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Nonexistence of global weak solutions of a system of wave equations on the Heisenberg group

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Abstract

Sufficient conditions are obtained for the nonexistence of solutions to the nonlinear higher order pseudo-parabolic equation

$$u_{tt} - \Delta_{\mathbb{H}} u + (-\Delta_{\mathbb{H}})^{\delta/2} u = f(\eta, t) u^p, \quad (\eta, t) \in \mathbb{H} \times (0, \infty),\ p > 1,\ u \geq 0,$$

where $\Delta_{\mathbb{H}}$ is the Kohn–Laplace operator on the $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$ and $f(\eta, t)$ is a given function. Then, this result is extended to the case of a 2×2-system of the same type. Our technique of proof is based on a duality argument.

Keywords: Nonexistence, nonlinear hyperbolic equation, systems of hyperbolic equations, Heisenberg group.
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1. Introduction and preliminaries

In this paper, we are first concerned with the nonexistence of global weak solutions to the nonlinear hyperbolic equation

$$u_{tt} - \Delta_{\mathbb{H}} u + (-\Delta_{\mathbb{H}})^{\frac{\delta}{2}} u = f(\eta, t) u^p, \quad (\eta, t) \in \mathbb{H} \times (0, \infty),\ p > 1,\ u \geq 0, \tag{1.1}$$

equipped with the initial conditions

$$u(\eta, 0) = u_0(\eta),\ u_t(\eta, 0) = u_1(\eta),\ \eta \in \mathbb{H}, \tag{1.2}$$

where $(-\Delta_{\mathbb{H}})^{\delta/2}$, $0 < \delta < 2$ is the fractional power of the Kohn–Laplace operator $\Delta_{\mathbb{H}}$ on the $(2N+1)$-

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dimensional Heisenberg group $\mathbb{H}$. Then we extend our analysis to the 2×2 system

$$\begin{cases} u_{tt} - \Delta_{\mathbb{H}} u + (-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}} u = f(\eta,t) v^q, & (\eta,t) \in \mathbb{H} \times (0,\infty), \\ v_{tt} - \Delta_{\mathbb{H}} v + (-\Delta_{\mathbb{H}})^{\frac{\beta}{2}} v = g(\eta,t) u^p, & (\eta,t) \in \mathbb{H} \times (0,\infty), \\ u(\eta,0) = u_0(\eta),\ u_1(\eta,0) = u_1(\eta), & \eta \in \mathbb{H}, \\ v(\eta,0) = v_0(\eta), v(\eta,0) = v_1(\eta), & \eta \in \mathbb{H}, \end{cases} \tag{1.3}$$

where $u, v > 0$ and $p, q > 1$, and $f(\eta,t)$, $g(\eta,t)$ are given functions.

Before we state and prove our results, let us dwell a while on the existing literature. In the Euclidean case, hyperbolic equations and systems are well documented, one is referred to the valuable books [2], [16]. However works on hyperbolic equations (systems) on the Heisenberg group are scarce. The study of evolution equations on the Heisenberg group started with the paper of Nachman [11] and then followed by Müller [10] , Zhang [17], Pascucci [12], Véron and Pokhozhaev [14], Elhamidi and Kirane [4], Elhamidi and Obeid [5], D'Ambrosio [3], Goldstein and Kombe [7] and Mokrani [9].

For the reader convenience, we recall some background facts that will be used in the sequel.

The $(2N+1)$-dimensional Heisenberg group $\mathbb{H}$ is the space $\mathbb{R}^{2N+1}$ equipped with the group operation

$$\eta \circ \eta' = (x + x', y + y', \tau + \tau' + 2(x \cdot y' - x' \cdot y)),$$

for all $\eta = (x, y, \tau), \eta' = (x', y', \tau') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, where $\cdot$ denotes the standard scalar product in $\mathbb{R}^N$. This group operation endows $\mathbb{H}$ with the structure of a Lie group.

