Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2298–2304
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Schur-m power convexity for a mean of two variables with three parameters

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Communicated by F. Qi

Abstract

The Schur-m power convexity of a mean for two variables with three parameters is investigated and a judging condition about the Schur-m power convexity of a mean for two variables with three parameters is given.

Keywords: Mean of two variables, Schur convexity, Schur geometric convexity, Schur harmonic convexity, Schur-m power convexity, majorization.
2010 MSC: 26E60, 26D15, 26A51.

1. Introduction and Preliminaries

Throughout the paper we assume that the set of n-dimensional row vector on the real number field by $\mathbb{R}^n$.

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}.$$

In particular, $\mathbb{R}^1$ and $\mathbb{R}^1_+$ denoted by $\mathbb{R}$ and $\mathbb{R}_+$ respectively.

In 2009, Kuang [5]defined a mean of two variables with three parameters as follows:

$$K(\omega_1, \omega_2, p; a, b) = \left[\frac{\omega_1 A(a^p, b^p) + \omega_2 G(a^p, b^p)}{\omega_1 + \omega_2}\right]^{\frac{1}{p}}, \tag{1.1}$$

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Received 2015-10-05

where $A(a,b) = \frac{a+b}{2}$ and $G(a,b) = \sqrt{ab}$ respectively is the arithmetic mean and geometric mean of two positive numbers a and b, parameters $p \neq 0$, $\omega_1, \omega_2 \geq 0$ with $\omega_1 + \omega_2 \neq 0$.

In particular,

$$K\left(1, \frac{\omega}{2}, 1; a, b\right) = \frac{a + \omega\sqrt{ab} + b}{\omega + 2}$$

is the generalized Heron mean, which was introduced by Janous [4] in 2001.

$$K\left(1, \frac{\omega}{2}, p, a, b\right) = \frac{a^p + \omega(ab)^{p/2} + b^p}{\omega + 2}$$

is the generalized Heron mean with parameter.

For simplicity, sometimes we will show $K(\omega_1, \omega_2, p; a, b)$ by $K(\omega_1, \omega_2, p)$ or $K(a, b)$.

In recent years, the study on the properties of the mean with two variables by using theory of majorization is unusually active.

Yang [18],[19],[20] generalized the notion of Schur convexity to Schur f-convexity, which contains the Schur geometrical convexity, Schur harmonic convexity and so on. Moreover, he discussed Schur m-power convexity of Stolarsky means [18], Gini means [19] and Daróczy means [20]. Subsequently, many scholars have aroused the interest of Schur m-power convexity (see [2], [16], [17], [21]).

In this paper, the Schur-m power convexity of the mean $K(\omega_1, \omega_2, p)$ is discussed, a judging condition about the Schur-m power convexity of the mean $K(\omega_1, \omega_2, p)$ is given.

Our main result is as follows:

Theorem 1.1.

(I) *For* $m > 0$,

- (i) *if* $p \geq \max\{(1 + \frac{\omega_2}{\omega_1})m, 2m\}$, *then* $K(\omega_1, \omega_2, p)$ *is Schur-m power convex with* $(a, b) \in \mathbb{R}^2_+$,
- (ii) *if* $m \leq p \leq \min\{(1 + \frac{\omega_2}{\omega_1})m, 2m\}$, *then* $K(\omega_1, \omega_2, p)$ *is Schur-m power concave with* $(a, b) \in \mathbb{R}^2_+$,
- (iii) *if* $0 \leq p < m$, *then* $K(\omega_1, \omega_2, p)$ *is Schur-m power concave with* $(a, b) \in \mathbb{R}^2_+$,
- (iv) *if* $p < 0$, *then* $K(\omega_1, \omega_2, p)$ *is Schur-m power concave with* $(a, b) \in \mathbb{R}^2_+$.

(II) *For* $m < 0$,

- (i) *if* $p \geq 0$, *then* $K(\omega_1, \omega_2, p)$ *is Schur-m power convex with* $(a, b) \in \mathbb{R}^2_+$,
- (ii) *if* $m \leq p < 0$, *then* $K(\omega_1, \omega_2, p)$ *is Schur-m power convex with* $(a, b) \in \mathbb{R}^2_+$,
- (iii) *if* $2m \leq p < m$ *and* $p = (1 + \frac{\omega_2}{\omega_1})m$, $(0 < \frac{\omega_2}{\omega_1} < 1)$, *then* $K(\omega_1, \omega_2, p)$ *is Schur-m power convex with* $(a, b) \in \mathbb{R}^2_+$,
- (iv) *if* $p < 2m$ *and* $p = (1 + \frac{\omega_1}{\omega_2})m$, $(\frac{\omega_2}{\omega_1} > 1)$, *then* $K(\omega_1, \omega_2, p)$ *is Schur-m power concave with* $(a, b) \in \mathbb{R}^2_+$.

