha+\gamma}{Q\Gamma(\alpha+1)} + \frac{A_1(\eta^{\alpha-1}+\gamma)}{Q^2\Gamma(\alpha)} + \frac{MB_1(\eta^{\alpha-p}+\gamma)}{2Q^2\Gamma(\alpha-p+1)} \tag{2.9}$$

and

$$\Lambda_2 = \frac{A_1'(\eta^{\alpha-1}+\gamma)}{Q^2\Gamma(\alpha)} + \frac{MB_1'(\eta^{\alpha-p}+\gamma)}{2Q^2\Gamma(\alpha-p+1)}. \tag{2.10}$$

Theorem 2.4. *Let $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Suppose that:*

(H_1) *There exists a continuous function $L : [0,1] \to \mathbb{R}$ such that for each $t \in [0,1]$ and for all $u_i, v_i \in \mathbb{R}$, $i = 1,2$, we have*

$$|f(t,u_1,u_2) - f(t,v_1,v_2)| \le L(t)(|u_1-v_1| + |u_2-v_2|).$$

(H_2) *There exist a continuous function* $\mu : [0,1] \to \mathbb{R}^+$ *and a non-decreasing continuous function* $\psi : [0,1] \to \mathbb{R}^+$ *such that*

$$|f(t,u_1,u_2)| \le \mu(t)\psi(|u_1|+|u_2|), \qquad t \in [0,1],\ u_i \in \mathbb{R},\ i=1,2.$$

Then, the three-point boundary value problem (1.3)–(1.4) *has at least one solution on* $[0,1]$ *if*

$$k := \|L\|(\Lambda_1+\Lambda_2) < 1,$$

where $\|L\| = \sup_{t\in[0,1]} |L(t)|$ *and* Λ_1, Λ_2 *are given by* (2.9)–(2.10).

Proof. We define $\|\mu\| = \sup_{t\in[0,1]} |\mu(t)|$ and choose a suitable constant r such that

$$r \ge \psi(r)\|\mu\|\big\{\Delta_1+\Delta_2\big\}, \tag{2.11}$$

where Δ_i's are given by (2.7)–(2.8). We consider the set $B_r = \{u \in X : \ \|u\| \le r\}$, where r is given in (2.11). It is clear that B_r is a closed, bounded, convex and nonempty subset of the Banach space X. Now, we define two operators F and G on B_r as follows:

$$(Fu)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),u'(s))ds \tag{2.12}$$

and

$$\begin{aligned}(Gu)(t) = &\frac{1}{Q}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),u'(s))ds - \frac{\gamma}{Q}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),u'(s))ds\\ &+ \frac{A(t)}{Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s),u'(s))ds - \frac{\gamma A(t)}{Q^2}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s),u'(s))ds\\ &+ \frac{MB(t)}{2Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f(s,u(s),u'(s))ds\\ &- \frac{\gamma MB(t)}{2Q^2}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f(s,u(s),u'(s))ds\end{aligned} \tag{2.13}$$

for each $t \in [0,1]$.

For $u, v \in B_r$, we can write

$$\begin{aligned}|(Fu+Gv)(t)| \le &\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),u'(s))|ds + \frac{1}{Q}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,v(s),v'(s))|ds\\ &+ \frac{\gamma}{Q}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,v(s),v'(s))|ds\\ &+ \frac{|A(t)|}{Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,v(s),v'(s))|ds\\ &+ \frac{\gamma|A(t)|}{Q^2}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,v(s),v'(s))|ds\\ &+ \frac{M|B(t)|}{2Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|f(s,v(s),v'(s))|ds\\ &+ \frac{\gamma M|B(t)|}{2Q^2}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|f(s,v(s),v'(s))|ds\\ \le &\|\mu\|\psi(r)\Delta_1.\end{aligned}$$

Also,

$$\begin{aligned}|(F'u+G'v)(t)| \leq& \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s),u'(s))|ds + \frac{A_1'}{Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,v(s),v'(s))|ds \\ &+ \frac{\gamma A_1'}{Q^2}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,v(s),v'(s))|ds \\ &+ \frac{MB_1'}{2Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|f(s,v(s),v'(s))|ds \\ &+ \frac{\gamma MB_1'}{2Q^2}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|f(s,v(s),v'(s))|ds \\ \leq& \|\mu\|\psi(r)\Delta_2.\end{aligned}$$

Hence $\|Fu+Gv\| \leq r$ and so, $Fu+Gv \in B_r$.

