Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 5629–5647
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

# The form of solutions and periodic nature for some rational difference equations systems

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Communicated by N. Shahzad

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**Abstract**

In this paper, we investigate the expressions of solutions and the periodic nature of the following systems of rational difference equations with order four

$$x_{n+1}=\frac{y_{n-3}}{\pm1\pm y_n z_{n-1} x_{n-2} y_{n-3}},\ y_{n+1}=\frac{z_{n-3}}{\pm1\pm z_n x_{n-1} y_{n-2} z_{n-3}},\ z_{n+1}=\frac{x_{n-3}}{\pm1\pm x_n y_{n-1} z_{n-2} x_{n-3}},$$

with initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0,\ z_{-3},\ z_{-2},\ z_{-1}$ and $z_0$ which are arbitrary real numbers. 

*Keywords:* Difference equations, recursive sequences, stability, periodic solution, system of difference equations.
*2010 MSC:* 39A10.

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## 1. Introduction

The goal of this paper is to obtain the form of the solutions and the periodicity character of some systems of rational difference equations

$$x_{n+1}=\frac{y_{n-3}}{\pm1\pm y_n z_{n-1} x_{n-2} y_{n-3}},\ y_{n+1}=\frac{z_{n-3}}{\pm1\pm z_n x_{n-1} y_{n-2} z_{n-3}},\ z_{n+1}=\frac{x_{n-3}}{\pm1\pm x_n y_{n-1} z_{n-2} x_{n-3}},$$

with initial conditions, which are arbitrary real numbers.

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*Received 2016-08-04*

Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economy, physics, and so on. Although difference equations are very simple in form, it is extremely difficult to understand thoroughly the behaviors of their solution, see [1–4, 6–8, 11, 13, 15, 19, 20, 22, 24, 25, 27, 31, 32] and the references cited therein. Recently, a great effort has been made in studying the qualitative analysis of rational difference equations and rational difference equations system, see [4–31, 33, 34].

Din et al. [8] investigated the qualitative behavior of the following competitive system of rational difference equations

$$x_{n+1} = \frac{\alpha_1+\beta_1 x_{n-1}}{a_1+b_1 y_n},\ y_{n+1} = \frac{\alpha_2+\beta_2 y_{n-1}}{a_2+b_2 x_n}.$$

Elsayed et al. [15] studied the form of the solutions and the periodicity of the following rational systems of rational difference equations

$$x_{n+1} = \frac{x_{n-5}}{1-x_{n-5}y_{n-2}},\ y_{n+1} = \frac{y_{n-5}}{\pm1\pm y_{n-5}x_{n-2}}.$$

Grove et al. [19] studied the existence and behavior of solution of the rational system

$$x_{n+1} = \frac{a}{x_n} + \frac{b}{y_n},\ \ y_{n+1} = \frac{c}{x_n} + \frac{d}{y_n}.$$

The behavior of positive solutions of the following system

$$x_{n+1} = \frac{x_{n-1}}{1+x_{n-1}y_n},\ y_{n+1} = \frac{y_{n-1}}{1+y_{n-1}x_n},$$

were studied by Kurbanli et al. in [24].

Özban [25] has investigated the positive solution of the system of rational difference equations

$$x_{n+1} = \frac{a}{y_{n-3}},\ y_{n+1} = \frac{by_{n-3}}{x_{n-q}y_{n-q}}.$$

Also, Touafek et al. [27] studied the periodicity and gave the form of the solutions of the following systems

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm1\pm y_n)},\ y_{n+1} = \frac{x_n}{y_{n-1}(\pm1\pm x_n)}.$$

Elabbasy et al. [11] obtained the solution of particular cases of the following general system of difference equations

$$x_{n+1} = \frac{a_1+a_2 y_n}{a_3 z_n+a_4 x_{n-1}z_n},\ y_{n+1} = \frac{b_1 z_{n-1}+b_2 z_n}{b_3 x_n y_n+b_4 x_n y_{n-1}},\ z_{n+1} = \frac{c_1 z_{n-1}+c_2 z_n}{c_3 x_{n-1}y_{n-1}+c_4 x_{n-1}y_n+c_5 x_n y_n}.$$

## 2. On the systems: $x_{n+1} = \frac{y_{n-3}}{1\pm y_n z_{n-1}x_{n-2}y_{n-3}},\ y_{n+1} = \frac{z_{n-3}}{1\pm z_n x_{n-1}y_{n-2}z_{n-3}},\ z_{n+1} = \frac{x_{n-3}}{1\pm x_n y_{n-1}z_{n-2}x_{n-3}}$

In this section, we study the solutions of the system of three difference equations in the following form

$$\begin{aligned} x_{n+1} &= \frac{y_{n-3}}{1+y_n z_{n-1}x_{n-2}y_{n-3}},\ y_{n+1} = \frac{z_{n-3}}{1+z_n x_{n-1}y_{n-2}z_{n-3}},\\ z_{n+1} &= \frac{x_{n-3}}{1+x_n y_{n-1}z_{n-2}x_{n-3}}, \qquad n=0,1,..., \end{aligned} \tag{2.1}$$

with nonzero real initial conditions $x_{-3}, x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0,\ z_{-3},\ z_{-2},\ z_{-1},\ z_0$.

**Theorem 2.1.** *Suppose that $\{x_n, y_n, z_n\}$ are solutions of system (2.1), then for $n = 0, 1, 2, ...,$ we see that*

$$x_{12n-3} = a\prod_{i=0}^{n-1}\frac{(1+12iadgl)(1+(12i+4)adgl)(1+(12i+8)adgl)}{(1+(12i+1)adgl)(1+(12i+5)adgl)(1+(12i+9)adgl)},$$

$$x_{12n-2} = b\prod_{i=0}^{n-1}\frac{(1+(12i+1)behm)(1+(12i+5)behm)(1+(12i+9)behm)}{(1+(12i+2)behm)(1+(12i+6)behm)(1+(12i+10)behm)},$$

$$x_{12n-1} = c \prod_{i=0}^{n-1} \frac{(1+(12i+2)cfko)(1+(12i+6)cfko)(1+(12i+10)cfko)}{(1+(12i+3)cfko)(1+(12i+7)cfko)(1+(12i+11)cfko)},$$

$$x_{12n} = d \prod_{i=0}^{n-1} \frac{(1+(12i+3)adgl)(1+(12i+7)adgl)(1+(12i+11)adgl)}{(1+(12i+4)adgl)(1+(12i+8)adgl)(1+(12i+12)adgl)},$$

$$x_{12n+1} = \frac{e}{(1+behm)} \prod_{i=0}^{n-1} \frac{(1+(12i+4)behm)(1+(12i+8)behm)(1+(12i+12)behm)}{(1+(12i+5)behm)(1+(12i+9)behm)(1+(12i+13)behm)},$$

$$x_{12n+2} = \frac{f\ (1+cfko)}{(1+2cfko)} \prod_{i=0}^{n-1} \frac{(1+(12i+5)cfko)(1+(12i+9)cfko)(1+(12i+13)cfko)}{(1+(12i+6)cfko)(1+(12i+10)cfko)(1+(12i+14)cfko)},$$

$$x_{12n+3} = \frac{g\ (1+2adgl)}{(1+3adgl)} \prod_{i=0}^{n-1} \frac{(1+(12i+6)adgl)(1+(12i+10)adgl)(1+(12i+14)adgl)}{(1+(12i+7)adgl)(1+(12i+11)adgl)(1+(12i+15)adgl)},$$

$$x_{12n+4} = \frac{h\ (1+3behm)}{(1+4behm)} \prod_{i=0}^{n-1} \frac{(1+(12i+7)behm)(1+(12i+11)behm)(1+(12i+15)behm)}{(1+(12i+8)behm)(1+(12i+12)behm)(1+(12i+16)behm)},$$

$$x_{12n+5} = \frac{k\ (1+4cfko)}{(1+cfko)(1+5cfko)} \prod_{i=0}^{n-1} \frac{(1+(12i+8)cfko)(1+(12i+12)cfko)(1+(12i+16)cfko)}{(1+(12i+9)cfko)(1+(12i+13)cfko)(1+(12i+17)cfko)},$$

$$x_{12n+6} = \frac{l\ (1+adgl)(1+5adgl)}{(1+2adgl)(1+6adgl)} \prod_{i=0}^{n-1} \frac{(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)}{(1+(12i+10)adgl)(1+(12i+14)adgl)(1+(12i+18)adgl)},$$

$$x_{12n+7} = \frac{m\ (1+2behm)(1+6behm)}{(1+3behm)(1+7behm)} \prod_{i=0}^{n-1} \frac{(1+(12i+10)behm)(1+(12i+14)behm)(1+(12i+18)behm)}{(1+(12i+11)behm)(1+(12i+15)behm)(1+(12i+19)behm)},$$

$$x_{12n+8} = \frac{o\ (1+3cfko)(1+7cfko)}{(1+4cfko)(1+8cfko)} \prod_{i=0}^{n-1} \frac{(1+(12i+11)cfko)(1+(12i+15)cfko)(1+(12i+19)cfko)}{(1+(12i+12)cfko)(1+(12i+16)cfko)(1+(12i+20)cfko)},$$

$$y_{12n-3} = e \prod_{i=0}^{n-1} \frac{(1+(12i)behm)(1+(12i+4)behm)(1+(12i+8)behm)}{(1+(12i+1)behm)(1+(12i+5)behm)(1+(12i+9)behm)},$$

$$y_{12n-2} = f \prod_{i=0}^{n-1} \frac{(1+(12i+1)cfko)(1+(12i+5)cfko)(1+(12i+9)cfko)}{(1+(12i+2)cfko)(1+(12i+6)cfko)(1+(12i+10)cfko)},$$

$$y_{12n-1} = g \prod_{i=0}^{n-1} \frac{(1+(12i+2)adgl)(1+(12i+6)adgl)(1+(12i+10)adgl)}{(1+(12i+3)adgl)(1+(12i+7)adgl)(1+(12i+11)adgl)},$$

$$y_{12n} = h \prod_{i=0}^{n-1} \frac{(1+(12i+3)behm)(1+(12i+7)behm)(1+(12i+11)behm)}{(1+(12i+4)behm)(1+(12i+8)behm)(1+(12i+12)behm)},$$

$$y_{12n+1} = \frac{k}{(1+cfko)} \prod_{i=0}^{n-1} \frac{(1+(12i+4)cfko)(1+(12i+8)cfko)(1+(12i+12)cfko)}{(1+(12i+5)cfko)(1+(12i+9)cfko)(1+(12i+13)cfko)},$$

$$y_{12n+2} = \frac{l\ (1+adgl)}{(1+2adgl)} \prod_{i=0}^{n-1} \frac{(1+(12i+5)adgl)(1+(12i+9)adgl)(1+(12i+13)adgl)}{(1+(12i+6)adgl)(1+(12i+10)adgl)(1+(12i+14)adgl)},$$

$$y_{12n+3} = \frac{m\ (1+2behm)}{(1+3behm)} \prod_{i=0}^{n-1} \frac{(1+(12i+6)behm)(1+(12i+10)behm)(1+(12i+14)behm)}{(1+(12i+7)behm)(1+(12i+11)behm)(1+(12i+15)behm)},$$

$$y_{12n+4} = \frac{o\ (1+3cfko)}{(1+4cfko)} \prod_{i=0}^{n-1} \frac{(1+(12i+7)cfko)(1+(12i+11)cfko)(1+(12i+15)cfko)}{(1+(12i+8)cfko)(1+(12i+12)cfko)(1+(12i+16)cfko)},$$

$$y_{12n+5} = \frac{a\ (1+4adgl)}{(1+adgl)(1+5adgl)} \prod_{i=0}^{n-1} \frac{(1+(12i+8)adgl)(1+(12i+12)adgl)(1+(12i+16)adgl)}{(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)},$$

$$y_{12n+6} = \frac{b\ (1+behm)(1+5behm)}{(1+2behm)(1+6behm)} \prod_{i=0}^{n-1} \frac{(1+(12i+9)behm)(1+(12i+13)behm)(1+(12i+17)behm)}{(1+(12i+10)behm)(1+(12i+14)behm)(1+(12i+18)behm)},$$

$$y_{12n+7} = \frac{c\ (1+2cfko)(1+6cfko)}{(1+3cfko)(1+7cfko)} \prod_{i=0}^{n-1} \frac{(1+(12i+10)cfko)(1+(12i+14)cfko)(1+(12i+18)cfko)}{(1+(12i+11)cfko)(1+(12i+15)cfko)(1+(12i+19)cfko)},$$

$$y_{12n+8}=\frac{d\ (1+3adgl)(1+7adgl)}{(1+4adgl)(1+8adgl)}\prod_{i=0}^{n-1}\frac{(1+(12i+11)adgl)(1+(12i+15)adgl)(1+(12i+19)adgl)}{(1+(12i+12)adgl)(1+(12i+16)adgl)(1+(12i+20)adgl)},$$