On $\mathbb{H}$ it is natural to define a distance from $\eta = (x, y, \tau) =: (z, \tau)$ to the origin by

$$|\eta|_{\mathbb{H}} = \left(\tau^2 + \left(\sum_{i=1}^{N} (x_i^2 + y_i^2) \right)^2 \right)^{1/4} = \left(\tau^2 + |z|^4 \right)^{1/4},$$

where $x = (x_1, \ldots, x_N)$ and $y = (y_1, \ldots, y_N)$.

The Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}$ can be defined from the vectors fields

$$X_i = \partial_{x_i} + 2y_i \partial_\tau \quad \text{and} \quad Y_i = \partial_{y_i} - 2x_i \partial_\tau$$

for $i = 1, \ldots, N$, as follows

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} (X_i^2 + Y_i^2).$$

A simple computation gives the expression

$$\Delta_{\mathbb{H}} u = \sum_{i=1}^{N} \left(\partial^2_{x_i x_i} u + \partial^2_{y_i y_i} u + 4y_i \partial^2_{x_i \tau} u - 4x_i \partial^2_{y_i \tau} u + 4(x_i^2 + y_i)^2 \partial^2_{\tau\tau} u \right).$$

The operator $\Delta_{\mathbb{H}}$ has the following properties:

- It is invariant with respect to the left multiplication in the group, i.e., for all $\eta, \eta' \in \mathbb{H}$, we have
$$\Delta_{\mathbb{H}}(u(\eta \circ \eta')) = \Delta_{\mathbb{H}} u(\eta \circ \eta');$$
- It is homogeneous with respect to dilatation. More precisely, for $\lambda \in \mathbb{R}$ and $(x, y, \tau) \in \mathbb{H}$, we have
$$\Delta_{\mathbb{H}}(u(\lambda x, \lambda y, \lambda^2 \tau)) = \lambda^2 (\Delta_{\mathbb{H}} u)(\lambda x, \lambda y, \lambda^2 \tau);$$
- If $u(\eta) = v(|\eta|_{\mathbb{H}})$, then
$$\Delta_{\mathbb{H}} v(\rho) = a(\eta) \left(\frac{d^2 v}{d\rho^2} + \frac{Q-1}{\rho} \frac{dv}{d\rho} \right),$$
where $\rho = |\eta|_{\mathbb{H}}$, $a(\eta) = \rho^{-2} \sum_{i=1}^{N} (x_i^2 + y_i^2)$ and $Q = 2N + 2$ is the homogeneous dimension of $\mathbb{H}$.

For more details on Heisenberg groups, we refer to [6]. We also need the following lemmas.

Lemma 1.1 ([1])**.** *Consider a convex function $F \in C^2(\mathbb{R})$ and assume that $0 \le \varphi \in C_0^\infty(\mathbb{R}^{2N+1})$. Then*

$$F'(\varphi)(-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}\varphi \ge (-\Delta_{\mathbb{H}})^{\frac{\alpha}{2}}F(\varphi) \quad for \quad 0<\alpha<2. \tag{1.4}$$

Remark 1.2. When $\alpha = 2$, the relation (1.4) is still valid as

$$\Delta_{\mathbb{H}}\varphi^\sigma = \sigma\phi^{\sigma-1}\Delta_{\mathbb{H}}\varphi + \sigma(\sigma-1)\varphi^{\sigma-2}\sum_{i=1}^{N}(|X_i\varphi|^2+|Y_i\varphi|^2) \ge \sigma\varphi^{\sigma-1}\Delta_{\mathbb{H}}\varphi.$$

Lemma 1.3 ([13])**.** *Let $f \in L^1(\mathbb{R}^{2N+1})$ be such that $\int_{\mathbb{R}^{2N+1}} f\,d\eta \ge 0$. Then there exists a test function $0 \le \varphi \le 1$ such that*

$$\int_{\mathbb{R}^{2N+1}} f\varphi\,d\eta \ge 0.$$

2. Results and proofs

Let $\mathcal{H}_\mathcal{T} = \mathbb{H} \times (0,T)$, $\mathcal{H} = \mathbb{H}\times(0,\infty)$. For $R>0$, let

$$\mathcal{C}_R = \{(x,y,\tau,t) \in \mathcal{H} :\ R^4 \le t^2 + |x|^4 + |y|^4 + \tau^2 \le 2R^4\}.$$

Define $L^p_{loc}(\mathcal{H}_\mathcal{T},\, f(\eta,t)\,d\eta dt) = \big\{u : \mathcal{H}_\mathcal{T} \mapsto \mathbb{R} \mid \int_K |u|^p f(\eta,t)\,d\eta dt < +\infty \text{ for any compact } K \subset \mathcal{H}_\mathcal{T}\big\}$.