Since f is continuous, then the operator F is continuous. Also, for each $u \in B_r$, we have

$$|(Fu)(t)| \leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),u'(s))|ds \leq \frac{1}{\Gamma(\alpha+1)}\|\mu\|\psi(r)$$

and

$$|(F'u)(t)| \leq \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s),u'(s))|ds \leq \frac{1}{\Gamma(\alpha)}\|\mu\|\psi(r).$$

Thus $\|Fu\| \leq \left\{\frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)}\right\}\|\mu\|\psi(r)$. This shows that the operator F is uniformly bounded on B_r.

Now, we prove that the operator F is compact on B_r. For each $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, one can write

$$\begin{aligned}|(Fu)(t_2)-(Fu)(t_1)| =& \Big|\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),u'(s))ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),u'(s))ds\Big| \\ \leq& \Big|\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),u'(s))ds\Big| \\ &+ \Big|\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),u'(s))ds\Big| \\ \leq& \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),u'(s))|ds \\ &+ \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),u'(s))|ds \\ \leq& \left\{\frac{t_2^\alpha - t_1^\alpha - (t_2-t_1)^\alpha}{\Gamma(\alpha+1)} + \frac{(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}\right\}\|\mu\|\psi(r).\end{aligned}$$

It is seen that $|(Fu)(t_2)-(Fu)(t_1)| \to 0$ as $t_2 \to t_1$. Also, we have

$$\begin{aligned}|(F'u)(t_2)-(F'u)(t_1)| =& \Big|\int_0^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s),u'(s))ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s),u'(s))ds\Big| \\ \leq& \Big|\int_0^{t_1} \frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s),u'(s))ds\Big| \\ &+ \Big|\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,u(s),u'(s))ds\Big| \\ \leq& \int_0^{t_1} \frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s),u'(s))|ds\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s),u'(s))|ds\\
&\leq\left\{\frac{t_2^{\alpha-1}-t_1^{\alpha-1}-(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(t_2-t_1)^{\alpha-1}}{\Gamma(\alpha)}\right\}\|\mu\|\psi(r).
\end{aligned}
$$

Again, we see that $|(F'u)(t_2)-(F'u)(t_1)|\to 0$ as $t_2\to t_1$. Hence $\|(Fu)(t_2)-(Fu)(t_1)\|$ tends to zero as $t_2\to t_1$. Thus, F is equicontinuous and so F is relatively compact on B_r. Consequently, the Arzelá-Ascoli theorem implies that F is a compact operator on B_r.

Finally, we show that G is a contraction mapping. For every $u,v\in B_r$, we have

$$
\begin{aligned}
|(Gu)(t)-(Gv)(t)|\leq&\frac{1}{Q}\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds\\
&+\frac{\gamma}{Q}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds\\
&+\frac{|A(t)|}{Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds\\
&+\frac{\gamma A(t)}{Q^2}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds\\
&+\frac{M|B(t)|}{2Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds\\
&+\frac{\gamma M|B(t)|}{2Q^2}\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|f(s,u(s),u'(s))-f(s,v(s),v'(s))|ds\\
\leq&\frac{1}{Q}\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)\Big(|u(s)-v(s)|+|u'(s)-v'(s)|\Big)ds\\
&+\frac{\gamma}{Q}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}L(s)\Big(|u(s)-v(s)|+|u'(s)-v'(s)|\Big)ds\\
&+\frac{|A(t)|}{Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)\Big(|u(s)-v(s)|+|u'(s)-v'(s)|\Big)ds\\
&+\frac{\gamma A(t)}{Q^2}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)\Big(|u(s)-v(s)|+|u'(s)-v'(s)|\Big)ds\\
&+\frac{M|B(t)|}{2Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}L(s)\Big(|u(s)-v(s)|+|u'(s)-v'(s)|\Big)ds\\
&+\frac{\gamma M|B(t)|}{2Q^2}\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}L(s)\Big(|u(s)-v(s)|+|u'(s)-v'(s)|\Big)ds.
\end{aligned}
$$