*and*

$$z_{12n-3}=k\prod_{i=0}^{n-1}\frac{(1+(12i)cfko)(1+(12i+4)cfko)(1+(12i+8)cfko)}{(1+(12i+1)cfko)(1+(12i+5)cfko)(1+(12i+9)cfko)},$$
$$z_{12n-2}=l\prod_{i=0}^{n-1}\frac{(1+(12i+1)adgl)(1+(12i+5)adgl)(1+(12i+9)adgl)}{(1+(12i+2)adgl)(1+(12i+6)adgl)(1+(12i+10)adgl)},$$
$$z_{12n-1}=m\prod_{i=0}^{n-1}\frac{(1+(12i+2)behm)(1+(12i+6)behm)(1+(12i+10)behm)}{(1+(12i+3)behm)(1+(12i+7)behm)(1+(12i+11)behm)},$$
$$z_{12n}=o\prod_{i=0}^{n-1}\frac{(1+(12i+3)cfko)(1+(12i+7)cfko)(1+(12i+11)cfko)}{(1+(12i+4)cfko)(1+(12i+8)cfko)(1+(12i+12)cfko)},$$
$$z_{12n+1}=\frac{a}{(1+adgl)}\prod_{i=0}^{n-1}\frac{(1+(12i+4)adgl)(1+(12i+8)adgl)(1+(12i+12)adgl)}{(1+(12i+5)adgl)(1+(12i+9)adgl)(1+(12i+13)adgl)},$$
$$z_{12n+2}=\frac{b\ (1+behm)}{(1+2behm)}\prod_{i=0}^{n-1}\frac{(1+(12i+5)behm)(1+(12i+9)behm)(1+(12i+13)behm)}{(1+(12i+6)behm)(1+(12i+10)behm)(1+(12i+14)behm)},$$
$$z_{12n+3}=\frac{c\ (1+2cfko)}{(1+3cfko)}\prod_{i=0}^{n-1}\frac{(1+(12i+6)cfko)(1+(12i+10)cfko)(1+(12i+14)cfko)}{(1+(12i+7)cfko)(1+(12i+11)cfko)(1+(12i+15)cfko)},$$
$$z_{12n+4}=\frac{d\ (1+3adgl)}{(1+4adgl)}\prod_{i=0}^{n-1}\frac{(1+(12i+7)adgl)(1+(12i+11)adgl)(1+(12i+15)adgl)}{(1+(12i+8)adgl)(1+(12i+12)adgl)(1+(12i+16)adgl)},$$
$$z_{12n+5}=\frac{e\ (1+4behm)}{(1+behm)(1+5behm)}\prod_{i=0}^{n-1}\frac{(1+(12i+8)behm)(1+(12i+12)behm)(1+(12i+16)behm)}{(1+(12i+9)behm)(1+(12i+13)behm)(1+(12i+17)behm)},$$
$$z_{12n+6}=\frac{f\ (1+cfko)(1+5cfko)}{(1+2cfko)(1+6cfko)}\prod_{i=0}^{n-1}\frac{(1+(12i+9)cfko)(1+(12i+13)cfko)(1+(12i+17)cfko)}{(1+(12i+10)cfko)(1+(12i+14)cfko)(1+(12i+18)cfko)},$$
$$z_{12n+7}=\frac{g\ (1+2adgl)(1+6adgl)}{(1+3adgl)(1+7adgl)}\prod_{i=0}^{n-1}\frac{(1+(12i+10)adgl)(1+(12i+14)adgl)(1+(12i+18)adgl)}{(1+(12i+11)adgl)(1+(12i+15)adgl)(1+(12i+19)adgl)},$$
$$z_{12n+8}=\frac{h\ (1+3behm)(1+7behm)}{(1+4behm)(1+8behm)}\prod_{i=0}^{n-1}\frac{(1+(12i+11)behm)(1+(12i+15)behm)(1+(12i+19)behm)}{(1+(12i+12)behm)(1+(12i+16)behm)(1+(12i+20)behm)},$$

*where* $x_{-3}=a,\ x_{-2}=b,\ x_{-1}=c,\ x_0=d,\ y_{-3}=e,\ y_{-2}=f,\ y_{-1}=g,\ y_0=h,\ z_{-3}=k,\ z_{-2}=l,\ z_{-1}=m,\ z_0=o$ *and* $\prod_{i=0}^{-1}A_i=1$.

*Proof.* For $n=0$, the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. Then we have

$$x_{12n-7}=\frac{k\ (1+4cfko)}{(1+cfko)(1+5cfko)}\prod_{i=0}^{n-2}\frac{(1+(12i+8)cfko)(1+(12i+12)cfko)(1+(12i+16)cfko)}{(1+(12i+9)cfko)(1+(12i+13)cfko)(1+(12i+17)cfko)},$$
$$x_{12n-6}=\frac{l\ (1+adgl)(1+5adgl)}{(1+2adgl)(1+6adgl)}\prod_{i=0}^{n-2}\frac{(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)}{(1+(12i+10)adgl)(1+(12i+14)adgl)(1+(12i+18)adgl)},$$
$$x_{12n-5}=\frac{m\ (1+2behm)(1+6behm)}{(1+3behm)(1+7behm)}\prod_{i=0}^{n-2}\frac{(1+(12i+10)behm)(1+(12i+14)behm)(1+(12i+18)behm)}{(1+(12i+11)behm)(1+(12i+15)behm)(1+(12i+19)behm)},$$
$$x_{12n-4}=\frac{o\ (1+3cfko)(1+7cfko)}{(1+4cfko)(1+8cfko)}\prod_{i=0}^{n-2}\frac{(1+(12i+11)cfko)(1+(12i+15)cfko)(1+(12i+19)cfko)}{(1+(12i+12)cfko)(1+(12i+16)cfko)(1+(12i+20)cfko)},$$
$$y_{12n-7}=\frac{a\ (1+4adgl)}{(1+adgl)(1+5adgl)}\prod_{i=0}^{n-2}\frac{(1+(12i+8)adgl)(1+(12i+12)adgl)(1+(12i+16)adgl)}{(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)},$$

$$y_{12n-6} = \frac{b\ (1+behm)(1+5behm)}{(1+2behm)(1+6behm)} \prod_{i=0}^{n-2} \frac{(1+(12i+9)behm)(1+(12i+13)behm)(1+(12i+17)behm)}{(1+(12i+10)behm)(1+(12i+14)behm)(1+(12i+18)behm)},$$

$$y_{12n-5} = \frac{c\ (1+2cfko)(1+6cfko)}{(1+3cfko)(1+7cfko)} \prod_{i=0}^{n-2} \frac{(1+(12i+10)cfko)(1+(12i+14)cfko)(1+(12i+18)cfko)}{(1+(12i+11)cfko)(1+(12i+15)cfko)(1+(12i+19)cfko)},$$

$$y_{12n-4} = \frac{d\ (1+3adgl)(1+7adgl)}{(1+4adgl)(1+8adgl)} \prod_{i=0}^{n-2} \frac{(1+(12i+11)adgl)(1+(12i+15)adgl)(1+(12i+19)adgl)}{(1+(12i+12)adgl)(1+(12i+16)adgl)(1+(12i+20)adgl)},$$

$$z_{12n-7} = \frac{e\ (1+4behm)}{(1+behm)(1+5behm)} \prod_{i=0}^{n-2} \frac{(1+(12i+8)behm)(1+(12i+12)behm)(1+(12i+16)behm)}{(1+(12i+9)behm)(1+(12i+13)behm)(1+(12i+17)behm)},$$

$$z_{12n-6} = \frac{f\ (1+cfko)(1+5cfko)}{(1+2cfko)(1+6cfko)} \prod_{i=0}^{n-2} \frac{(1+(12i+9)cfko)(1+(12i+13)cfko)(1+(12i+17)cfko)}{(1+(12i+10)cfko)(1+(12i+14)cfko)(1+(12i+18)cfko)},$$

$$z_{12n-5} = \frac{g\ (1+2adgl)(1+6adgl)}{(1+3adgl)(1+7adgl)} \prod_{i=0}^{n-2} \frac{(1+(12i+10)adgl)(1+(12i+14)adgl)(1+(12i+18)adgl)}{(1+(12i+11)adgl)(1+(12i+15)adgl)(1+(12i+19)adgl)},$$

$$z_{12n-4} = \frac{h\ (1+3behm)(1+7behm)}{(1+4behm)(1+8behm)} \prod_{i=0}^{n-2} \frac{(1+(12i+11)behm)(1+(12i+15)behm)(1+(12i+19)behm)}{(1+(12i+12)behm)(1+(12i+16)behm)(1+(12i+20)behm)}.$$

Now, it follows from Eq. (2.1) that

$$\begin{aligned}
x_{12n-3} &= \frac{y_{12n-7}}{1+y_{12n-7}z_{12n-5}x_{12n-6}y_{12n-4}} \\
&= \frac{\left(\frac{a\ (1+4adgl)}{(1+adgl)(1+5adgl)} \prod_{i=0}^{n-2} \frac{(1+(12i+8)adgl)(1+(12i+12)adgl)(1+(12i+16)adgl)}{(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)}\right)}{\left(\begin{array}{l} 1+\frac{a\ (1+4adgl)}{(1+adgl)(1+5adgl)} \prod_{i=0}^{n-2} \frac{(1+(12i+8)adgl)(1+(12i+12)adgl)(1+(12i+16)adgl)}{(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)} \\ \frac{g\ (1+2adgl)(1+6adgl)}{(1+3adgl)(1+7adgl)} \prod_{i=0}^{n-2} \frac{(1+(12i+10)adgl)(1+(12i+14)adgl)(1+(12i+18)adgl)}{(1+(12i+11)adgl)(1+(12i+15)adgl)(1+(12i+19)adgl)} \\ \frac{l\ (1+adgl)(1+5adgl)}{(1+2adgl)(1+6adgl)} \prod_{i=0}^{n-2} \frac{(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)}{(1+(12i+10)adgl)(1+(12i+14)adgl)(1+(12i+18)adgl)} \\ \frac{d\ (1+3adgl)(1+7adgl)}{(1+4adgl)(1+8adgl)} \prod_{i=0}^{n-2} \frac{(1+(12i+11)adgl)(1+(12i+15)adgl)(1+(12i+19)adgl)}{(1+(12i+12)adgl)(1+(12i+16)adgl)(1+(12i+20)adgl)} \end{array}\right)} \\
&= \frac{a(1+4adgl)(1+8adgl)\left(\prod_{i=0}^{n-2}(1+(12i+8)adgl)(1+(12i+12)adgl)(1+(12i+16)adgl)\right)}{(1+adgl)(1+5adgl)\left(\prod_{i=0}^{n-2}(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)\right)} \\
&\quad \times \frac{\left(\prod_{i=0}^{n-2}(1+(12i+20)adgl)\right)}{(1+8adgl)\left(\prod_{i=0}^{n-2}(1+(12i+20)adgl)\right)+adgl\left(\prod_{i=0}^{n-2}(1+(12i+8)adgl)\right)} \\
&= \frac{a(1+4adgl)(1+8adgl)\left(\prod_{i=0}^{n-2}(1+(12i+12)adgl)(1+(12i+16)adgl)(1+(12i+20)adgl)\right)}{(1+adgl)(1+5adgl)\left(\prod_{i=0}^{n-2}(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)\right)} \frac{1}{(1+(12n-4)adgl)+adgl} \\
&= \frac{\left(a(1+4adgl)(1+8adgl)\left(\prod_{i=0}^{n-2}(1+(12i+12)adgl)(1+(12i+16)adgl)(1+(12i+20)adgl)\right)\right)}{\left((1+adgl)(1+5adgl)(1+(12n-3)adgl)\left(\prod_{i=0}^{n-2}(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)\right)\right)} \\
&= a\prod_{i=0}^{n-1} \frac{(1+12iadgl)(1+(12i+4)adgl)(1+(12i+8)adgl)}{(1+(12i+1)adgl)(1+(12i+5)adgl)(1+(12i+9)adgl)}.
\end{aligned}$$