2.1. The case of a single equation.

Here we consider the problem (1.1)–(1.2). Let us start with the following definition.

Definition 2.1. A local weak solution u of the equation (1.1) in $\mathbb{R}^{2N+1}_+$, equipped with the initial data $u(.,0) = u_0(\ .\ )$ and $u_t(.,0) = u_1(\ .\ )$, is a function

$$u \in C^1_B\big([0,T];(L^1_{loc}(\mathcal{H}B)\big) \cap C((0,T);L^p_{loc}(\mathcal{H})) \cap L^1L^p_{loc}(\mathcal{H}, f(\eta,t)\,d\eta dt)),$$

satisfying

$$\begin{aligned} -\int_{\mathcal{H}_\mathcal{T}} u\Delta_{\mathbb{H}}\varphi\,d\eta dt + \int_{\mathcal{H}_\mathcal{T}} (-\Delta_{\mathbb{H}})^{\delta/2}\varphi u\ d\eta dt + \int_{\mathcal{H}_\mathcal{T}} u\varphi_{tt}\,d\eta dt \\ = \int_{\mathcal{H}_\mathcal{T}} fu^p\varphi\,d\eta dt + \int_{\mathbb{R}^{2N+1}} u_1(\eta)\varphi(\eta,0)\,d\eta - \int_{\mathbb{R}^{2N+1}} u_0(\eta)\varphi_t(\eta,0)\,d\eta \end{aligned} \tag{2.1}$$

for any test function $\varphi \in C^2_0(\mathcal{H}_\mathcal{T})$. We say that the solution is global when $T=+\infty$.

Theorem 2.2. *Let $p>1$ and*

$$\liminf_{R\to\infty}\int_{|\eta|<R} u_1(\eta)\,d\eta > 0.$$

Then the problem (1.1)–(1.2) does not admit a global weak solution defined on $\mathbb{R}^{2N+1}_+$ if

$$1 < p \le \frac{Q+\delta-1+\delta(2\lambda+1)}{Q-1} = p^*.$$

Proof. Let u be a global weak solution and φ a regular nonnegative test function chosen so that $\varphi_t(\eta,0)=0$.

From (2.1), we have

$$\int_{\mathbb{R}^{2N+1}} u_1(\eta)\varphi(\eta,0)\,d\eta + \int_{\mathcal{H}} f.u^p\varphi\,d\eta dt = \int_{\mathcal{H}} u\varphi_{tt}\,d\eta dt - \int_{\mathcal{H}} u\Delta_{\mathbb{H}}\varphi\,d\eta dt + \int_{\mathcal{H}_{\mathcal{T}}} u(-\Delta_{\mathbb{H}})^{\delta/2}\varphi\,d\eta dt. \tag{2.2}$$

Let us take φ^σ, $\sigma \gg 1$, rather than φ (this will be clear later) in (2.2); by using Young's inequality, we have

$$\int_{\mathcal{H}} u(\varphi^\sigma)_{tt}\,d\eta dt \le \frac{1}{6}\int_{\mathcal{H}} fu^p\varphi^\sigma + \; C_1\int_{\mathcal{H}} f^{-\frac{p'}{p}}\left(\varphi^{(\sigma-1-\frac{\sigma}{p})p'}|\varphi_{tt}|^{p'} + \varphi^{(\sigma-2-\frac{\sigma}{p})p'}|\varphi_t|^{2p'}\right)\,d\eta dt, \tag{2.3}$$