2. Definitions and Lemmas

We need the following definitions and lemmas.

Definition 2.1 ([8, 15])**.** Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

- (i) x is said to be majorized by y (in symbols $x \prec y$) if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of x and y in a descending order,
- (ii) $\Omega \subset \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \Omega$ for any x and $y \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$,

(*iii*) let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \leq \varphi(y)$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function.

Definition 2.2 ([11, 22])**.** Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$.

(*i*) $\Omega \subset \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for any x and $y \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$,

(*ii*) let $\Omega \subset \mathbb{R}^n_+$, φ: $\Omega \to \mathbb{R}_+$ is said to be a Schur-geometrically convex function on Ω if $(\log x_1, \ldots, \log x_n) \prec (\log y_1, \ldots, \log y_n)$ on Ω implies $\varphi(x) \leq \varphi(y)$. φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is Schur-geometrically convex function.

Definition 2.3 ([1, 9])**.** Let $\Omega \subset \mathbb{R}^n_+$.

(*i*) A set Ω is said to be a harmonically convex set if $\left(\frac{x_1 y_1}{\lambda x_1 + (1-\lambda) y_1}, \cdots, \frac{x_n y_n}{\lambda x_n + (1-\lambda) y_n}\right) \in \Omega$ for every $x, y \in \Omega$ and $\lambda \in [0, 1]$.

(*ii*) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur harmonically convex function on Ω if $\left(\frac{1}{x_1}, \cdots, \frac{1}{x_n}\right) \prec \left(\frac{1}{y_1}, \cdots, \frac{1}{y_n}\right)$ implies $\varphi(x) \leq \varphi(y)$. A function φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur harmonically convex function.

Definition 2.4 ([18])**.** Let $f : \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{x^m - 1}{m}, & m \neq 0; \\ \log x, & m = 0. \end{cases} \tag{2.1}$$

Then a function $\phi : \Omega \subset \mathbb{R}^n_+ \to \mathbb{R}$ is said to be Schur m-power convex on Ω if

$$(f(x_1), f(x_2), \ldots, f(x_n)) \prec (f(y_1), f(y_2), \ldots, f(y_n))$$

for all $(x_1, x_2, \ldots, x_n) \in \Omega$ and $(y_1, y_2, \ldots, y_n) \in \Omega$ implies $\phi(x) \leq \phi(y)$.

If $-\phi$ is Schur m-power convex, then we say that ϕ is Schur m-power concave.

If putting $f(x) = x, \log x, \frac{1}{x}$ in Definition 2.4, then definitions of the Schur-convex, Schur-geometrically convex and Schur-harmonically convex functions can be deduced respectively.

Lemma 2.5 ([8, 15])**.** *Let $\Omega \subset \mathbb{R}^n$ is convex set and has a nonempty interior set Ω^0 . Let $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω^0. Then φ is the $Schur-convex(Schur-concave)function$ if and only if it is symmetric on Ω and if*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 (\leq 0)$$

holds for any $x = (x_1, \cdots, x_n) \in \Omega^0$.

Lemma 2.6 ([11, 22])**.** *Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric geometrically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω^0. Then φ is a Schur geometrically convex (Schur geometrically concave) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (\leq 0) \tag{2.2}$$

holds for any $x = (x_1, \cdots, x_n) \in \Omega^0$.

Lemma 2.7 ([1, 9]). *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric harmonically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable on Ω^0. Then φ is a Schur harmonically convex (Schur harmonically concave) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (\leq 0) \tag{2.3}$$

holds for any $x = (x_1, \cdots, x_n) \in \Omega^0$.

Lemma 2.8 ([18]). *Let $\Omega \subset \mathbb{R}_+^n$ be a symmetric set with nonempty interior Ω° and $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. Then φ is Schur m-power convex on Ω if and only if φ is symmetric on Ω and*

$$\frac{x_1^m - x_2^m}{m}\left[x_1^{1-m}\frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2^{1-m}\frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0, \quad \text{if } m \neq 0 \tag{2.4}$$

and

$$(\log x_1 - \log x_2)\left[x_1\frac{\partial \varphi(\boldsymbol{x})}{\partial x_1} - x_2\frac{\partial \varphi(\boldsymbol{x})}{\partial x_2}\right] \geq 0, \quad \text{if } m = 0 \tag{2.5}$$

for all $x \in \Omega^\circ$.

Lemma 2.9. *Let*

$$g(x) = \omega_1(p-m)x^{\frac{p}{2}} - \omega_2\frac{p}{2}x^m + \omega_2(\frac{p}{2} - m), \quad x \in [1, \infty), \tag{2.6}$$

where $\omega_1 \geq 0, \omega_2 \geq 0, m \in \mathbb{R}$ and $m \neq 0$.