Also, we see that

$$y_{4n-2}=\frac{z_{12n-7}}{1+z_{12n-7}x_{12n-5}y_{12n-6}z_{12n-4}},$$

$$y_{4n-2}=\frac{\left(\frac{e\ (1+4behm)}{(1+behm)(1+5behm)}\prod_{i=0}^{n-2}\frac{(1+(12i+8)behm)(1+(12i+12)behm)(1+(12i+16)behm)}{(1+(12i+9)behm)(1+(12i+13)behm)(1+(12i+17)behm)}\right)}{1+\left(\begin{array}{c}\left(\frac{e\ (1+4behm)}{(1+behm)(1+5behm)}\prod_{i=0}^{n-2}\frac{(1+(12i+8)behm)(1+(12i+12)behm)(1+(12i+16)behm)}{(1+(12i+9)behm)(1+(12i+13)behm)(1+(12i+17)behm)}\right)\\ \left(\frac{m\ (1+2behm)(1+6behm)}{(1+3behm)(1+7behm)}\prod_{i=0}^{n-2}\frac{(1+(12i+10)behm)(1+(12i+14)behm)(1+(12i+18)behm)}{(1+(12i+11)behm)(1+(12i+15)behm)(1+(12i+19)behm)}\right)\\ \left(\frac{b\ (1+behm)(1+5behm)}{(1+2behm)(1+6behm)}\prod_{i=0}^{n-2}\frac{(1+(12i+9)behm)(1+(12i+13)behm)(1+(12i+17)behm)}{(1+(12i+10)behm)(1+(12i+14)behm)(1+(12i+18)behm)}\right)\\ \left(\frac{h\ (1+3behm)(1+7behm)}{(1+4behm)(1+8behm)}\prod_{i=0}^{n-2}\frac{(1+(12i+11)behm)(1+(12i+15)behm)(1+(12i+19)behm)}{(1+(12i+12)behm)(1+(12i+16)behm)(1+(12i+20)behm)}\right)\end{array}\right)}$$

$$=\frac{(e(1+4behm)(1+8behm))\left(\prod_{i=0}^{n-2}(1+(12i+12)behm)(1+(12i+16)behm)(1+(12i+20)behm)\right)}{(1+behm)(1+5behm)\left(\prod_{i=0}^{n-2}(1+(12i+9)behm)(1+(12i+13)behm)(1+(12i+17)behm)\right)}$$

$$\times\frac{\left(\prod_{i=0}^{n-2}(1+(12i+8)behm)\right)}{((1+8behm))\left(\prod_{i=0}^{n-2}(1+(12i+20)behm)\right)+behm\left(\prod_{i=0}^{n-2}(1+(12i+8)behm)\right)}$$

$$=\frac{(e(1+4behm)(1+8behm))\left(\prod_{i=0}^{n-2}(1+(12i+12)behm)(1+(12i+16)behm)(1+(12i+20)behm)\right)}{(1+behm)(1+5behm)\left(\prod_{i=0}^{n-2}(1+(12i+9)behm)(1+(12i+13)behm)(1+(12i+17)behm)\right)}$$

$$\times\frac{\left(\prod_{i=0}^{n-2}(1+(12i+8)behm)\right)}{\left(\prod_{i=0}^{n-1}(1+(12i+8)behm)\right)+behm\left(\prod_{i=0}^{n-2}(1+(12i+8)behm)\right)}$$

$$=\frac{(e(1+4behm)(1+8behm))\left(\prod_{i=0}^{n-2}(1+(12i+12)behm)(1+(12i+16)behm)(1+(12i+20)behm)\right)}{(1+behm)(1+5behm)\left(\prod_{i=0}^{n-2}(1+(12i+9)behm)(1+(12i+13)behm)(1+(12i+17)behm)\right)}$$

$$\times\frac{\left(\prod_{i=0}^{n-2}(1+(12i+8)behm)\right)}{\left(\prod_{i=0}^{n-2}(1+(12i+8)behm)\right)((1+(12n-4)behm)+behm)}$$

$$=\frac{(e(1+4behm)(1+8behm))\left(\prod_{i=0}^{n-2}(1+(12i+12)behm)(1+(12i+16)behm)(1+(12i+20)behm)\right)}{(1+behm)(1+5behm)(1+(12n-3)behm)\left(\prod_{i=0}^{n-2}(1+(12i+9)behm)(1+(12i+13)behm)(1+(12i+17)behm)\right)}$$

$$=e\prod_{i=0}^{n-1}\frac{(1+(12i)behm)(1+(12i+4)behm)(1+(12i+8)behm)}{(1+(12i+1)behm)(1+(12i+5)behm)(1+(12i+9)behm)},$$

and

$$z_{12n-3}=\frac{x_{12n-7}}{1+x_{12n-7}y_{12n-5}z_{12n-6}x_{12n-4}}$$

$$= \frac{\left(\frac{k\ (1+4cfko)}{(1+cfko)(1+5cfko)}\prod_{i=0}^{n-2}\frac{(1+(12i+8)cfko)(1+(12i+12)cfko)(1+(12i+16)cfko)}{(1+(12i+9)cfko)(1+(12i+13)cfko)(1+(12i+17)cfko)}\right)}{1+\left(\begin{array}{c}\left(\frac{k\ (1+4cfko)}{(1+cfko)(1+5cfko)}\prod_{i=0}^{n-2}\frac{(1+(12i+8)cfko)(1+(12i+12)cfko)(1+(12i+16)cfko)}{(1+(12i+9)cfko)(1+(12i+13)cfko)(1+(12i+17)cfko)}\right)\\ \left(\frac{c\ (1+2cfko)(1+6cfko)}{(1+3cfko)(1+7cfko)}\prod_{i=0}^{n-2}\frac{(1+(12i+10)cfko)(1+(12i+14)cfko)(1+(12i+18)cfko)}{(1+(12i+11)cfko)(1+(12i+15)cfko)(1+(12i+19)cfko)}\right)\\ \left(\frac{f\ (1+cfko)(1+5cfko)}{(1+2cfko)(1+6cfko)}\prod_{i=0}^{n-2}\frac{(1+(12i+9)cfko)(1+(12i+13)cfko)(1+(12i+17)cfko)}{(1+(12i+10)cfko)(1+(12i+14)cfko)(1+(12i+18)cfko)}\right)\\ \left(\frac{o\ (1+3cfko)(1+7cfko)}{(1+4cfko)(1+8cfko)}\prod_{i=0}^{n-2}\frac{(1+(12i+11)cfko)(1+(12i+15)cfko)(1+(12i+19)cfko)}{(1+(12i+12)cfko)(1+(12i+16)cfko)(1+(12i+20)cfko)}\right)\end{array}\right)}$$

$$= \frac{(k(1+4cfko)(1+8cfko))\left(\prod_{i=0}^{n-2}(1+(12i+12)cfko)(1+(12i+16)cfko)(1+(12i+20)cfko)\right)}{(1+cfko)(1+5cfko)\left(\prod_{i=0}^{n-2}(1+(12i+9)cfko)(1+(12i+13)cfko)(1+(12i+17)cfko)\right)}$$

$$\times \frac{\left(\prod_{i=0}^{n-2}(1+(12i+8)cfko)\right)}{\left(\prod_{i=0}^{n-1}(1+(12i+8)cfko)\right)+(cfko)\left(\prod_{i=0}^{n-2}(1+(12i+8)cfko)\right)}$$

$$= \frac{(k(1+4cfko)(1+8cfko))\left(\prod_{i=0}^{n-2}(1+(12i+12)cfko)(1+(12i+16)cfko)(1+(12i+20)cfko)\right)}{(1+cfko)(1+5cfko)\left(\prod_{i=0}^{n-2}(1+(12i+9)cfko)(1+(12i+13)cfko)(1+(12i+17)cfko)\right)}$$

$$\times \frac{\left(\prod_{i=0}^{n-2}(1+(12i+8)cfko)\right)}{(1+(12n-4)cfko)\left(\prod_{i=0}^{n-2}(1+(12i+8)cfko)\right)+(cfko)\left(\prod_{i=0}^{n-2}(1+(12i+8)cfko)\right)}$$

$$= \frac{(k(1+4cfko)(1+8cfko))\left(\prod_{i=0}^{n-2}(1+(12i+12)cfko)(1+(12i+16)cfko)(1+(12i+20)cfko)\right)}{(1+cfko)(1+5cfko)\left(\prod_{i=0}^{n-2}(1+(12i+9)cfko)(1+(12i+13)cfko)(1+(12i+17)cfko)\right)}$$

$$\times \frac{\left(\prod_{i=0}^{n-2}(1+(12i+8)cfko)\right)}{\left(\prod_{i=0}^{n-2}(1+(12i+8)cfko)\right)(1+(12n-4)cfko+cfko)}$$

$$= \frac{(k(1+4cfko)(1+8cfko))\left(\prod_{i=0}^{n-2}(1+(12i+12)cfko)(1+(12i+16)cfko)(1+(12i+20)cfko)\right)}{(1+cfko)(1+5cfko)(1+(12n-3)cfko)\left(\prod_{i=0}^{n-2}(1+(12i+9)cfko)(1+(12i+13)cfko)(1+(12i+17)cfko)\right)},$$

$$z_{12n-3} = \frac{k\prod_{i=0}^{n-1}(1+(12i)cfko)(1+(12i+4)cfko)(1+(12i+8)cfko)}{\prod_{i=0}^{n-1}(1+(12i+1)cfko)(1+(12i+5)cfko)(1+(12i+9)cfko)}.$$

Also, we can prove the other relations. This completes the proof. □

**Lemma 2.2.** *Let $\{x_n,\ y_n\ , z_n\}$ be positive solutions of system* (2.1)*, then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are bounded and converges to zero.*

*Proof.* It follows from Eq. (2.1) that

$$x_{n+1} = \frac{y_{n-3}}{1+y_n z_{n-1} x_{n-2} y_{n-3}} < y_{n-3},\ y_{n+1} = \frac{z_{n-3}}{1+z_n x_{n-1} y_{n-2} z_{n-3}} < z_{n-3},$$

$$z_{n+1} = \frac{x_{n-3}}{1 + x_n y_{n-1} z_{n-2} x_{n-3}} < x_{n-3}.$$

Thus

$$\begin{aligned} x_{n+5} < y_{n+1},\ y_{n+5} < z_{n+1},\ z_{n+5} < x_{n+1} &\Rightarrow x_{n+5} < z_{n-3},\ y_{n+5} < x_{n-3},\ z_{n+5} < y_{n-3} \\ &\Rightarrow x_{n+9} < y_{n+5} < x_{n-3},\ y_{n+9} < z_{n+5} < y_{n-3},\quad z_{n+9} < x_{n+5} < z_{n-3}. \end{aligned}$$

Then the subsequences $\{x_{12n+i}\}_{n=0}^{\infty}$, $i = -3. -2, -1, 0, 1, 2, ..., 8$, are decreasing and bounded from above by $M = \max\{x_{-3}, x_{-2}, x_{-1}, x_0, ..., x_8\}$. Also, the subsequences $\{y_{12n+i}\}_{n=0}^{\infty}$ and $\{z_{12n+i}\}_{n=0}^{\infty}$, $i = -3. -2, -1, 0, 1, 2, ..., 8$, are decreasing and bounded from above by $L = \max\{y_{-3}, y_{-2}, ..., y_8\}$ and $N = \max\{z_{-3}, z_{-2}, ..., z_8\}$, respectively. This completes the proof. □

**Lemma 2.3.** *If $x_i,\ y_i,\ z_i,\ i = -3, -2, -1, 0,$ are arbitrary real numbers and let $\{x_n, y_n, z_n\}$ be solutions of system* (2.1)*, then the following statements are true.*

(i) *If $x_{-3} = a = 0$, then we have $x_{12n-3} = y_{12n+5} = z_{12n+1} = 0$, $x_{12n} = y_{12n+8} = z_{12n+4} = d$, $x_{12n+6} = y_{12n+2} = z_{12n-2} = l$ and $x_{12n+3} = y_{12n-1} = z_{12n+7} = g$.*

(ii) *If $x_{-2} = b = 0$, then we have $x_{12n-2} = y_{12n+6} = z_{12n+2} = 0$, $x_{12n+1} = y_{12n-3} = z_{12n+5} = e$, $x_{12n+4} = y_{12n} = z_{12n+8} = h$ and $x_{12n+7} = y_{12n+3} = z_{12n-1} = m$.*

(iii) *If $x_{-1} = c = 0$, then we have $x_{12n-1} = y_{12n+7} = z_{12n+3} = 0$, $x_{12n+2} = y_{12n-2} = z_{12n+6} = f$, $x_{12n+5} = y_{12n+1} = z_{12n-3} = k$ and $x_{12n+8} = y_{12n+4} = z_{12n} = o$.*

(iv) *If $x_0 = d = 0$, then we have $x_{12n} = y_{12n+8} = z_{12n+4} = 0$, $x_{12n-3} = y_{12n+5} = z_{12n+1} = a$, $x_{12n+3} = y_{12n-1} = z_{12n+7} = g$ and $x_{12n+6} = y_{12n+2} = z_{12n-2} = l$.*

(v) *If $y_{-3} = e = 0$, then we have $x_{12n+1} = y_{12n-3} = z_{12n+5} = 0$, $x_{12n+7} = y_{12n+3} = z_{12n-1} = m$, $x_{12n-2} = y_{12n+6} = z_{12n+2} = b$ and $x_{12n+4} = y_{12n} = z_{12n+8} = h$.*

(vi) *If $y_{-2} = f = 0$, then we have $x_{12n+2} = y_{12n-2} = z_{12n+6} = 0$, $x_{12n+5} = y_{12n+1} = z_{12n-3} = k$, $x_{12n+8} = y_{12n+4} = z_{12n} = o$ and $x_{12n+7} = y_{12n+3} = z_{12n-1} = m$.*

(vii) *If $y_{-1} = g = 0$, then we have $x_{12n+3} = y_{12n-1} = z_{12n+7} = 0$, $x_{12n+6} = y_{12n+2} = z_{12n-2} = l$, $x_{12n-3} = y_{12n+5} = z_{12n+1} = a$ and $x_{12n} = y_{12n+8} = z_{12n+4} = d$.*

(viii) *If $y_0 = h = 0$, then we have $x_{12n+4} = y_{12n} = z_{12n+8} = 0$, $x_{12n+7} = y_{12n+3} = z_{12n-1} = m$, $x_{12n-2} = y_{12n+6} = z_{12n+2} = b$ and $x_{12n+1} = y_{12n-3} = z_{12n+5} = e$.*

(ix) *If $z_{-3} = k = 0$, then we have $x_{12n+5} = y_{12n+1} = z_{12n-3} = 0$, $x_{12n+2} = y_{12n-2} = z_{12n+6} = f$, $x_{12n+8} = y_{12n+4} = z_{12n} = o$ and $x_{12n-1} = y_{12n+7} = z_{12n+3} = c$.*