where $p+p' = pp'$. Now, using Lemma 1.3, we obtain the estimate

$$-\int_{\mathcal{H}} u\Delta_{\mathbb{H}}\varphi^\sigma d\eta dt \le \frac{1}{6}\int_{\mathcal{H}} fu^p\varphi^\sigma + C_2\int_{\mathcal{H}} \varphi^{(\sigma-1-\frac{\sigma}{p})p'}|\Delta_{\mathbb{H}}\varphi|^{p'} f^{-\frac{p'}{p}},$$

and similarly,

$$\int_{\mathcal{H}} u(-\Delta_{\mathbb{H}})^{\delta/2}\varphi^\sigma d\eta dt \le \frac{1}{6}\int_{\mathcal{H}} fu^p\varphi^\sigma + \; C_3\int_{\mathcal{H}} \varphi^{(\sigma-1-\frac{\sigma}{p})p'}|(-\Delta_{\mathbb{H}})^{\delta/2}\varphi|^{p'} f^{-\frac{p'}{p}}.$$

Collecting the above inequalities, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2N+1}} u_1(\eta)\varphi(\eta\;,\;0)\;d\eta + \int_{\mathcal{H}} fu^p\varphi\,d\eta\;dt \le C\Big\{ &\int_{\mathcal{H}} f^{\frac{-p'}{p}}(\varphi^{(\sigma-1-\frac{\sigma}{p})p'}|\varphi_{tt}|^{p'} + \varphi^{(\sigma-2-\frac{\sigma}{p})p'}|\varphi_t|^{2p'})\,d\eta\;dt \\ &+ \int_{\mathcal{H}} f^{-p'/p}(\varphi^{(\sigma-1-\frac{\sigma}{p})p'}|\Delta_{\mathbb{H}}\varphi|^{p'})\,d\eta\;dt \\ &+ \int_{\mathcal{H}} \varphi^{(\sigma-1-\frac{\sigma}{p})p'} f^{-p'/p}|(-\Delta)^{\frac{\delta}{2}}\varphi|^{p'}\,d\eta\;dt\Big\}.\end{aligned}$$

At this stage, we take

$$\varphi(\eta\;,t) = \Phi\Big(\frac{\tau^2+|x|^4+|y|^4+t^\alpha}{R^4}\Big) \quad , \quad \alpha = \frac{2}{\delta},$$

where $\Phi \in C_0^2(\mathbb{R}^+)$ is nonincreasing, $0 \le \Phi \le 1$, and

$$\Phi(r) = \begin{cases} 0 & if \quad\;\; r \ge 2, \\ 1 & if \quad 0 \le r \le 1. \end{cases}$$

We have

$$\begin{aligned}\Delta_{\mathbb{H}}\varphi(\eta\;,\;t) = &\frac{4(N+2)}{R^4}\Phi'(\varrho)\Big(|x|^2+|y|^2\Big) + \frac{16}{R^8}\Phi''(\varrho)\Big(|x|^6+|y|^6\Big) \\ &+ \tau^2\Big(|x|^2+|y|^2\Big) + 2\tau\langle x,y\rangle\Big(|x|^2-|y|^2\Big),\end{aligned}$$

where

$$\varrho = \frac{1}{R^4}\Big(\tau^2+|x|^4+|y|^4+t^\alpha\Big).$$

Passing now to the scaled variables

$$\widetilde{t} = R^{\frac{-4}{\alpha}}t, \quad \widetilde{\tau} = R^{-2}\tau, \quad \widetilde{x} = R^{-1}x, \quad \widetilde{y} = R^{-1}y$$

and setting

$$\begin{aligned}&\widetilde{\varrho} = \widetilde{\tau}^2 + |\widetilde{x}|^4 + |\widetilde{y}|^4 + \widetilde{t}^{\,\alpha}, \\ &\Omega = \{(\widetilde{\tau}\;,\;\widetilde{t})\;,\;\widetilde{\tau}^2 + |\widetilde{x}|^4 + |\widetilde{y}|^4 + \widetilde{t}^{\,\alpha} \le 2\}, \\ &\mathcal{T} = \{(\widetilde{\tau}\;,\;\widetilde{t})\;,\;1 \le \widetilde{\tau}^2 + |\widetilde{x}|^4 + |\widetilde{y}|^4 + \widetilde{t}^{\,\alpha} \le 2\},\end{aligned}$$