(I) For $m > 0$,

- *(i) if $p \geq \max\{(1 + \frac{\omega_2}{\omega_1})m, 2m\}$, then $g(x) \geq 0$,*
- *(ii) if $m \leq p \leq \min\{(1 + \frac{\omega_2}{\omega_1})m, 2m\}$, then $g(x) \leq 0$,*
- *(iii) if $0 \leq p < m$, then $g(x) \leq 0$,*
- *(iv) if $p < 0$, then the symbol of $g(x)$ is not fixed (from negative to positive).*

(II) For $m < 0$,

- *(i) if $p \geq 0$, then $g(x) > 0$,*
- *(ii) if $m \leq p < 0$, then $g(x) \geq 0$,*
- *(iii) if $2m \leq p < m$ and $p = (1 + \frac{\omega_2}{\omega_1})m$, $(0 < \frac{\omega_2}{\omega_1} < 1)$, then $g(x) \geq 0$,*
- *(iv) if $p < 2m$ and $p = (1 + \frac{\omega_1}{\omega_2})m$, $(\frac{\omega_2}{\omega_1} > 1)$, then $g(x) \leq 0$.*

Proof. From (2.6), we have

$$g(1) = \omega_1(p-m) - \omega_2\frac{p}{2} + \omega_2(\frac{p}{2} - m) = \omega_1[p - (1 + \frac{\omega_2}{\omega_1})m] \tag{2.7}$$

and

$$g^{'}(x) = \omega_1(p-m)\frac{p}{2}x^{\frac{p}{2}-1} - \omega_2 m\frac{p}{2}x^{m-1} = \frac{p}{2}x^{m-1}h(x), \tag{2.8}$$

where

$$h(x) = \omega_1(p-m)x^{\frac{p}{2}-m} - \omega_2 m. \tag{2.9}$$

(I) For $m > 0$,

(i) if $p \geq \max\{(1 + \frac{\omega_2}{\omega_1})m, 2m\}$, then $p > 0$ and

$$h(x) \geq \omega_1(p-m) - \omega_2 m = g(1) \geq 0, \tag{2.10}$$

so for $x \in [1, \infty)$, we have $g^{'}(x) \geq 0$ and then, $g(x) \geq g(1) \geq 0$,

(ii) if $m \le p \le \min\{(1+\frac{\omega_2}{\omega_1})m, 2m\}$, then it is easy to see that the inequality (2.10) is reversed, so for $x \in [1,\infty)$, we have $g^{'}(x) \le 0$ and then, $g(x) \le g(1) \le 0$,

(iii) if $0 \le p < m$, then $h(x) \le 0$ and $g(1) \le 0$, so for $x \in [1,\infty)$, we have $g^{'}(x) \le 0$ and then, $g(x) \le g(1) \le 0$,

(iv) if $p < 0$, then $h(x) \le 0$, so for $x \in [1,\infty)$, we have $g^{'}(x) \ge 0$, this means that $g(x)$ is increasing on $[1,\infty)$. For $p < 0$, notice that $\lim_{x\to+\infty} x^{\frac{p}{2}} = 0$, from (2.6), it is easy to see that $\lim_{x\to+\infty} g(x) = +\infty$, but $g(1) < 0$, so the symbol of $g(x)$ is not fixed (from negative to positive) on $[1,\infty)$.

(II) For $m < 0$,

(i) if $p \ge 0$, then $h(x) \ge 0$, so for $x \in [1,\infty)$, we have $g^{'}(x) \ge 0$ and then, $g(x) \ge g(1) > 0$,

(ii) if $m \le p < 0$, then $h(x) \ge 0$, so for $x \in [1,\infty)$, we have $g^{'}(x) \le 0$ and then, $g(x) \ge \lim_{x\to+\infty} g(x) = \omega_2(\frac{p}{2} - m) \ge 0$,

(iii) if $2m \le p < m$ and $p = (1+\frac{\omega_2}{\omega_1})m$, $\quad (0 < \frac{\omega_2}{\omega_1} < 1)$, then

$$h(x) \le \omega_1(p-m) - \omega_2 m = g(1) = 0, \tag{2.11}$$

so for $x \in [1,\infty)$, we have $g^{'}(x) \ge 0$ and then, $g(x) \ge g(1) = 0$,

(iv) if $p < 2m$ and $p = (1+\frac{\omega_2}{\omega_1})m$, $(\frac{\omega_2}{\omega_1} > 1)$, then

$$h(x) \ge \omega_1(p-m) - \omega_2 m = g(1) = 0, \tag{2.12}$$

so for $x \in [1,\infty)$, we have $g^{'}(x) \le 0$ and then, $g(x) \le g(1) = 0$.