(x) *If $z_{-2} = l = 0$, then we have $x_{12n+6} = y_{12n+2} = z_{12n-2} = 0$, $x_{12n+3} = y_{12n-1} = z_{12n+7} = g$, $x_{12n-3} = y_{12n+5} = z_{12n+1} = a$ and $x_{12n} = y_{12n+8} = z_{12n+4} = d$.*

(xi) *If $z_{-1} = m = 0$, then we have $x_{12n+7} = y_{12n+3} = z_{12n-1} = 0$, $x_{12n+1} = y_{12n-3} = z_{12n+5} = e$, $x_{12n+4} = y_{12n} = z_{12n+8} = h$ and $x_{12n-2} = y_{12n+6} = z_{12n+2} = b$.*

(xii) *If $z_0 = o = 0$, then we have $x_{12n+8} = y_{12n+4} = z_{12n} = 0$ and $x_{12n+2} = y_{12n-2} = z_{12n+6} = f$, $x_{12n+5} = y_{12n+1} = z_{12n-3} = k$, $x_{12n-1} = y_{12n+7} = z_{12n+3} = c$.*

*Proof.* The proof follows from the form of the solutions of system (2.1). □

**Theorem 2.4.** *The solutions of the system*

$$x_{n+1}=\frac{y_{n-3}}{1-y_{n-3}z_{n-1}x_{n-2}y_{n}},\ y_{n+1}=\frac{z_{n-3}}{1-z_{n-3}x_{n-1}y_{n-2}z_{n}},\ z_{n+1}=\frac{x_{n-3}}{1-x_{n-3}y_{n-1}z_{n-2}x_{n}}, \tag{2.2}$$

*are given by the following equations*

$$x_{12n-3}=(-1)^{n}a\prod_{i=0}^{n-1}\frac{(-1+12iadgl)(-1+(12i+4)adgl)(-1+(12i+8)adgl)}{(-1+(12i+1)adgl)(-1+(12i+5)adgl)(-1+(12i+9)adgl)},$$

$$x_{12n-2}=b\prod_{i=0}^{n-1}\frac{(-1+(12i+1)behm)(-1+(12i+5)behm)(-1+(12i+9)behm)}{(-1+(12i+2)behm)(-1+(12i+6)behm)(-1+(12i+10)behm)},$$

$$x_{12n-1}=c\prod_{i=0}^{n-1}\frac{(-1+(12i+2)cfko)(-1+(12i+6)cfko)(-1+(12i+10)cfko)}{(-1+(12i+3)cfko)(-1+(12i+7)cfko)(-1+(12i+11)cfko)},$$

$$x_{12n}=d\prod_{i=0}^{n-1}\frac{(-1+(12i+3)adgl)(-1+(12i+7)adgl)(-1+(12i+11)adgl)}{(-1+(12i+4)adgl)(-1+(12i+8)adgl)(-1+(12i+12)adgl)},$$

$$x_{12n+1}=-\frac{e}{(-1+behm)}\prod_{i=0}^{n-1}\frac{(-1+(12i+4)behm)(-1+(12i+8)behm)(-1+(12i+12)behm)}{(-1+(12i+5)behm)(-1+(12i+9)behm)(-1+(12i+13)behm)},$$

$$x_{12n+2}=\frac{f\ (-1+cfko)}{(-1+2cfko)}\prod_{i=0}^{n-1}\frac{(-1+(12i+5)cfko)(-1+(12i+9)cfko)(-1+(12i+13)cfko)}{(-1+(12i+6)cfko)(-1+(12i+10)cfko)(-1+(12i+14)cfko)},$$

$$x_{12n+3}=\frac{g\ (-1+2adgl)}{(-1+3adgl)}\prod_{i=0}^{n-1}\frac{(-1+(12i+6)adgl)(-1+(12i+10)adgl)(-1+(12i+14)adgl)}{(-1+(12i+7)adgl)(-1+(12i+11)adgl)(-1+(12i+15)adgl)},$$

$$x_{12n+4}=\frac{h\ (-1+3behm)}{(-1+4behm)}\prod_{i=0}^{n-1}\frac{(-1+(12i+7)behm)(-1+(12i+11)behm)(-1+(12i+15)behm)}{(-1+(12i+8)behm)(-1+(12i+12)behm)(-1+(12i+16)behm)},$$

$$x_{12n+5}=-\frac{k\ (-1+4cfko)}{(-1+cfko)(-1+5cfko)}\prod_{i=0}^{n-1}\frac{(-1+(12i+8)cfko)(-1+(12i+12)cfko)(-1+(12i+16)cfko)}{(-1+(12i+9)cfko)(-1+(12i+13)cfko)(-1+(12i+17)cfko)},$$

$$x_{12n+6}=\frac{l\ (-1+adgl)(-1+5adgl)}{(-1+2adgl)(-1+6adgl)}\prod_{i=0}^{n-1}\frac{(-1+(12i+9)adgl)(-1+(12i+13)adgl)(-1+(12i+17)adgl)}{(-1+(12i+10)adgl)(-1+(12i+14)adgl)(-1+(12i+18)adgl)},$$

$$x_{12n+7}=\frac{m\ (-1+2behm)(-1+6behm)}{(-1+3behm)(-1+7behm)}\prod_{i=0}^{n-1}\frac{(-1+(12i+10)behm)(-1+(12i+14)behm)(-1+(12i+18)behm)}{(-1+(12i+11)behm)(-1+(12i+15)behm)(-1+(12i+19)behm)},$$

$$x_{12n+8}=\frac{o\ (-1+3cfko)(-1+7cfko)}{(-1+4cfko)(-1+8cfko)}\prod_{i=0}^{n-1}\frac{(-1+(12i+11)cfko)(-1+(12i+15)cfko)(-1+(12i+19)cfko)}{(-1+(12i+12)cfko)(-1+(12i+16)cfko)(-1+(12i+20)cfko)},$$

$$y_{12n-3}=(-1)^{n}e\prod_{i=0}^{n-1}\frac{(-1+(12i)behm)(-1+(12i+4)behm)(-1+(12i+8)behm)}{(-1+(12i+1)behm)(-1+(12i+5)behm)(-1+(12i+9)behm)},$$

$$y_{12n-2}=f\prod_{i=0}^{n-1}\frac{(-1+(12i+1)cfko)(-1+(12i+5)cfko)(-1+(12i+9)cfko)}{(-1+(12i+2)cfko)(-1+(12i+6)cfko)(-1+(12i+10)cfko)},$$

$$y_{12n-1}=g\prod_{i=0}^{n-1}\frac{(-1+(12i+2)adgl)(1+(12i+6)adgl)(-1+(12i+10)adgl)}{(-1+(12i+3)adgl)(-1+(12i+7)adgl)(-1+(12i+11)adgl)},$$

$$y_{12n}=h\prod_{i=0}^{n-1}\frac{(-1+(12i+3)behm)(-1+(12i+7)behm)(-1+(12i+11)behm)}{(-1+(12i+4)behm)(-1+(12i+8)behm)(-1+(12i+12)behm)},$$

$$y_{12n+1}=-\frac{k}{(-1+cfko)}\prod_{i=0}^{n-1}\frac{(-1+(12i+4)cfko)(-1+(12i+8)cfko)(-1+(12i+12)cfko)}{(-1+(12i+5)cfko)(-1+(12i+9)cfko)(-1+(12i+13)cfko)},$$

$$y_{12n+2}=\frac{l\ (-1+adgl)}{(-1+2adgl)}\prod_{i=0}^{n-1}\frac{(-1+(12i+5)adgl)(-1+(12i+9)adgl)(-1+(12i+13)adgl)}{(-1+(12i+6)adgl)(-1+(12i+10)adgl)(-1+(12i+14)adgl)},$$

$$y_{12n+3}=\frac{m\ (-1+2behm)}{(-1+3behm)}\prod_{i=0}^{n-1}\frac{(-1+(12i+6)behm)(-1+(12i+10)behm)(-1+(12i+14)behm)}{(-1+(12i+7)behm)(-1+(12i+11)behm)(-1+(12i+15)behm)},$$

$$y_{12n+4}=\frac{o\ (-1+3cfko)}{(-1+4cfko)}\prod_{i=0}^{n-1}\frac{(-1+(12i+7)cfko)(-1+(12i+11)cfko)(-1+(12i+15)cfko)}{(-1+(12i+8)cfko)(-1+(12i+12)cfko)(-1+(12i+16)cfko)},$$

$$y_{12n+5} = -\frac{a\ (-1+4adgl)}{(-1+adgl)(-1+5adgl)} \prod_{i=0}^{n-1} \frac{(-1+(12i+8)adgl)(-1+(12i+12)adgl)(-1+(12i+16)adgl)}{(-1+(12i+9)adgl)(-1+(12i+13)adgl)(-1+(12i+17)adgl)},$$

$$y_{12n+6} = \frac{b\ (-1+behm)(-1+5behm)}{(-1+2behm)(-1+6behm)} \prod_{i=0}^{n-1} \frac{(-1+(12i+9)behm)(-1+(12i+13)behm)(-1+(12i+17)behm)}{(-1+(12i+10)behm)(-1+(12i+14)behm)(-1+(12i+18)behm)},$$

$$y_{12n+7} = \frac{c\ (-1+2cfko)(-1+6cfko)}{(-1+3cfko)(-1+7cfko)} \prod_{i=0}^{n-1} \frac{(-1+(12i+10)cfko)(-1+(12i+14)cfko)(-1+(12i+18)cfko)}{(-1+(12i+11)cfko)(-1+(12i+15)cfko)(-1+(12i+19)cfko)},$$

$$y_{12n+8} = \frac{d\ (-1+3adgl)(-1+7adgl)}{(-1+4adgl)(-1+8adgl)} \prod_{i=0}^{n-1} \frac{(-1+(12i+11)adgl)(-1+(12i+15)adgl)(-1+(12i+19)adgl)}{(-1+(12i+12)adgl)(-1+(12i+16)adgl)(-1+(12i+20)adgl)},$$

$$z_{12n-3} = (-1)^n k \prod_{i=0}^{n-1} \frac{(-1+(12i)cfko)(-1+(12i+4)cfko)(-1+(12i+8)cfko)}{(-1+(12i+1)cfko)(-1+(12i+5)cfko)(-1+(12i+9)cfko)},$$

$$z_{12n-2} = l \prod_{i=0}^{n-1} \frac{(-1+(12i+1)adgl)(-1+(12i+5)adgl)(-1+(12i+9)adgl)}{(-1+(12i+2)adgl)(-1+(12i+6)adgl)(-1+(12i+10)adgl)},$$

$$z_{12n-1} = m \prod_{i=0}^{n-1} \frac{(-1+(12i+2)behm)(-1+(12i+6)behm)(-1+(12i+10)behm)}{(-1+(12i+3)behm)(-1+(12i+7)behm)(-1+(12i+11)behm)},$$

$$z_{12n} = o \prod_{i=0}^{n-1} \frac{(-1+(12i+3)cfko)(-1+(12i+7)cfko)(-1+(12i+11)cfko)}{(-1+(12i+4)cfko)(-1+(12i+8)cfko)(-1+(12i+12)cfko)},$$

$$z_{12n+1} = -\frac{a}{(-1+adgl)} \prod_{i=0}^{n-1} \frac{(-1+(12i+4)adgl)(-1+(12i+8)adgl)(-1+(12i+12)adgl)}{(-1+(12i+5)adgl)(-1+(12i+9)adgl)(-1+(12i+13)adgl)},$$

$$z_{12n+2} = \frac{b\ (-1+behm)}{(-1+2behm)} \prod_{i=0}^{n-1} \frac{(-1+(12i+5)behm)(-1+(12i+9)behm)(-1+(12i+13)behm)}{(-1+(12i+6)behm)(-1+(12i+10)behm)(-1+(12i+14)behm)},$$

$$z_{12n+3} = \frac{c\ (-1+2cfko)}{(-1+3cfko)} \prod_{i=0}^{n-1} \frac{(-1+(12i+6)cfko)(-1+(12i+10)cfko)(-1+(12i+14)cfko)}{(-1+(12i+7)cfko)(-1+(12i+11)cfko)(-1+(12i+15)cfko)},$$

$$z_{12n+4} = \frac{d\ (-1+3adgl)}{(-1+4adgl)} \prod_{i=0}^{n-1} \frac{(-1+(12i+7)adgl)(-1+(12i+11)adgl)(-1+(12i+15)adgl)}{(-1+(12i+8)adgl)(-1+(12i+12)adgl)(-1+(12i+16)adgl)},$$

$$z_{12n+5} = -\frac{e\ (-1+4behm)}{(-1+behm)(-1+5behm)} \prod_{i=0}^{n-1} \frac{(-1+(12i+8)behm)(-1+(12i+12)behm)(-1+(12i+16)behm)}{(-1+(12i+9)behm)(-1+(12i+13)behm)(-1+(12i+17)behm)},$$

$$z_{12n+6} = \frac{f\ (-1+cfko)(-1+5cfko)}{(-1+2cfko)(-1+6cfko)} \prod_{i=0}^{n-1} \frac{(-1+(12i+9)cfko)(-1+(12i+13)cfko)(-1+(12i+17)cfko)}{(-1+(12i+10)cfko)(-1+(12i+14)cfko)(-1+(12i+18)cfko)},$$

$$z_{12n+7} = \frac{g\ (-1+2adgl)(-1+6adgl)}{(-1+3adgl)(-1+7adgl)} \prod_{i=0}^{n-1} \frac{(-1+(12i+10)adgl)(-1+(12i+14)adgl)(-1+(12i+18)adgl)}{(-1+(12i+11)adgl)(-1+(12i+15)adgl)(-1+(12i+19)adgl)},$$

$$z_{12n+8} = \frac{h\ (-1+3behm)(-1+7behm)}{(-1+4behm)(-1+8behm)} \prod_{i=0}^{n-1} \frac{(-1+(12i+11)behm)(-1+(12i+15)behm)(-1+(12i+19)behm)}{(-1+(12i+12)behm)(-1+(12i+16)behm)(-1+(12i+20)behm)},$$

*where* $\prod_{i=0}^{-1} A_i = 1$.