Also,

$$
\begin{aligned}
|(G'u)(t)-(G'v)(t)|\leq&\frac{A_1'}{Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)\Big(|u(s)-v(s)|+|u'(s)-v'(s)|\Big)ds\\
&+\frac{\gamma A_1'}{Q^2}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}L(s)\Big(|u(s)-v(s)|+|u'(s)-v'(s)|\Big)ds\\
&+\frac{MB_1'}{2Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}L(s)\Big(|u(s)-v(s)|+|u'(s)-v'(s)|\Big)ds\\
&+\frac{\gamma MB_1'}{2Q^2}\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}L(s)\Big(|u(s)-v(s)|+|u'(s)-v'(s)|\Big)ds.
\end{aligned}
$$

Hence, letting $\sup_{t\in[0,1]}|L(t)|=\|L\|$, we obtain

$$
\sup_{t\in[0,1]}|(Gu)(t)-(Gv)(t)|\leq\|L\|\Lambda_1\|u-v\|,
$$

$$\sup_{t\in[0,1]} |(G'u)(t) - (G'v)(t)| \leq \|L\|\Lambda_2\|u - v\|.$$

Thus, $\|Gu - Gv\| \leq \|L\|(\Lambda_1 + \Lambda_2)\|u - v\|$ or $\|Gu - Gv\| \leq k\|u - v\|$. Since $k < 1$, thus G is contraction on B_r. Hence, all the assumptions of Theorem 1.4 are satisfied. Consequently, Theorem 1.4 implies that the three-point boundary value problem (1.3)–(1.4) has at least one solution on $[0, 1]$. □

Theorem 2.5. *Let $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assume that:*

(H_3) *There exist functions $h_1, h_2 \in L^1([0, 1], \mathbb{R}^+)$ and a non-decreasing and continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*
$$|f(t, u_1, u_2)| \leq h_1(t)\psi(|u_1| + |u_2|) + h_2(t)$$
for $t \in [0, 1]$ and $u_i \in \mathbb{R}$, $i = 1, 2$.

(H_4) *There exists a positive constant M such that $\dfrac{M}{\psi(M)[\delta_{11} + \delta_{21}] + [\delta_{12} + \delta_{22}]} > 1$, where for $i = 1, 2$,*

$$\begin{aligned}\delta_{1i} :=& I^\alpha h_i(1) + \frac{1}{Q} I^\alpha h_i(\eta) + \frac{\gamma}{Q} I^\alpha h_i(1) + \frac{A_1}{Q^2} I^{\alpha-1} h_i(\eta) + \frac{\gamma A_1}{Q^2} I^{\alpha-1} h_i(1)\\ &+ \frac{MB_1}{2Q^2} I^{\alpha-p} h_i(\eta) + \frac{\gamma MB_1}{2Q^2} I^{\alpha-p} h_i(1),\end{aligned} \tag{2.14}$$

$$\begin{aligned}\delta_{2i} :=& I^{\alpha-1} h_i(1) + \frac{A'_1}{Q^2} I^{\alpha-1} h_i(\eta) + \frac{\gamma A'_1}{Q^2} I^{\alpha-1} h_i(1)\\ &+ \frac{MB'_1}{2Q^2} I^{\alpha-p} h_i(\eta) + \frac{\gamma MB'_1}{2Q^2} I^{\alpha-p} h_i(1).\end{aligned} \tag{2.15}$$

Then the three-point boundary value problem (1.3)–(1.4) *has at least one solution on* $[0, 1]$.

Proof. We define the operator $F : X \to X$ by

$$\begin{aligned}(Fu)(t) =& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), u'(s))ds + \frac{1}{Q}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), u'(s))ds\\ &- \frac{\gamma}{Q}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), u'(s))ds + \frac{A(t)}{Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, u(s), u'(s))ds\\ &- \frac{\gamma A(t)}{Q^2}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s, u(s), u'(s))ds + \frac{MB(t)}{2Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f(s, u(s), u'(s))ds\\ &- \frac{\gamma MB(t)}{2Q^2}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} f(s, u(s), u'(s))ds.\end{aligned}$$