we obtain

$$\begin{aligned}
\left|\Delta_{\mathbb{H}}\varphi(\widetilde{\tau},\ \widetilde{t}\ )\right| &\leq CR^{-2},\ \text{for}\ \ (\widetilde{\tau},\ \widetilde{t}\ )\in\mathcal{T},\\
\left|(-\Delta_{\mathbb{H}})^{\frac{\delta}{2}}\varphi(\widetilde{\tau},\ \widetilde{t}\ )\right| &\leq CR^{-\delta},\ \text{for}\ (\widetilde{\tau},\ \widetilde{t}\ )\in\mathbb{R}_{+}^{2N+1},\\
\left|\varphi_t(\widetilde{\tau},\ \widetilde{t}\ )\right| &\leq CR^{-\frac{4}{\alpha}},\ \text{for}\ (\widetilde{\tau},\ \widetilde{t}\ )\in\mathcal{T},\\
\left|\varphi_{tt}(\widetilde{\tau},\ \widetilde{t}\ )\right| &\leq CR^{-\frac{2}{\alpha}},\ \text{for}\ (\widetilde{\tau},\ \widetilde{t}\ )\in\mathcal{T},
\end{aligned}$$

which, in turn, allow us to obtain the estimates

$$\begin{aligned}
&\int_{\mathcal{H}} f^{-\frac{p'}{p}}\left(\varphi^{(\sigma-1-\frac{\sigma}{p})p'}|\varphi_{tt}|^{p'}+\ \varphi^{(\sigma-2-\frac{\sigma}{p})p'}|\varphi_t|^{2p'}\right)d\eta\ dt\leq CR^{-(\gamma+\frac{4\lambda}{\alpha})\frac{p'}{p}-\frac{2}{\alpha}p'+2N+1+\frac{4}{\alpha}},\\
&\int_{\mathcal{H}} f^{-\frac{p'}{p}}\varphi^{(\sigma-1-\frac{\sigma}{p})p'}|\Delta_{\mathbb{H}}\varphi|^{p'}\,d\eta dt\leq CR^{-(\gamma+\frac{4\lambda}{\alpha})\frac{p'}{p}-2p'+2N+1+\frac{4}{\alpha}},\\
&\int_{\mathcal{H}} f^{-\frac{p'}{p}}\varphi^{(\sigma-1-\frac{\sigma}{p})p'}|(-\Delta_{\mathbb{H}})^{\delta/2}\varphi|^{p'}\,d\eta dt\leq CR^{-(\gamma+\frac{4\lambda}{\alpha})\frac{p'}{p}-\delta p'+2N+1+\frac{4}{\alpha}}.
\end{aligned}$$

For $\alpha=\frac{2}{\delta}$, we have the estimate

$$0<\int_{\mathbb{R}^{2N+1}}u_1(\eta)\varphi(\eta,0)d\eta+\int_{\mathcal{H}} fu^p\varphi\,d\eta dt\leq CR^{-(\gamma+2\delta\lambda)\frac{p'}{p}-\delta p'+2N+2\delta}. \tag{2.4}$$

Now, if $p<p_*$, then passing to the limit in (2.4) when $R\to\infty$, we obtain

$$0<\liminf_{R\to\infty}\int_{|\eta|<R}u_1(\eta)\,d\eta+\int_{\mathcal{H}} fu^p\ d\eta dt=0,$$

which is a contradiction.

The case of $p=p^*$ may be treated as in [8]. □

2.2. The case of the system (1.3)

First, we introduce the definition of a weak solution of the system (1.3).