## 3. On the systems: $x_{n+1}=\frac{y_{n-3}}{\pm 1+y_n z_{n-1} x_{n-2} y_{n-3}}$, $y_{n+1}=\frac{z_{n-3}}{\pm 1-z_n x_{n-1} y_{n-2} z_{n-3}}$, $z_{n+1}=\frac{x_{n-3}}{\pm 1-x_n y_{n-1} z_{n-2} x_{n-3}}$

In this section, we study the solutions of the system of three difference equations in the following form

$$x_{n+1} = \frac{y_{n-3}}{1+y_n z_{n-1} x_{n-2} y_{n-3}},\ y_{n+1} = \frac{z_{n-3}}{1-z_n x_{n-1} y_{n-2} z_{n-3}},\ z_{n+1} = \frac{x_{n-3}}{-1-x_n y_{n-1} z_{n-2} x_{n-3}},\ n = 0, 1, ..., \tag{3.1}$$

with nonzero real initial conditions.

**Theorem 3.1.** *Suppose that* $\{x_n, y_n, z_n\}$ *are solutions of system* (3.1), *we see that*

$$x_{12n-3} = \frac{a\ (1+2adgl)^n}{(-1+adgl)^n\ (1+adgl)^{2n}}, \qquad x_{12n-2} = \frac{(-1)^n b\ (1+behm)^{2n}\ (1+3behm)^n}{(1+2behm)^{2n}},$$

$$x_{12n-1}=\frac{c\ (1+2cfko)^{n}}{(-1+cfko)^{n}\ (1+cfko)^{2n}},\qquad x_{12n}=\frac{d\ (-1+adgl)^{n}\ (1+adgl)^{2n}}{(1+2adgl)^{n}},$$
$$x_{12n+1}=\frac{(-1)^{n}e\ (1+2behm)^{2n}}{(1+behm)^{2n+1}\ (1+3behm)^{n}},\qquad x_{12n+2}=-\frac{f\ (-1+cfko)^{n+1}\ (1+cfko)^{2n}}{(1+2cfko)^{n}},$$
$$x_{12n+3}=\frac{g\ (1+2adgl)^{n+1}}{(-1+adgl)^{n}\ (1+adgl)^{2n+1}},\qquad x_{12n+4}=\frac{(-1)^{n}h\ (1+behm)^{2n}\ (1+3behm)^{n+1}}{(1+2behm)^{2n+1}},$$
$$x_{12n+5}=-\frac{k\ (1+2cfko)^{n+1}}{(-1+cfko)^{n+1}\ (1+cfko)^{2n+1}},\qquad x_{12n+6}=-\frac{l\ (-1+adgl)^{n+1}\ (1+adgl)^{2n+1}}{(1+2adgl)^{n+1}},$$
$$x_{12n+7}=\frac{(-1)^{n}m\ (1+2behm)^{2n+1}}{(1+behm)^{2n+1}\ (1+3behm)^{n+1}},\qquad x_{12n+8}=-\frac{o\ (-1+cfko)^{n+1}\ (1+cfko)^{2n+1}}{(1+2cfko)^{n+1}},$$
$$y_{12n-3}=\frac{(-1)^{n}\ e\ (1+2behm)^{2n}}{(1+behm)^{2n}(1+3behm)^{n}},\qquad y_{12n-2}=\frac{f\ (-1+cfko)^{n}(1+cfko)^{2n}}{(1+2cfko)^{n}},$$
$$y_{12n-1}=\frac{g\ (1+2adgl)^{n}}{(-1+adgl)^{n}(1+adgl)^{2n}},\qquad y_{12n}=\frac{(-1)^{n}h\ (1+behm)^{2n}(1+3behm)^{n}}{(1+2behm)^{2n}},$$
$$y_{12n+1}=-\frac{k(1+2cfko)^{n}}{(-1+cfko)^{n+1}(1+cfko)^{2n}},\qquad y_{12n+2}=\frac{l\ (1+adgl)^{2n+1}(-1+adgl)^{n}}{(1+2adgl)^{n+1}},$$
$$y_{12n+3}=\frac{(-1)^{n}m\ (1+2behm)^{2n+1}}{(1+behm)^{2n}(1+3behm)^{n+1}},\qquad y_{12n+4}=\frac{o\ (-1+cfko)^{n}(1+cfko)^{2n+1}}{(1+2cfko)^{n+1}},$$
$$y_{12n+5}=\frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{n+1}(1+adgl)^{2n+1}},\qquad y_{12n+6}=\frac{(-1)^{n+1}b\ (1+behm)^{2n+2}(1+3behm)^{n}}{(1+2behm)^{2n+1}},$$
$$y_{12n+7}=\frac{c\ (1+2cfko)^{n}}{(-1+cfko)^{n+1}(1+cfko)^{2n+1}},\qquad y_{12n+8}=-\frac{d\ (-1+adgl)^{n}(1+adgl)^{2n+2}}{(1+2adgl)^{n+1}},$$
$$z_{12n-3}=\frac{k\ (1+2cfko)^{n}}{(-1+cfko)^{n}(1+cfko)^{2n}},\qquad z_{12n-2}=\frac{l\ (-1+adgl)^{n}(1+adgl)^{2n}}{(1+2adgl)^{n}},$$
$$z_{12n-1}=\frac{(-1)^{n}\ m\ (1+2behm)^{2n}}{(1+behm)^{2n}(1+3behm)^{n}},\qquad z_{12n}=\frac{o\ (-1+cfko)^{n}(1+cfko)^{2n}}{(1+2cfko)^{n}},$$
$$z_{12n+1}=-\frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{n}\ (1+adgl)^{2n+1}},\qquad z_{12n+2}=\frac{(-1)^{n+1}b\ (1+behm)^{2n+1}(1+3behm)^{n}}{(1+2behm)^{2n+1}},$$
$$z_{12n+3}=-\frac{c\ (1+2cfko)^{n}}{(-1+cfko)^{n}(1+cfko)^{2n+1}},\qquad z_{12n+4}=-\frac{d\ (1+adgl)^{2n+1}(-1+adgl)^{n}}{(1+2adgl)^{n}},$$
$$z_{12n+5}=\frac{(-1)^{n+1}e\ (1+2behm)^{2n+1}}{(1+behm)^{2n+2}(1+3behm)^{n}},\qquad z_{12n+6}=\frac{f\ (-1+cfko)^{n+1}(1+cfko)^{2n+1}}{(1+2cfko)^{n}},$$
$$z_{12n+7}=-\frac{g\ (1+2adgl)^{n+1}}{(-1+adgl)^{n}(1+adgl)^{2n+2}},\qquad z_{12n+8}=\frac{(-1)^{n+1}h\ (1+behm)^{2n+1}(1+3behm)^{n+1}}{(1+2behm)^{2n+2}},$$
$$n=0,1,2,\ldots .$$

*Proof.* For $n=0$, the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. We have

$$x_{12n-7}=-\frac{k\ (1+2cfko)^{n}}{(-1+cfko)^{n}\ (1+cfko)^{2n-1}},\qquad x_{12n-6}=-\frac{l\ (-1+adgl)^{n}\ (1+adgl)^{2n-1}}{(1+2adgl)^{n}},$$
$$x_{12n-5}=\frac{(-1)^{n-1}m\ (1+2behm)^{2n-1}}{(1+behm)^{2n-1}\ (1+3behm)^{n}},\qquad x_{12n-4}=-\frac{o\ (1+cfko)^{n}\ (1+cfko)^{2n-1}}{(1+2cfko)^{n}},$$
$$y_{12n-7}=\frac{a\ (1+2adgl)^{n-1}}{(-1+adgl)^{n}(1+adgl)^{2n-1}},\qquad y_{12n-6}=\frac{(-1)^{n}b\ (1+behm)^{2n}(1+3behm)^{n-1}}{(1+2behm)^{2n-1}},$$
$$y_{12n-5}=\frac{c\ (1+2cfko)^{n-1}}{(-1+cfko)^{n}(1+cfko)^{2n-1}},\qquad y_{12n-4}=-\frac{d\ (-1+adgl)^{n-1}(1+adgl)^{2n}}{(1+2adgl)^{n}},$$
$$z_{12n-7}=\frac{(-1)^{n}e\ (1+2behm)^{2n-1}}{(1+behm)^{2n}(1+3behm)^{n-1}},\qquad z_{12n-6}=\frac{f\ (-1+cfko)^{n}(1+cfko)^{2n-1}}{(1+2cfko)^{n-1}},$$
$$z_{12n-5}=-\frac{g\ (1+2adgl)^{n}}{(-1+adgl)^{n-1}(1+adgl)^{2n}},\qquad z_{12n-4}=\frac{(-1)^{n}h\ (1+behm)^{2n-1}(1+3behm)^{n}}{(1+2behm)^{2n}}.$$

Now, it follows from Eq. (3.1) that

$$\begin{aligned}x_{12n}&=\frac{y_{12n-4}}{1+y_{12n-4}x_{12n-3}z_{12n-2}y_{12n-1}}\\&=\frac{\left(\frac{-d\ (-1+adgl)^{n-1}(1+adgl)^{2n}}{(1+2adgl)^{n}}\right)}{\left[1+\left(\begin{array}{c}\left(\frac{-d\ (-1+adgl)^{n-1}(1+adgl)^{2n}}{(1+2adgl)^{n}}\right)\left(\frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{n}\ (1+adgl)^{2n}}\right)\\ \left(\frac{l\ (-1+adgl)^{n}(1+adgl)^{2n}}{(1+2adgl)^{n}}\right)\left(\frac{g\ (1+2adgl)^{n}}{(-1+adgl)^{n}(1+adgl)^{2n}}\right)\end{array}\right)\right]}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(\frac{-d\ (-1+adgl)^{n-1}(1+adgl)^{2n}}{(1+2adgl)^{n}}\right)}{\left[1+\left(\frac{-adgl}{(-1+adgl)}\right)\right]} = \frac{\left(\frac{-d\ (-1+adgl)^{n-1}(1+adgl)^{2n}}{(1+2adgl)^{n}}\right)}{\left(\frac{-1+adgl-adgl}{-1+adgl}\right)}\\
&= \frac{d\ (-1+adgl)^{n}(1+adgl)^{2n}}{(1+2adgl)^{n}},
\end{aligned}
$$

$$
\begin{aligned}
y_{12n+1} &= \frac{z_{12n-3}}{1-z_{12n-3}y_{12n-2}x_{12n-1}z_{12n}}\\
&= \frac{\left(\frac{k\ (1+2cfko)^{n}}{(-1+cfko)^{n}(1+cfko)^{2n}}\right)}{\left[1-\left(\begin{array}{c}\left(\frac{k\ (1+2cfko)^{n}}{(-1+cfko)^{n}(1+cfko)^{2n}}\right)\left(\frac{f\ (-1+cfko)^{n}(1+cfko)^{2n}}{(1+2cfko)^{n}}\right)\\ \left(\frac{c\ (1+2cfko)^{n}}{(-1+cfko)^{n}\ (1+cfko)^{2n}}\right)\left(\frac{o\ (-1+cfko)^{n}(1+cfko)^{2n}}{(1+2cfko)^{n}}\right)\end{array}\right)\right]}\\
&= \frac{\left(\frac{(-1)^{n}k\ (1+2cfko)^{n}}{(1-cfko)^{n}(1+cfko)^{2n}}\right)}{(1-cfko)} = \frac{(-1)^{n}k\ (1+2cfko)^{n}}{(1-cfko)^{n+1}(1+cfko)^{2n}},
\end{aligned}
$$

$$
\begin{aligned}
z_{12n+1} &= \frac{x_{12n-3}}{-1-x_{12n-3}z_{12n-2}y_{12n-1}x_{12n}}\\
&= \frac{\left(\frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{n}\ (1+adgl)^{2n}}\right)}{\left[-1-\left(\begin{array}{c}\left(\frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{n}\ (1+adgl)^{2n}}\right)\left(\frac{l\ (-1+adgl)^{n}(1+adgl)^{2n}}{(1+2adgl)^{n}}\right)\\ \left(\frac{g\ (1+2adgl)^{n}}{(-1+adgl)^{n}(1+adgl)^{2n}}\right)\left(\frac{d\ (-1+adgl)^{n}\ (1+adgl)^{2n}}{(1+2adgl)^{n}}\right)\end{array}\right)\right]}\\
&= -\frac{\left(\frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{n}\ (1+adgl)^{2n}}\right)}{(1+adgl)} = -\frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{n}\ (1+adgl)^{2n+1}}.
\end{aligned}
$$