□

3. Proof of Theorem 1.1

Proof. From the definition of $K(\omega_1,\omega_2,p)$, we have

$$K(\omega_1,\omega_2,p) = \left(\frac{\omega_1\frac{a^p+a^p}{2} + \omega_2 a^{\frac{p}{2}} b^{\frac{p}{2}}}{\omega_1+\omega_2}\right)^{\frac{1}{p}}.$$

It is clear that $K(\omega_1,\omega_2,p)$ is symmetric with $(a,b) \in \mathbb{R}^2_+$.

Write

$$s(a,b) := \left[\frac{\omega_1(a^p+a^p) + 2\omega_2 a^{\frac{p}{2}} b^{\frac{p}{2}}}{2(\omega_1+\omega_2)}\right]^{\frac{1}{p}-1}.$$

Then

$$\frac{\partial K}{\partial a} = s(a,b)\left(\frac{\omega_1 a^{p-1} + \omega_2 a^{\frac{p}{2}-1} b^{\frac{p}{2}}}{\omega_1+\omega_2}\right),$$

$$\frac{\partial K}{\partial b} = s(a,b)\left(\frac{\omega_1 b^{p-1} + \omega_2 a^{\frac{p}{2}} b^{\frac{p}{2}-1}}{\omega_1+\omega_2}\right),$$

and then

$$\Delta := \frac{a^m-b^m}{m}\left(a^{1-m}\frac{\partial K}{\partial a} - b^{1-m}\frac{\partial K}{\partial b}\right) = \frac{s(a,b)}{2(\omega_1+\omega_2)} f(a,b),$$

where

$$f(a,b) := \frac{a^m-b^m}{m}[\omega_1(a^{p-m}-b^{p-m}) + \omega_2(a^{\frac{p}{2}-m}b^{\frac{p}{2}} - a^{\frac{p}{2}}b^{\frac{p}{2}-m})].$$

Without loss of generality, we may assume that $a \geq b$, then $z := \frac{a}{b} \geq 1$ and then

$$\Delta = \frac{s(a,b)b^p}{2(\omega_1+\omega_2)} \cdot \frac{z^m-1}{m} q(z), \tag{3.1}$$

where

$$q(z) = \omega_1(z^{p-m}-1) + \omega_2(z^{\frac{p}{2}-m} - z^{\frac{p}{2}}) = \omega_1 z^{p-m} + \omega_2 z^{\frac{p}{2}-m} - \omega_2 z^{\frac{p}{2}} - \omega_1,$$
$$q^{'}(z) = \omega_1(p-m)z^{p-m-1} + \omega_2(\frac{p}{2}-m)z^{\frac{p}{2}-m-1} - \omega_2\frac{p}{2}z^{\frac{p}{2}-1} = z^{\frac{p}{2}-m-1}g(z).$$

(I) For $m > 0$,

- (i) if $p \geq \max\{(1+\frac{\omega_2}{\omega_1})m, 2m\}$, by $(I)(i)$ from Lemma 2.9, it follows that $q^{'}(z) \geq 0$, so $q(z) \geq q(1) = 0$. Notice that
$$\frac{s(a,b)b^p}{2(\omega_1+\omega_2)} > 0, \qquad \frac{z^m-1}{m} \geq 0,$$
from (3.1), we have $\Delta \geq 0$ and by Lemma 2.8, it follows that $K(\omega_1, \omega_2, p)$ is Schur-m power convex with $(a,b) \in \mathbb{R}^2_+$.
By the same arguments, from $(I)(ii)$ and $(I)(iii)$ in Lemma 2.9 we can prove $(I)(ii)$ and $(I)(iii)$ in Theorem 1.1, respectively,
- (iv) if $p < 0$, then from $(I)(iv)$ in Lemma 2.9, it follows that the symbol of $q^{'}(z)$ is not fixed (from negative to positive). This means that $q(x)$ first decreases and then increases, but $q(1) = 0$ and
$$\lim_{z\to+\infty} q(z) = \lim_{z\to+\infty} (\omega_1 z^{p-m} + \omega_2 z^{\frac{p}{2}-m} - \omega_2 z^{\frac{p}{2}} - \omega_1) = -\omega_1 < 0,$$
hence $q(z) \leq 0$ and then $\Delta \leq 0$, by Lemma 2.8, it follows that $K(\omega_1, \omega_2, p)$ is Schur-m power concave with $(a,b) \in \mathbb{R}^2_+$.

By analogous discussing with case (I), from $(II)(i)$, $(II)(ii)$, $(II)(iii)$ and $(II)(iv)$ in Lemma 2.9, we can prove $(II)(i)$, $(II)(ii)$, $(II)(iii)$ and $(II)(iv)$ in Theorem 1.1, respectively. The detailed proofs are left to the reader.

The proof of Theorem 1.1 is complete.

□

In recent years, the study on the properties of the mean with two variables by using theory of majorization is unusually active, interested readers can also refer to the references [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

Acknowledgements

The authors are indebted to the referees for their helpful suggestions.

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