Definition 2.3. A local weak solution of the system (1.3) is a couple of functions

$$\begin{aligned}
(u,v)\in C_B^1\big([0,T];(L^1_{loc}(\mathcal{H}B)\big)\cap C((0,T);L^p_{loc}(\mathcal{H}))\cap L^1L^p_{loc}(\mathcal{H},g(\eta,t)\,d\eta dt))\times&\\
C_B^1\big([0,T];(L^1_{loc}(\mathcal{H}B)\big)\cap C((0,T);L^p_{loc}(\mathcal{H}))\cap L^1L^p_{loc}(\mathcal{H},f(\eta,t)\,d\eta dt))&
\end{aligned}$$

satisfying the equations

$$\begin{aligned}
-\int_{\mathcal{H}_{\mathcal{T}}}u\Delta_{\mathbb{H}}\varphi+\int_{\mathcal{H}_{\mathcal{T}}}u(-\Delta_{\mathbb{H}})^{\delta/2}\varphi+\int_{\mathcal{H}_{\mathcal{T}}}u\varphi_{tt}&\\
=\int_{\mathcal{H}_{\mathcal{T}}}gv^q+\int_{\mathcal{H}}u_1(\eta)\varphi(\eta,0)d\eta-\int_{\mathbb{R}^{2N+1}}u_0(\eta)\varphi_t(\eta,0)\,d\eta&
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
-\int_{\mathcal{H}_{\mathcal{T}}}v\Delta_{\mathbb{H}}\varphi+\int_{\mathcal{H}_{\mathcal{T}}}v(-\Delta_{\mathbb{H}})^{\varrho/2}\varphi+\int_{\mathcal{H}_{\mathcal{T}}}v\varphi_{tt}&\\
=\int_{\mathcal{H}_{\mathcal{T}}}fu^p+\int_{\mathbb{R}^{2N+1}}v_1(\eta)\varphi(\eta,0)\,d\eta-\int_{\mathbb{R}^{2N+1}}v_0(\eta)\varphi_t(\eta,0)\,d\eta&
\end{aligned} \tag{2.6}$$

for any test function $\varphi\in C_0^2(\mathcal{H}_{\mathcal{T}})$. We say the the solution is global when $T=+\infty$.

Theorem 2.4. *Let $p, q > 1$ be two real numbers. If*

$$\liminf_{R\to\infty}\int_{|\eta|} u_1(\eta)d\eta > 0 \quad \textit{and} \quad \liminf_{R\to\infty}\int_{|\eta|} v_1(\eta)\,d\eta > 0,$$

then the system (1.3) *does not admit a global weak solution defined on $\mathcal{H}$ if*

$$(Q+1)(pq-1) \le \max\{\Lambda_1, \Lambda_2\},$$

where

$$\Lambda_1 = (k+1+2\mu)p + \min\{1,\varrho\}pq + \gamma + 2\lambda,$$

and

$$\Lambda_2 = (\gamma+1+2\lambda)p + \min\{1,\delta\}pq + k + 2\mu.$$

Proof. Let (u,v) be a global weak nontrivial solution and φ a regular nonnegative test function chosen so that $\varphi_t(\eta,0)=0$. From (2.5), we have

$$\int_{\mathbb{R}^{2N+1}} u_1(\eta)\varphi(\eta,0)\,d\eta + \int_{\mathcal{H}_\mathcal{T}} gv^q\,d\eta dt = \int_{\mathcal{H}_\mathcal{T}} u\varphi_{tt}\,d\eta dt - \int_{\mathcal{H}_\mathcal{T}} u\Delta_{\mathbb{H}}\varphi\,d\eta dt + \int_{\mathcal{H}_\mathcal{T}} u(-\Delta)^{\delta/2}\varphi\,d\eta dt.$$