Also, we can see that

$$
\begin{aligned}
x_{12n+1} &= \frac{y_{12n-3}}{1+y_{12n-3}x_{12n-2}z_{12n-1}y_{12n}}\\
&= \frac{\left(\frac{(-1)^{n}\ e\ (1+2behm)^{2n}}{(1+behm)^{2n}(1+3behm)^{n}}\right)}{\left[1+\left(\begin{array}{c}\left(\frac{(-1)^{n}\ e\ (1+2behm)^{2n}}{(1+behm)^{2n}(1+3behm)^{n}}\right)\left(\frac{(-1)^{n}b\ (1+behm)^{2n}\ (1+3behm)^{n}}{(1+2behm)^{2n}}\right)\\ \left(\frac{(-1)^{n}\ m\ (1+2bhm)^{2n}}{(1+bhm)^{2n}(1+3bhm)^{n}}\right)\left(\frac{(-1)^{n}h\ (1+behm)^{2n}(1+3behm)^{n}}{(1+2behm)^{2n}}\right)\end{array}\right)\right]}\\
&= \frac{\left(\frac{(-1)^{n}\ e\ (1+2behm)^{2n}}{(1+behm)^{2n}(1+3behm)^{n}}\right)}{(1+behm)} = \frac{(-1)^{n}\ e\ (1+2behm)^{2n}}{(1+behm)^{2n+1}(1+3behm)^{n}},
\end{aligned}
$$

$$
\begin{aligned}
y_{12n+5} &= \frac{z_{12n+1}}{1-z_{12n+4}x_{12n+3}y_{12n+2}z_{12n+1}}\\
&= \frac{\left(-\frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{n}\ (1+adgl)^{2n+1}}\right)}{\left[1-\left(\begin{array}{c}\left(-\frac{d\ (1+adgl)^{2n+1}(-1+adgl)^{n}}{(1+2adgl)^{n}}\right)\left(\frac{g\ (1+2adgl)^{n+1}}{(-1+adgl)^{n}\ (1+adgl)^{2n+1}}\right)\\ \left(\frac{l\ (1+adgl)^{2n+1}(-1+adgl)^{n}}{(1+2adgl)^{n+1}}\right)\left(-\frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{n}\ (1+adgl)^{2n+1}}\right)\end{array}\right)\right]}\\
&= \frac{\left(-\frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{n}\ (1+adgl)^{2n+1}}\right)}{(1-adgl)} = \frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{n+1}\ (1+adgl)^{2n+1}}\\
&= \frac{a\ (1+2adgl)^{n}}{(-1+adgl)^{2n}(1+adgl)^{2n+1}},
\end{aligned}
$$

$$
\begin{aligned}
z_{12n+6} &= \frac{x_{12n+2}}{-1-x_{12n+5}y_{12n+4}z_{12n+3}x_{12n+2}}\\
&= \frac{\left(-\frac{f\ (-1+cfko)^{n+1}\ (1+cfko)^{2n}}{(1+2cfko)^{n}}\right)}{-1-\left[\begin{array}{c}\left(-\frac{k\ (1+2cfko)^{n+1}}{(-1+cfko)^{n+1}\ (1+cfko)^{2n+1}}\right)\left(\frac{o\ (-1+cfko)^{n}(1+cfko)^{2n+1}}{(1+2cfko)^{n+1}}\right)\\ \left(-\frac{c\ (1+2cfko)^{n}}{(-1+cfko)^{n}(1+cfko)^{2n+1}}\right)\left(-\frac{f\ (-1+cfko)^{n+1}\ (1+cfko)^{2n}}{(1+2cfko)^{n}}\right)\end{array}\right]}
\end{aligned}
$$

$$= \frac{\left(-\frac{f\ (-1+cfko)^{n+1}\ (1+cfko)^{2n}}{(1+2cfko)^{n}}\right)}{-1+\left(\frac{cfko}{1+cfko}\right)} = \frac{\left(-\frac{f\ (-1+cfko)^{n+1}\ (1+cfko)^{2n}}{(1+2cfko)^{n}}\right)}{\frac{-1-cfko+cfko}{1+cfko}}$$
$$= \frac{f\ (-1+cfko)^{n+1}(1+cfko)^{2n+1}}{(1+2cfko)^{n}}.$$

Also, we can prove the other relations. This completes the proof. □

**Theorem 3.2.** *Assume that* $\{x_n,\ y_n,\ z_n\}$ *are solutions of the system*

$$x_{n+1} = \frac{y_{n-3}}{-1+y_{n-3}z_{n-1}x_{n-2}y_n},\ y_{n+1} = \frac{z_{n-3}}{1-z_{n-3}x_{n-1}y_{n-2}z_n},\ z_{n+1} = \frac{x_{n-3}}{1-x_{n-3}y_{n-1}z_{n-2}x_n},$$

*then for* $n = 0, 1, 2, \ldots$, *we see that*

$$x_{12n-3} = \frac{a\ (1-2adgl)^{2n}}{(-1+adgl)^{2n}(-1+3adgl)^{n}}, \qquad x_{12n-2} = \frac{b\ (-1+behm)^{n}\ (1+behm)^{2n}}{(1+2behm)^{n}},$$
$$x_{12n-1} = \frac{c\ (-1+2cfko)^{n}}{(-1+cfko)^{2n}\ (1+cfko)^{n}}, \qquad x_{12n} = \frac{d\ (-1+adgl)^{2n}\ (-1+3adgl)^{n}}{(1-2adgl)^{2n}},$$
$$x_{12n+1} = \frac{e\ (1+2behm)^{n}}{(-1+behm)^{n+1}\ (1+behm)^{2n}}, \qquad x_{12n+2} = -\frac{f\ (-1+cfko)^{2n+1}\ (1+cfko)^{n}}{(-1+2cfko)^{n+1}},$$
$$x_{12n+3} = -\frac{g\ (-1+2adgl)^{2n+1}}{(-1+3adgl)^{n+1}\ (-1+adgl)^{2n}}, \qquad x_{12n+4} = -\frac{h\ (1+behm)^{2n+1}\ (-1+behm)^{n}}{(1+2behm)^{n+1}},$$
$$x_{12n+5} = \frac{k\ (-1+2cfko)^{n}}{(-1+cfko)^{n+1}\ (1+cfko)^{2n+1}}, \qquad x_{12n+6} = \frac{l\ (-1+adgl)^{2n+2}\ (-1+3adgl)^{n}}{(-1+2adgl)^{2n+1}},$$
$$x_{12n+7} = \frac{m\ (1+2behm)^{n}}{(-1+behm)^{n+1}\ (1+behm)^{2n+1}}, \qquad x_{12n+8} = \frac{o\ (-1+cfko)^{2n+2}(1+cfko)^{n}}{(-1+2cfko)^{n+1}},$$
$$y_{12n-3} = \frac{e\ (1+2behm)^{n}}{(-1+behm)^{n}(1+behm)^{2n}}, \qquad y_{12n-2} = \frac{f\ (-1+cfko)^{2n}(1+cfko)^{n}}{(-1+2cfko)^{n}},$$
$$y_{12n-1} = \frac{g\ (1-2adgl)^{2n}}{(-1+adgl)^{2n}(-1+3adgl)^{n}}, \qquad y_{12n} = \frac{h\ (-1+behm)^{n}(1+behm)^{2n}}{(1+2behm)^{n}},$$
$$y_{12n+1} = -\frac{k(-1+2cfko)^{n}}{(-1+cfko)^{2n+1}(1+cfko)^{n}}, \qquad y_{12n+2} = \frac{l\ (-1+adgl)^{2n+1}(-1+3adgl)^{n}}{(-1+2adgl)^{2n+1}},$$
$$y_{12n+3} = \frac{m\ (1+2bem)^{n}}{(-1+bem)^{n}(1+bem)^{2n+1}}, \qquad y_{12n+4} = -\frac{o\ (-1+cfko)^{2n+1}(1+cfko)^{n}}{(-1+2cfko)^{n}},$$
$$y_{12n+5} = -\frac{a\ (-1+2adgl)^{2n+1}}{(-1+adgl)^{2n+2}(-1+3adgl)^{n}}, \qquad y_{12n+6} = -\frac{b\ (-1+behm)^{n+1}(1+behm)^{2n+1}}{(1+2behm)^{n}},$$
$$y_{12n+7} = -\frac{c\ (-1+2cfko)^{n+1}}{(-1+cfko)^{2n}(1+cfko)^{n}}, \qquad y_{12n+8} = \frac{d\ (-1+adgl)^{n+1}(-1+3adgl)^{2n+1}}{(1-2adgl)^{2n+2}},$$
$$z_{12n-3} = \frac{k\ (-1+2cfko)^{n}}{(-1+cfko)^{2n}(1+cfko)^{n}}, \qquad z_{12n-2} = \frac{l\ (-1+adgl)^{2n}(-1+3adgl)^{n}}{(1-2adgl)^{2n}},$$
$$z_{12n-1} = \frac{m\ (1+2behm)^{n}}{(-1+behm)^{n}(1+behm)^{2n}}, \qquad z_{12n} = \frac{o\ (-1+cfko)^{2n}(1+cfko)^{n}}{(-1+2cfko)^{n}},$$
$$z_{12n+1} = -\frac{a\ (1-2adgl)^{2n}}{(-1+adgl)^{2n+1}\ (-1+3adgl)^{n}}, \qquad z_{12n+2} = -\frac{b\ (-1+behm)^{n+1}(1+behm)^{2n}}{(1+2behm)^{n}},$$
$$z_{12n+3} = \frac{c\ (-1+2cfko)^{n+1}}{(-1+cfko)^{2n+1}(1+cfko)^{n}}, \qquad z_{12n+4} = \frac{d\ (-1+3adgl)^{n+1}(-1+adgl)^{2n}}{(-1+2adgl)^{2n+1}},$$
$$z_{12n+5} = \frac{e\ (1+2behm)^{n+1}}{(-1+behm)^{n+1}(1+behm)^{2n+1}}, \qquad z_{12n+6} = -\frac{f\ (-1+cfko)^{2n+1}(1+cfko)^{n+1}}{(-1+2cfko)^{n+1}},$$
$$z_{12n+7} = \frac{g\ (-1+2adgl)^{2n+1}}{(-1+adgl)^{2n+1}(-1+3adgl)^{n+1}}, \qquad z_{12n+8} = \frac{h\ (-1+behm)^{n+1}(1+behm)^{2n+1}}{(1+2behm)^{n+1}}.$$

**Theorem 3.3.** *Let* $\{x_n,\ y_n,\ z_n\}$ *be solutions of the system*

$$x_{n+1} = \frac{y_{n-3}}{-1+y_{n-3}z_{n-1}x_{n-2}y_n},\ y_{n+1} = \frac{z_{n-3}}{-1-z_{n-3}x_{n-1}y_{n-2}z_n},\ z_{n+1} = \frac{x_{n-3}}{-1-x_{n-3}y_{n-1}z_{n-2}x_n},$$

*then we get*

$$x_{12n-3} = \frac{a\ (-1+2adgl)^{n}}{(-1+adgl)^{2n}\ (1+adgl)^{n}}, \qquad x_{12n-2} = \frac{b\ (-1+behm)^{2n}\ (1+behm)^{n}}{(-1+2behm)^{n}},$$

$$x_{12n-1}=\frac{(-1)^{n}\ c\ (1+2cfko)^{2n}}{(1+cfko)^{2n}\ (1+3cfko)^{n}},\qquad x_{12n}=\frac{d\ (-1+adgl)^{2n}\ (1+adgl)^{n}}{(-1+2adgl)^{n}},$$

$$x_{12n+1}=\frac{e\ (-1+2behm)^{n}}{(-1+behm)^{2n+1}\ (1+behm)^{n}},\qquad x_{12n+2}=\frac{(-1)^{n+1}f\ (1+cfko)^{2n+1}\ (1+3cfko)^{n}}{(1+2cfko)^{2n+1}},$$

$$x_{12n+3}=\frac{g\ (-1+2adgl)^{n}}{(-1+adgl)^{2n+1}\ (1+adgl)^{n}},\qquad x_{12n+4}=\frac{h\ (-1+behm)^{2n+1}\ (1+behm)^{n}}{(-1+2behm)^{n}},$$

$$x_{12n+5}=\frac{(-1)^{n}k\ (1+2cfko)^{2n+1}}{(1+cfko)^{2n+2}\ (1+3cfko)^{n}},\qquad x_{12n+6}=-\frac{l\ (-1+adgl)^{2n+1}\ (1+adgl)^{n+1}}{(-1+2adgl)^{n}},$$

$$x_{12n+7}=\frac{(-1)^{n}m\ (1-2behm)^{n+1}}{(-1+behm)^{2n+2}\ (1+behm)^{n}},\qquad x_{12n+8}=\frac{(-1)^{n}o\ (1+cfko)^{2n+1}\ (1+3cfko)^{n+1}}{(1+2cfko)^{2n+2}},$$

$$y_{12n-3}=\frac{e\ (-1+2behm)^{n}}{(1+behm)^{2n}(1+behm)^{n}},\qquad y_{12n-2}=\frac{(-1)^{n}f\ (1+cfko)^{2n}(1+3cfko)^{n}}{(1+2cfko)^{2n}},$$

$$y_{12n-1}=\frac{g\ (-1+2adgl)^{n}}{(-1+adgl)^{2n}(1+adgl)^{n}},\qquad y_{12n}=\frac{h\ (-1+behm)^{2n}(1+behm)^{n}}{(-1+2behm)^{n}},$$