It is clear that F is a continuous operator. Now, we prove that F maps bounded sets into bounded subsets of X. For this purpose, let $B_r := \{u \in X : \ \|u\| \leq r\}$ be a bounded set in X for a positive number r. By (H_3), we have

$$\begin{aligned}|(Fu)(t)| \leq& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [h_1(s)\psi(\|u\|) + h_2(s)]ds + \frac{1}{Q}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} [h_1(s)\psi(\|u\|) + h_2(s)]ds\\ &+ \frac{\gamma}{Q}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} [h_1(s)\psi(\|u\|) + h_2(s)]ds + \frac{A_1}{Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} [h_1(s)\psi(\|u\|) + h_2(s)]ds\\ &+ \frac{\gamma A_1}{Q^2}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} [h_1(s)\psi(\|u\|) + h_2(s)]ds\\ &+ \frac{MB_1}{2Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} [h_1(s)\psi(\|u\|) + h_2(s)]ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{\gamma MB_1}{2Q^2}\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}[h_1(s)\psi(\|u\|)+h_2(s)]ds\\
\leq&\psi(r)\Bigg\{\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds+\frac{1}{Q}\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds+\frac{\gamma}{Q}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds\\
&+\frac{A_1}{Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}h_1(s)ds+\frac{\gamma A_1}{Q^2}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}h_1(s)ds\\
&+\frac{MB_1}{2Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}h_1(s)ds+\frac{\gamma MB_1}{2Q^2}\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}h_1(s)ds\Bigg\}\\
&+\Bigg\{\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h_2(s)ds+\frac{1}{Q}\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h_2(s)ds+\frac{\gamma}{Q}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h_2(s)ds\\
&+\frac{A_1}{Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}h_2(s)ds+\frac{\gamma A_1}{Q^2}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}h_2(s)ds\\
&+\frac{MB_1}{2Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}h_2(s)ds+\frac{\gamma MB_1}{2Q^2}\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}h_2(s)ds\Bigg\}\\
=&\psi(r)\Bigg\{I^\alpha h_1(1)+\frac{1}{Q}I^\alpha h_1(\eta)+\frac{\gamma}{Q}I^\alpha h_1(1)+\frac{A_1}{Q^2}I^{\alpha-1}h_1(\eta)+\frac{\gamma A_1}{Q^2}I^{\alpha-1}h_1(1)\\
&+\frac{MB_1}{2Q^2}I^{\alpha-p}h_1(\eta)+\frac{\gamma MB_1}{2Q^2}I^{\alpha-p}h_1(1)\Bigg\}+\Bigg\{I^\alpha h_2(1)+\frac{1}{Q}I^\alpha h_2(\eta)+\frac{\gamma}{Q}I^\alpha h_2(1)\\
&+\frac{A_1}{Q^2}I^{\alpha-1}h_2(\eta)+\frac{\gamma A_1}{Q^2}I^{\alpha-1}h_2(1)+\frac{MB_1}{2Q^2}I^{\alpha-p}h_2(\eta)+\frac{\gamma MB_1}{2Q^2}I^{\alpha-p}h_2(1)\Bigg\}\\
=&\psi(r)\delta_{11}+\delta_{12}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
|(F'u)(t)|\leq&\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}[h_1(s)\psi(\|u\|)+h_2(s)]ds\\
&+\frac{A_1'}{Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}[h_1(s)\psi(\|u\|)+h_2(s)]ds\\
&+\frac{\gamma A_1'}{Q^2}\int_0^1\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}[h_1(s)\psi(\|u\|)+h_2(s)]ds\\
&+\frac{MB_1'}{2Q^2}\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}[h_1(s)\psi(\|u\|)+h_2(s)]ds\\
&+\frac{\gamma MB_1'}{2Q^2}\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}[h_1(s)\psi(\|u\|)+h_2(s)]ds\\
\leq&\psi(r)\Bigg\{I^{\alpha-1}h_1(1)+\frac{A_1'}{Q^2}I^{\alpha-1}h_1(\eta)+\frac{\gamma A_1'}{Q^2}I^{\alpha-1}h_1(1)+\frac{MB_1'}{2Q^2}I^{\alpha-p}h_1(\eta)\\
&+\frac{\gamma MB_1'}{2Q^2}I^{\alpha-p}h_1(1)\Bigg\}+\Bigg\{I^{\alpha-1}h_2(1)+\frac{A_1'}{Q^2}I^{\alpha-1}h_2(\eta)+\frac{\gamma A_1'}{Q^2}I^{\alpha-1}h_2(1)\\
&+\frac{MB_1'}{2Q^2}I^{\alpha-p}h_2(\eta)+\frac{\gamma MB_1'}{2Q^2}I^{\alpha-p}h_2(1)\Bigg\}\\
=&\psi(r)\delta_{21}+\delta_{22}.
\end{aligned}
$$