Using φ^σ, $\sigma \gg 1$ and Hölder's inequality instead of Young's one, we have

$$\int_{\mathcal{H}_\mathcal{T}} u(\varphi^\sigma)_{tt}\,d\eta dt \le C\Big(\int_{\mathcal{H}_\mathcal{T}} fu^p\varphi^\sigma\Big)^{\frac{1}{p}}\mathcal{A}(f,p),$$

where

$$\mathcal{A}(f,p) = \Big(\int_{\mathcal{H}_\mathcal{T}} f^{\frac{-p'}{p}}\varphi^{(\sigma-1-\frac{\sigma}{p})p'}|\varphi_{tt}|^{p'}\,d\eta dt\Big)^{\frac{1}{p'}} + \Big(\int_{\mathcal{H}_\mathcal{T}} f^{-p'/p}\varphi^{(\sigma-2-\frac{\sigma}{p})p'}|\varphi_t|^{2p'}\,d\eta dt\Big)^{\frac{1}{p'}}.$$

Similarly, by using the Hölder inequality, we obtain the estimate

$$-\int_{\mathcal{H}_\mathcal{T}} u\Delta_{\mathbb{H}}\varphi^\sigma\,d\eta dt \le C\,\Big(\int_{\mathcal{H}_\mathcal{T}} fu^p\varphi^\sigma\,d\eta dt\Big)^{1/p}\mathcal{B}(f,p),$$

where

$$\mathcal{B}(f,p) = \Big(\int_{\mathcal{H}_\mathcal{T}} \varphi^{(\sigma-1-\frac{\sigma}{p})p'}|\Delta_{\mathcal{H}}\varphi|^{p'}f^{-p'/p}\,d\eta dt\Big)^{\frac{1}{p'}}.$$

Also, by Lemma 1.1 we have

$$\int_{\mathcal{H}_\mathcal{T}} u(-\Delta_{\mathcal{H}})^{\delta/2}\varphi^\sigma\,d\eta dt \le C\,\Big(\int_{\mathcal{H}_\mathcal{T}} fu^p\varphi^\sigma\,d\eta dt\Big)^{1/p}\mathcal{T}(f,p,\delta),$$

where

$$\mathcal{T}(f,p,\delta) = \Big(\int_{\mathcal{H}_\mathcal{T}} \varphi^{(\sigma-1-\frac{\sigma}{p})p'}|(-\Delta_{\mathcal{C}_R})^{-\delta/2}\varphi|^{p'}f^{-p'/p}\,d\eta dt\Big)^{\frac{1}{p'}}.$$

So we get

$$\int_{\mathbb{R}^{2N+1}} u_1(\eta)\varphi(\eta,0)\,d\eta + \int_{\mathcal{H}_\mathcal{T}} gv^q\varphi^\sigma\,d\eta dt \le C\,\Big(\Big(\int_{\mathcal{C}_R} fu^p\varphi^\sigma\Big)^{1/p}\Big(\mathcal{A}(f,p)+\mathcal{B}(f,p)+\mathcal{T}(f,p,\delta)\Big)\Big). \tag{2.7}$$

Proceeding in the same manner, we obtain the estimate

$$\int_{\mathbb{R}^{2N+1}} v_1(\eta)\varphi(\eta,0)\,d\eta + \int_{\mathcal{H}_\mathcal{T}} fu^p\varphi^\sigma\,d\eta dt \le C\,\Big(\Big(\int_{\mathcal{C}_R} gv^q\varphi^\sigma\Big)^{1/q}\Big(\mathcal{A}(g,q)+\mathcal{B}(g,q)+\mathcal{T}(g,q,\varrho)\Big)\Big). \tag{2.8}$$

To simplify, let us set

$$I=\int_{\mathcal{H}_{\mathcal{T}}} fu^p\varphi^\sigma\, d\eta dt \quad \text{and} \quad J=\int_{\mathcal{H}_{\mathcal{T}}} gv^q\varphi^\sigma\, d\eta dt.$$

By using

$$\int_{\mathbb{R}^{2N+1}} u_1(\eta)\varphi(\eta,0)\, d\eta>0 \quad \text{and} \quad \int_{\mathbb{R}^{2N+1}} v_1(\eta)\varphi(\eta,0)\, d\eta>0,$$

and Lemma 1.3 the inequalities (2.7) and (2.8) may be written as

$$I\le CJ^{\frac{1}{q}}\big(\mathcal{A}(g,q)+\mathcal{B}(g,q)+\mathcal{T}(g,q,\varrho)\big),$$

and

$$J\le CI^{\frac{1}{p}}\big(\mathcal{A}(f,p)+\mathcal{B}(f,p)+\mathcal{T}(f,p,\delta)\big). \tag{2.9}$$