$$y_{12n+1}=\frac{(-1)^{n+1}k(1+2cfko)^{2n}}{(1+cfko)^{2n+1}(1+3cfko)^{n}},\qquad y_{12n+2}=-\frac{l\ (1+adgl)^{n+1}(-1+adgl)^{2n}}{(-1+2adgl)^{n}},$$

$$y_{12n+3}=\frac{(-1)^{n}m\ (1-2bem)^{n+1}}{(-1+bem)^{2n+1}(1+bem)^{n}},\qquad y_{12n+4}=\frac{(-1)^{n+1}o\ (1+cfko)^{2n}(1+3cfko)^{n+1}}{(1+2cfko)^{2n+1}},$$

$$y_{12n+5}=\frac{a\ (-1+2adgl)^{n+1}}{(-1+adgl)^{2n+1}(1+adgl)^{n+1}},\qquad y_{12n+6}=\frac{b\ (-1+behm)^{2n+1}(1+behm)^{n+1}}{(-1+2behm)^{n+1}},$$

$$y_{12n+7}=\frac{(-1)^{n}c\ (1+2cfko)^{2n+1}}{(1+cfko)^{2n+1}(1+3cfko)^{n+1}},\qquad y_{12n+8}=\frac{d\ (-1+adgl)^{2n+1}(1+adgl)^{n+1}}{(-1+2adgl)^{n+1}},$$

$$z_{12n-3}=\frac{(-1)^{n}k\ (1+2cfko)^{2n}}{(1+cfko)^{2n}(1+3cfko)^{n}},\qquad z_{12n-2}=\frac{l\ (-1+adgl)^{2n}(1+adgl)^{n}}{(-1+2adgl)^{n}},$$

$$z_{12n-1}=\frac{m\ (-1+2behm)^{n}}{(-1+behm)^{2n}(1+behm)^{n}},\qquad z_{12n}=\frac{(-1)^{n}o\ (1+cfko)^{2n}(1+3cfko)^{n}}{(1+2cfko)^{2n}},$$

$$z_{12n+1}=-\frac{a\ (-1+2adgl)^{n}}{(-1+adgl)^{2n}\ (1+adgl)^{n+1}},\qquad z_{12n+2}=-\frac{b\ (-1+behm)^{2n+1}(1+behm)^{n}}{(-1+2behm)^{n+1}},$$

$$z_{12n+3}=\frac{(-1)^{n+1}c\ (1+2cfko)^{2n+1}}{(1+cfko)^{2n}(1+3cfko)^{n+1}},\qquad z_{12n+4}=-\frac{d\ (-1+adgl)^{2n+1}(1+adgl)^{n}}{(-1+2adgl)^{n+1}},$$

$$z_{12n+5}=-\frac{e\ (-1+2behm)^{n}}{(-1+behm)^{2n+1}(1+behm)^{n+1}},\qquad z_{12n+6}=\frac{(-1)^{n}f\ (1+cfko)^{2n+2}(1+3cfko)^{n}}{(1+2cfko)^{2n+1}},$$

$$z_{12n+7}=-\frac{g\ (-1+2adgl)^{n}}{(-1+adgl)^{2n+1}(1+adgl)^{n+1}},\qquad z_{12n+8}=-\frac{h\ (-1+behm)^{2n+2}(1+behm)^{n}}{(-1+2behm)^{n+1}}.$$

**Theorem 3.4.** *Let $\{x_n,\ y_n,\ z_n\}$ be solutions of the system*

$$x_{n+1}=\frac{y_{n-3}}{-1+y_{n-3}z_{n-1}x_{n-2}y_n},\ y_{n+1}=\frac{z_{n-3}}{1-z_{n-3}x_{n-1}y_{n-2}z_n},\ z_{n+1}=\frac{x_{n-3}}{-1-x_{n-3}y_{n-1}z_{n-2}x_n},$$

*then we see that*

$$x_{12n-3}=a\prod_{i=0}^{n-1}\frac{(1+12iadgl)(1+(12i+4)adgl)(1+(12i+8)adgl)}{(1+(12i+1)adgl)(1+(12i+5)adgl)(1+(12i+9)adgl)},$$

$$x_{12n-2}=b\prod_{i=0}^{n-1}\frac{(-1+(12i+1)behm)(-1+(12i+5)behm)(-1+(12i+9)behm)}{(-1+(12i+2)behm)(-1+(12i+6)behm)(-1+(12i+10)behm)},$$

$$x_{12n-1}=c\prod_{i=0}^{n-1}\frac{(-1+(12i+2)cfko)(-1+(12i+6)cfko)(-1+(12i+10)cfko)}{(-1+(12i+3)cfko)(-1+(12i+7)cfko)(-1+(12i+11)cfko)},$$

$$x_{12n}=d\prod_{i=0}^{n-1}\frac{(1+(12i+3)adgl)(1+(12i+7)adgl)(1+(12i+11)adgl)}{(1+(12i+4)adgl)(1+(12i+8)adgl)(1+(12i+12)adgl)},$$

$$x_{12n+1}=\frac{e}{(-1+behm)}\prod_{i=0}^{n-1}\frac{(-1+(12i+4)behm)(-1+(12i+8)behm)(-1+(12i+12)behm)}{(-1+(12i+5)behm)(-1+(12i+9)behm)(-1+(12i+13)behm)},$$

$$x_{12n+2}=-\frac{f\ (-1+cfko)}{(-1+2cfko)}\prod_{i=0}^{n-1}\frac{(-1+(12i+5)cfko)(-1+(12i+9)cfko)(-1+(12i+13)cfko)}{(-1+(12i+6)cfko)(-1+(12i+10)cfko)(-1+(12i+14)cfko)},$$

$$x_{12n+3}=-\frac{g\ (1+2adgl)}{(1+3adgl)}\prod_{i=0}^{n-1}\frac{(1+(12i+6)adgl)(1+(12i+10)adgl)(1+(12i+14)adgl)}{(1+(12i+7)adgl)(1+(12i+11)adgl)(1+(12i+15)adgl)},$$

$$x_{12n+4} = -\frac{h\ (-1+3behm)}{(-1+4behm)}\prod_{i=0}^{n-1}\frac{(-1+(12i+7)behm)(-1+(12i+11)behm)(-1+(12i+15)behm)}{(-1+(12i+8)behm)(-1+(12i+12)behm)(-1+(12i+16)behm)},$$

$$x_{12n+5} = \frac{k\ (-1+4cfko)}{(-1+cfko)(-1+5cfko)}\prod_{i=0}^{n-1}\frac{(-1+(12i+8)cfko)(-1+(12i+12)cfko)(-1+(12i+16)cfko)}{(-1+(12i+9)cfko)(-1+(12i+13)cfko)(-1+(12i+17)cfko)},$$

$$x_{12n+6} = -\frac{l\ (1+adgl)(1+5adgl)}{(1+2adgl)(1+6adgl)}\prod_{i=0}^{n-1}\frac{(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)}{(1+(12i+10)adgl)(1+(12i+14)adgl)(1+(12i+18)adgl)},$$

$$x_{12n+7} = -\frac{m\ (-1+2behm)(-1+6behm)}{(-1+3behm)(-1+7behm)}\prod_{i=0}^{n-1}\frac{(-1+(12i+10)behm)(-1+(12i+14)behm)(-1+(12i+18)behm)}{(-1+(12i+11)behm)(-1+(12i+15)behm)(-1+(12i+19)behm)},$$

$$x_{12n+8} = -\frac{o\ (-1+3cfko)(-1+7cfko)}{(-1+4cfko)(-1+8cfko)}\prod_{i=0}^{n-1}\frac{(-1+(12i+11)cfko)(-1+(12i+15)cfko)(-1+(12i+19)cfko)}{(-1+(12i+12)cfko)(-1+(12i+16)cfko)(-1+(12i+20)cfko)},$$

$$y_{12n-3} = -e\prod_{i=0}^{n-1}\frac{(-1+(12i)behm)(-1+(12i+4)behm)(-1+(12i+8)behm)}{(-1+(12i+1)behm)(-1+(12i+5)behm)(-1+(12i+9)behm)},$$

$$y_{12n-2} = f\prod_{i=0}^{n-1}\frac{(-1+(12i+1)cfko)(-1+(12i+5)cfko)(-1+(12i+9)cfko)}{(-1+(12i+2)cfko)(-1+(12i+6)cfko)(-1+(12i+10)cfko)},$$

$$y_{12n-1} = g\prod_{i=0}^{n-1}\frac{(1+(12i+2)adgl)(1+(12i+6)adgl)(1+(12i+10)adgl)}{(1+(12i+3)adgl)(1+(12i+7)adgl)(1+(12i+11)adgl)},$$

$$y_{12n} = h\prod_{i=0}^{n-1}\frac{(-1+(12i+3)behm)(-1+(12i+7)behm)(-1+(12i+11)behm)}{(-1+(12i+4)behm)(-1+(12i+8)behm)(-1+(12i+12)behm)},$$

$$y_{12n+1} = -\frac{k}{(-1+cfko)}\prod_{i=0}^{n-1}\frac{(-1+(12i+4)cfko)(-1+(12i+8)cfko)(-1+(12i+12)cfko)}{(-1+(12i+5)cfko)(-1+(12i+9)cfko)(-1+(12i+13)cfko)},$$

$$y_{12n+2} = \frac{l\ (1+adgl)}{(1+2adgl)}\prod_{i=0}^{n-1}\frac{(1+(12i+5)adgl)(1+(12i+9)adgl)(1+(12i+13)adgl)}{(1+(12i+6)adgl)(1+(12i+10)adgl)(1+(12i+14)adgl)},$$

$$y_{12n+3} = \frac{m\ (-1+2behm)}{(-1+3behm)}\prod_{i=0}^{n-1}\frac{(-1+(12i+6)behm)(-1+(12i+10)behm)(-1+(12i+14)behm)}{(-1+(12i+7)behm)(-1+(12i+11)behm)(-1+(12i+15)behm)},$$

$$y_{12n+4} = \frac{o\ (-1+3cfko)}{(-1+4cfko)}\prod_{i=0}^{n-1}\frac{(-1+(12i+7)cfko)(-1+(12i+11)cfko)(-1+(12i+15)cfko)}{(-1+(12i+8)cfko)(-1+(12i+12)cfko)(-1+(12i+16)cfko)},$$

$$y_{12n+5} = -\frac{a\ (1+4adgl)}{(1+adgl)(1+5adgl)}\prod_{i=0}^{n-1}\frac{(1+(12i+8)adgl)(1+(12i+12)adgl)(1+(12i+16)adgl)}{(1+(12i+9)adgl)(1+(12i+13)adgl)(1+(12i+17)adgl)},$$

$$y_{12n+6} = -\frac{b\ (-1+behm)(-1+5behm)}{(-1+2behm)(-1+6behm)}\prod_{i=0}^{n-1}\frac{(-1+(12i+9)behm)(-1+(12i+13)behm)(-1+(12i+17)behm)}{(-1+(12i+10)behm)(-1+(12i+14)behm)(-1+(12i+18)behm)},$$

$$y_{12n+7} = -\frac{c\ (-1+2cfko)(-1+6cfko)}{(-1+3cfko)(-1+7cfko)}\prod_{i=0}^{n-1}\frac{(-1+(12i+10)cfko)(-1+(12i+14)cfko)(-1+(12i+18)cfko)}{(-1+(12i+11)cfko)(-1+(12i+15)cfko)(-1+(12i+19)cfko)},$$

$$y_{12n+8} = -\frac{d\ (1+3adgl)(1+7adgl)}{(1+4adgl)(1+8adgl)}\prod_{i=0}^{n-1}\frac{(1+(12i+11)adgl)(1+(12i+15)adgl)(1+(12i+19)adgl)}{(1+(12i+12)adgl)(1+(12i+16)adgl)(1+(12i+20)adgl)},$$

$$z_{12n-3} = -k\prod_{i=0}^{n-1}\frac{(-1+(12i)cfko)(-1+(12i+4)cfko)(-1+(12i+8)cfko)}{(-1+(12i+1)cfko)(-1+(12i+5)cfko)(-1+(12i+9)cfko)},$$

$$z_{12n-2} = l\prod_{i=0}^{n-1}\frac{(1+(12i+1)adgl)(1+(12i+5)adgl)(1+(12i+9)adgl)}{(1+(12i+2)adgl)(1+(12i+6)adgl)(1+(12i+10)adgl)},$$

$$z_{12n-1} = m\prod_{i=0}^{n-1}\frac{(-1+(12i+2)behm)(-1+(12i+6)behm)(-1+(12i+10)behm)}{(-1+(12i+3)behm)(-1+(12i+7)behm)(-1+(12i+11)behm)},$$

$$z_{12n} = o\prod_{i=0}^{n-1}\frac{(-1+(12i+3)cfko)(-1+(12i+7)cfko)(-1+(12i+11)cfko)}{(-1+(12i+4)cfko)(-1+(12i+8)cfko)(-1+(12i+12)cfko)},$$

$$z_{12n+1} = -\frac{a}{(1+adgl)} \prod_{i=0}^{n-1} \frac{(1+(12i+4)adgl)(1+(12i+8)adgl)(1+(12i+12)adgl)}{(1+(12i+5)adgl)(1+(12i+9)adgl)(1+(12i+13)adgl)},$$