Hence

$$
\|Fu\|\leq\psi(r)[\delta_{11}+\delta_{21}]+[\delta_{12}+\delta_{22}], \tag{2.16}
$$

where δ_{ij}'s are given in (2.14)–(2.15) for $i, j = 1, 2$.

Next, we prove that F maps bounded sets into equicontinuous sets of X. Again, consider the bounded subset $B_r = \{u \in X : \|u\| \le r\}$ of X. Let $u \in X$ and $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$. Then, one can write

$$\begin{aligned}
|(Fu)(t_2) - (Fu)(t_1)| \le & \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s),u'(s))| ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s,u(s),u'(s))| ds \\
& + \frac{A(t_2)-A(t_1)}{Q^2} \int_0^{\eta} \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s),u'(s))| ds \\
& + \frac{\gamma(A(t_2)-A(t_1))}{Q^2} \int_0^{1} \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s),u'(s))| ds \\
& + \frac{M(B(t_2)-B(t_1))}{2Q^2} \int_0^{\eta} \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} |f(s,u(s),u'(s))| ds \\
& + \frac{\gamma M(B(t_2)-B(t_1))}{2Q^2} \int_0^{1} \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} |f(s,u(s),u'(s))| ds \\
\le & \int_0^{t_1} \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} [h_1(s)\psi(\|u\|) + h_2(s)] ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} [h_1(s)\psi(\|u\|) + h_2(s)] ds \\
& + \frac{A(t_2)-A(t_1)}{Q^2} \int_0^{\eta} \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} [h_1(s)\psi(\|u\|) + h_2(s)] ds \\
& + \frac{\gamma(A(t_2)-A(t_1))}{Q^2} \int_0^{1} \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} [h_1(s)\psi(\|u\|) + h_2(s)] ds \\
& + \frac{M(B(t_2)-B(t_1))}{2Q^2} \int_0^{\eta} \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} [h_1(s)\psi(\|u\|) + h_2(s)] ds \\
& + \frac{\gamma M(B(t_2)-B(t_1))}{2Q^2} \int_0^{1} \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} [h_1(s)\psi(\|u\|) + h_2(s)] ds.
\end{aligned}$$

It is clear that the right-hand side of the above inequilities converges to zero as $t_2 \to t_1$. Similarly, we have

$$\begin{aligned}
|(F'u)(t_2) - (F'u)(t_1)| \le & \int_0^{t_1} \frac{(t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s),u'(s))| ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s),u'(s))| ds \\
& + \frac{A'(t_2)-A'(t_1)}{Q^2} \int_0^{\eta} \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s),u'(s))| ds \\
& + \frac{\gamma(A'(t_2)-A'(t_1))}{Q^2} \int_0^{1} \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} |f(s,u(s),u'(s))| ds \\
& + \frac{M(B'(t_2)-B'(t_1))}{2Q^2} \int_0^{\eta} \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} |f(s,u(s),u'(s))| ds \\
& + \frac{\gamma M(B'(t_2)-B'(t_1))}{2Q^2} \int_0^{1} \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} |f(s,u(s),u'(s))| ds \\
\le & \int_0^{t_1} \frac{(t_2-s)^{\alpha-2} - (t_1-s)^{\alpha-2}}{\Gamma(\alpha-1)} [h_1(s)\psi(\|u\|) + h_2(s)] ds \\
& + \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)} [h_1(s)\psi(\|u\|) + h_2(s)] ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{A'(t_2)-A'(t_1)}{Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)}[h_1(s)\psi(\|u\|)+h_2(s)]ds\\
&+\frac{\gamma(A(t_2)-A(t_1))}{Q^2}\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}[h_1(s)\psi(\|u\|)+h_2(s)]ds\\
&+\frac{M(B'(t_2)-B'(t_1))}{2Q^2}\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}[h_1(s)\psi(\|u\|)+h_2(s)]ds\\
&+\frac{\gamma M(B'(t_2)-B'(t_1))}{2Q^2}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}[h_1(s)\psi(\|u\|)+h_2(s)]ds.
\end{aligned}
$$

Again, it is seen that the right-hand side of the above inequalities tends to zero as $t_2 \to t_1$. Thus, $\|(Fu)(t_2)-(Fu)(t_1)\| \to 0$ as $t_2 \to t_1$. This shows that the operator F is completely continuous, by the Arzelá-Ascoli theorem. Since all conditions of Theorem 1.5 hold about the operator F, so either condition (i) or condition (ii) holds.

Take $U := \{u \in X : \ \|u\| < M\}$ with $\psi(M)[\delta_{11}+\delta_{21}]+[\delta_{12}+\delta_{22}] < M$. In view of the condition (H_3) and by (2.16), we get

$$\|Fu\| \le \psi(r)[\delta_{11}+\delta_{21}]+[\delta_{12}+\delta_{22}] < M.$$

Now, suppose that there exists $u \in \partial U$ and $\lambda \in (0,1)$ such that $u = \lambda F u$. For such choice of u and the constant λ, we have

$$M = \|u\| = \lambda\|Fu\| < \psi(\|u\|)[\delta_{11}+\delta_{21}]+[\delta_{12}+\delta_{22}] = \psi(M)[\delta_{11}+\delta_{21}]+[\delta_{12}+\delta_{22}] < M.$$

This is impossible. Hence, by Theorem 1.5, it follows that the operator F has a fixed point in $\overline{U}$ which is a solution of the three-point boundary value problem (1.3)–(1.4) and the proof is completed. □

Now, we give an illustrative example.

Example 2.6. Consider the following fractional differential equation

$$
{}^cD_0^{\frac{5}{2}}u(t) = \frac{t}{2}|\text{Arctan}x(t)| + \frac{t|\sin y(t)|}{2+2|\sin y(t)|}, \qquad t\in[0,1] \tag{2.17}
$$

with three-point boundary conditions

$$
\begin{cases}
0.001u(0)+0.01u(1)=u(0.16),\\
0.001u'(0)+0.01u'(1)=u'(0.16),\\
0.001{}^cD^{1.33}u(0)+0.01{}^cD^{1.33}u(1)={}^c D^{1.33}u(0.16).
\end{cases} \tag{2.18}
$$

One can easily see that $\alpha = 2.5$, $\eta = 0.16$, $p = 1.33$, $\beta = 0.001$ and $\gamma = 0.01$. We define the function $f : [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$f(t,x,y) = \frac{t}{2}|\text{Arctan}x(t)| + \frac{t|\sin y(t)|}{2+2|\sin y(t)|}, \quad t\in[0,1].$$

In this case, for every $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have $|f(t,x_1,y_1)-f(t,x_2,y_2)| \le L(t)(|x_1-x_2|+|y_1-y_2|)$, where the function $L : [0,1]\to\mathbb{R}$ is defined by $L(t) = \dfrac{t}{2}$ and $\|L\| = 0.5$.

On the other hand, we have

$$|f(t,x,y)| = \left|\frac{t}{2}|\text{Arctan}x(t)| + \frac{t|\sin y(t)|}{2+2|\sin y(t)|}\right| \le \frac{t}{2}\psi(|x(t)|+|y(t)|).$$

Put $\psi(t) = t$ and $\mu(t) = \dfrac{t}{2}$. Clearly, $\|\mu\| = 0.5$ and the function ψ is nondecreasing and continuous on $[0,1]$. It can be easily found that $\Delta_1 = 0.6277$, $\Delta_2 = 1.2417$, $\Lambda_1 = 0.3277$ and $\Lambda_2 = 0.4897$. Finally, since $k = \|L\|(\Lambda_1+\Lambda_2) = 0.4087 < 1$, thus all assumptions and conditions of Theorem 2.4 are satisfied. Hence, Theorem 2.4 implies that the three-point boundary value problem (2.17)–(2.18) has a solution.

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