$$z_{12n+2} = -\frac{b\ (-1+behm)}{(-1+2behm)} \prod_{i=0}^{n-1} \frac{(-1+(12i+5)behm)(-1+(12i+9)behm)(-1+(12i+13)behm)}{(-1+(12i+6)behm)(-1+(12i+10)behm)(-1+(12i+14)behm)},$$

$$z_{12n+3} = -\frac{c\ (-1+2cfko)}{(-1+3cfko)} \prod_{i=0}^{n-1} \frac{(-1+(12i+6)cfko)(-1+(12i+10)cfko)(-1+(12i+14)cfko)}{(-1+(12i+7)cfko)(-1+(12i+11)cfko)(-1+(12i+15)cfko)},$$

$$z_{12n+4} = -\frac{d\ (1+3adgl)}{(1+4adgl)} \prod_{i=0}^{n-1} \frac{(1+(12i+7)adgl)(1+(12i+11)adgl)(1+(12i+15)adgl)}{(1+(12i+8)adgl)(1+(12i+12)adgl)(1+(12i+16)adgl)},$$

$$z_{12n+5} = -\frac{e\ (-1+4behm)}{(-1+behm)(-1+5behm)} \prod_{i=0}^{n-1} \frac{(-1+(12i+8)behm)(-1+(12i+12)behm)(-1+(12i+16)behm)}{(-1+(12i+9)behm)(-1+(12i+13)behm)(-1+(12i+17)behm)},$$

$$z_{12n+6} = \frac{f\ (-1+cfko)(-1+5cfko)}{(-1+2cfko)(-1+6cfko)} \prod_{i=0}^{n-1} \frac{(-1+(12i+9)cfko)(-1+(12i+13)cfko)(-1+(12i+17)cfko)}{(-1+(12i+10)cfko)(-1+(12i+14)cfko)(-1+(12i+18)cfko)},$$

$$z_{12n+7} = \frac{g\ (1+2adgl)(1+6adgl)}{(1+3adgl)(1+7adgl)} \prod_{i=0}^{n-1} \frac{(1+(12i+10)adgl)(1+(12i+14)adgl)(1+(12i+18)adgl)}{(1+(12i+11)adgl)(1+(12i+15)adgl)(1+(12i+19)adgl)},$$

$$z_{12n+8} = \frac{h\ (-1+3behm)(-1+7behm)}{(-1+4behm)(-1+8behm)} \prod_{i=0}^{n-1} \frac{(-1+(12i+11)behm)(-1+(12i+15)behm)(-1+(12i+19)behm)}{(-1+(12i+12)behm)(-1+(12i+16)behm)(-1+(12i+20)behm)},$$

*where* $\prod_{i=0}^{-1} A_i = 1$.

## 4. On the system: $x_{n+1} = \frac{y_{n-3}}{\pm 1 + y_n z_{n-1} x_{n-2} y_{n-3}}$, $y_{n+1} = \frac{z_{n-3}}{\pm 1 + z_n x_{n-1} y_{n-2} z_{n-3}}$, $z_{n+1} = \frac{x_{n-3}}{\pm 1 + x_n y_{n-1} z_{n-2} x_{n-3}}$

In this section, we study the solutions and periodicity of the following system

$$x_{n+1} = \frac{y_{n-3}}{-1+y_n z_{n-1} x_{n-2} y_{n-3}},\ y_{n+1} = \frac{z_{n-3}}{-1+z_n x_{n-1} y_{n-2} z_{n-3}},\ z_{n+1} = \frac{x_{n-3}}{-1+x_n y_{n-1} z_{n-2} x_{n-3}}, \tag{4.1}$$

with nonzero real initial conditions.

**Theorem 4.1.** *The solutions of system* (4.1) *are given by the following formulas*

$$x_{12n-3} = \frac{a}{(-1+adgl)^{3n}}, \quad x_{12n-2} = b\ (-1+behm)^{3n}, \quad x_{12n-1} = \frac{c}{(-1+cfko)^{3n}},$$
$$x_{12n} = d\ (-1+adgl)^{3n}, \quad x_{12n+1} = \frac{e}{(-1+behm)^{3n+1}}, \quad x_{12n+2} = f\ (-1+cfko)^{3n+1},$$
$$x_{12n+3} = \frac{g}{(-1+adgl)^{3n+1}}, \quad x_{12n+4} = h\ (-1+behm)^{3n+1}, \quad x_{12n+5} = \frac{k}{(-1+cfko)^{3n+2}},$$
$$x_{12n+6} = l\ (-1+adgl)^{3n+2}, \quad x_{12n+7} = \frac{m}{(-1+behm)^{3n+2}}, \quad x_{12n+8} = o\ (-1+cfko)^{3n+2},$$
$$y_{12n-3} = \frac{e}{(-1+behm)^{3n+1}}, \quad y_{12n-2} = f\ (-1+cfko)^{3n+1}, \quad y_{12n-1} = \frac{g}{(-1+adgl)^{3n}},$$
$$y_{12n} = h\ (-1+behm)^{3n}, \quad y_{12n+1} = \frac{k}{(-1+cfko)^{3n+1}}, \quad y_{12n+2} = l\ (-1+adgl)^{3n+1},$$
$$y_{12n+3} = \frac{m}{(-1+behm)^{3n+1}}, \quad y_{12n+4} = o\ (-1+cfko)^{3n+1}, \quad y_{12n+5} = \frac{a}{(-1+adgl)^{3n+2}},$$
$$y_{12n+6} = b\ (-1+behm)^{3n+2}, \quad y_{12n+7} = \frac{c}{(-1+cfko)^{3n+2}}, \quad y_{12n+8} = d\ (-1+adgl)^{3n+2},$$
$$z_{12n-3} = \frac{k}{(-1+cfko)^{3n}}, \quad z_{12n-2} = \ (-1+adgl)^{3n}, \quad z_{12n-1} = \frac{m}{(-1+behm)^{3n}},$$
$$z_{12n} = o\ (-1+cfko)^{3n}, \quad z_{12n+1} = \frac{a}{(-1+adgl)^{3n+1}}, \quad z_{12n+2} = b\ (-1+behm)^{3n+1},$$
$$z_{12n+3} = \frac{c}{(-1+cfko)^{3n+1}}, \quad z_{12n+4} = d\ (-1+adgl)^{3n+1}, \quad z_{12n+5} = \frac{e}{(-1+behm)^{3n+2}},$$
$$z_{12n+6} = f\ (-1+cfko)^{3n+2}, \quad z_{12n+7} = \frac{g}{(-1+adgl)^{3n+2}}, \quad z_{12n+8} = h\ (-1+behm)^{3n+2},$$

*for* $n = 0, 1, 2, \ldots$ .

**Lemma 4.2.** *The solutions of system* (4.1) *are unbounded except in the following two theorems.*

**Theorem 4.3.** *System* (4.1) *has a periodic solution of period twelve, iff* $adgl = behm = cfko = -2$ *and it will take the following form*

$$\{x_n\} = \{x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, ...\},$$
$$\{y_n\} = \{y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, ...\},$$
$$\{z_n\} = \{z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, ...\}.$$

**Theorem 4.4.** *System* (4.1) *has a periodic solution of period four, iff* $x_{-i} = y_{-i} = z_{-i}$, $i = 0, 1, 2, 3$, $adgl = -2$ *and will be in the form* $\{x_n\} = \{y_n\} = \{z_n\} = \{x_{-3}, x_{-2}, x_{-1}, x_0, x_{-3}, x_{-2}, ...\}$.

## 5. Numerical examples

Here we consider some numerical examples for the previous systems to illustrate the results.

**Example 5.1.** We consider numerical example for the difference equations system (2.1) with the initial conditions $x_{-3} = 0.5$, $x_{-2} = 0.11$, $x_{-1} = -0.28$, $x_0 = 1.3$, $y_{-3} = 0.3$, $y_{-2} = 2$, $y_{-1} = 0.2$, $y_0 = 5$, $z_{-3} = 0.8$, $z_{-2} = 0.4$, $z_{-1} = -0.1$ and $z_0 = 0.7$, (see Figure 1).

**Example 5.2.** We assume the system of difference equation (2.2) with the initial conditions $x_{-3} = 0.5$, $x_{-2} = 0.11$, $x_{-1} = 0.28$, $x_0 = -1.3$, $y_{-3} = 0.21$, $y_{-2} = 0.52$, $y_{-1} = 0.2$, $y_0 = 0.31$, $z_{-3} = -0.6$, $z_{-2} = 0.4$, $z_{-1} = -0.1$ and $z_0 = 0.7$, (see Figure 2).

**Example 5.3.** We consider the difference system (3.1) with the initial conditions $x_{-3} = 0.5$, $x_{-2} = 0.11$, $x_{-1} = -0.28$, $x_0 = 1.3$, $y_{-3} = 0.21$, $y_{-2} = 2$, $y_{-1} = 0.2$, $y_0 = 0.75$, $z_{-3} = 0.6$, $z_{-2} = 0.4$, $z_{-1} = -0.1$ and $z_0 = 0.7$, (see Figure 3).

**Example 5.4.** Suppose that the initial conditions are $x_{-3} = 5$, $x_{-2} = -6$, $x_{-1} = 0.8$, $x_0 = -3$, $y_{-3} = -7$, $y_{-2} = 4$, $y_{-1} = 0.7$, $y_0 = -3$, $z_{-3} = 5$, $z_{-2} = -20/105$, $z_{-1} = -1/63$ and $z_0 = 1/8$, for the system (4.1), (see Figure 4).

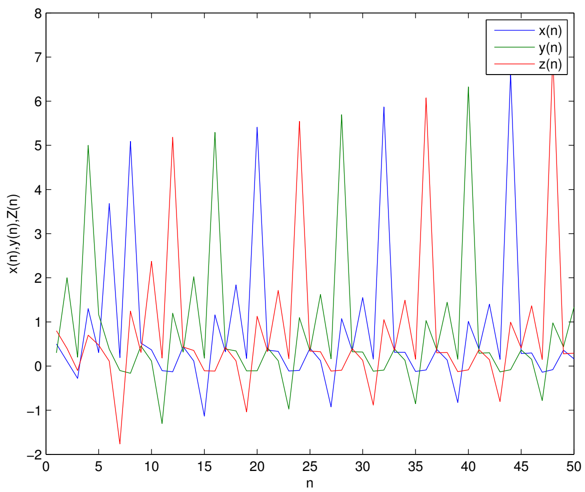


Figure 1: Plot of the system (2.1).

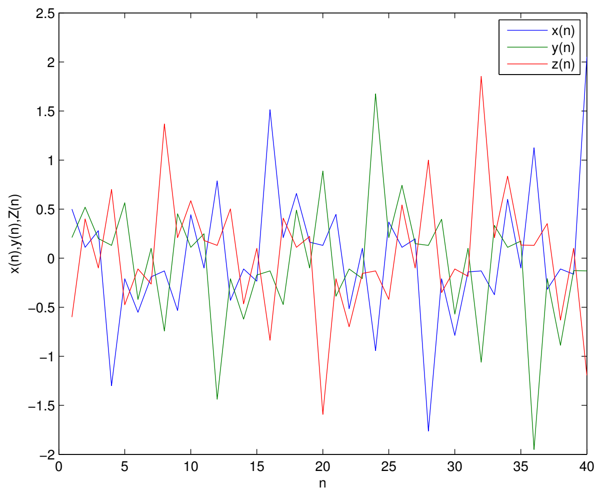


Figure 2: Plot of the system (2.2).

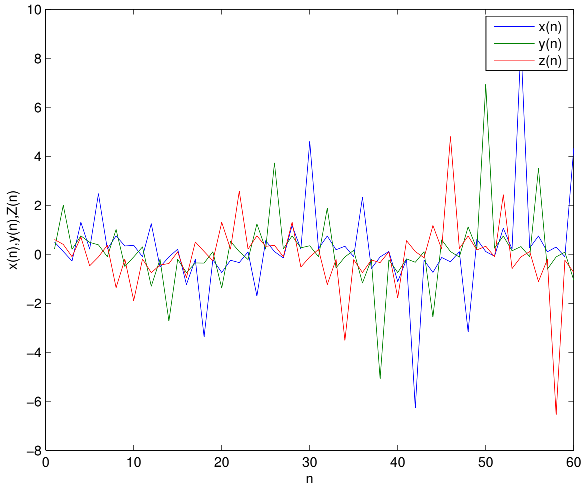


Figure 3: Plot of the system (3.1).

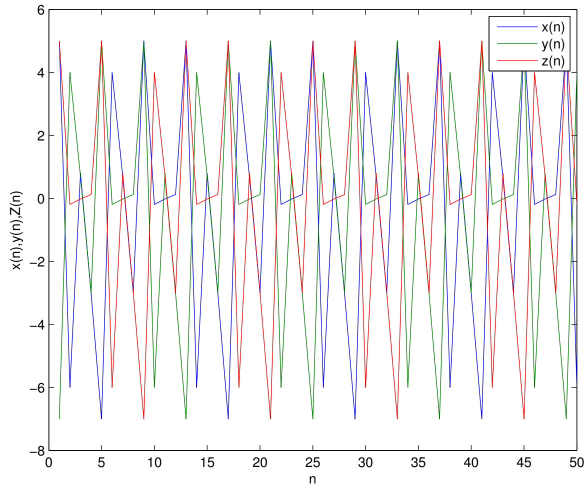


Figure 4: Plot of the system (4.1).

## Acknowledgment

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

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