So

$$I^{1-\frac{1}{pq}}\le C\big(\mathcal{A}(g,q)+\mathcal{B}(g,q)+\mathcal{T}(g,q,\varrho)\big)\times\big(\mathcal{A}^{\frac{1}{q}}(f,p)+\mathcal{B}^{\frac{1}{q}}(f,p)+\mathcal{T}^{\frac{1}{q}}(f,p,\delta)\big), \tag{2.10}$$

and

$$J^{1-\frac{1}{pq}}\le C\big(\mathcal{A}(f,p)+\mathcal{B}(f,p)+\mathcal{T}(f,p,\delta)\big)\times\big(\mathcal{A}^{\frac{1}{p}}(g,q)+\mathcal{B}^{\frac{1}{p}}(g,q)+\mathcal{T}^{\frac{1}{p}}(g,q,\delta)\big). \tag{2.11}$$

At this stage, we choose

$$\varphi(\eta,t)=\Phi\left(\frac{\tau^2+|x|^4+|y|^4+t^2}{R^4}\right), \tag{2.12}$$

where Φ is the same as in the proof of Theorem 2.2. Passing to the scaled variables

$$\widetilde{t}=R^{-2}t,\ \widetilde{\tau}=R^{-2}\tau,\ \widetilde{x}=R^{-1}x \text{ and } \widetilde{y}=R^{-1}y, \tag{2.13}$$

we obtain the estimates

$$\mathcal{A}(f,p)\le CR^{-(\gamma+2\lambda)\frac{1}{p}-1+(2N+3)\frac{1}{p'}}\le CR^{\sigma_1}, \tag{2.14}$$

$$\mathcal{B}(f,p)\le CR^{-(\gamma+2\lambda)\frac{1}{p}-2+(2N+3)\frac{1}{p'}}\le CR^{\sigma_2}, \tag{2.15}$$

$$\mathcal{T}(f,p,\delta)\le CR^{-(\gamma+2\lambda)\frac{1}{p}-\delta+(2N+3)\frac{1}{p'}}\le CR^{\sigma_3}, \tag{2.16}$$

$$\mathcal{A}(g,q)\le CR^{-(k+2\mu)\frac{1}{q}-1+(2N+3)\frac{1}{q'}}\le CR^{\theta_1}, \tag{2.17}$$

$$\mathcal{B}(g,q)\le CR^{-(k+2\mu)\frac{1}{q}-2+(2N+3)\frac{1}{q'}}\le CR^{\theta_2}, \tag{2.18}$$

and

$$\mathcal{T}(g,q,\varrho)\le CR^{-(k+2\mu)\frac{1}{q}-\varrho+(2N+3)\frac{1}{q'}}\le CR^{\theta_3}. \tag{2.19}$$

Using these inequalities in (2.8) and in (2.10), we obtain

$$I^{1-\frac{1}{pq}}\le C\sum_{i=1}^{9}R^{\omega_i}, \tag{2.20}$$

where

$$\omega_i=\theta_i+\frac{\sigma_1}{q},\quad \text{for}\quad 1\le i\le 3,$$

$$\omega_{i+3}=\theta_i+\frac{\sigma_2}{q},\quad \text{for}\quad 1\le i\le 3,$$

and

$$\omega_{i+6}=\theta_i+\frac{\sigma_3}{q},\quad \text{for}\quad 1\le i\le 3.$$

A similar inequality holds for $J^{1-\frac{1}{pq}}$.

Now, we require $\omega_i \leq 0$ for $1 \leq i \leq 9$. This is equivalent to

$$(Q+1)(pq-1) \leq (k+1+2\mu)p + \min\{1,\varrho\}pq + \gamma + 2\lambda = \Lambda_1.$$

Considering $J^{1-\frac{1}{pq}}$, we obtain

$$(Q+1)(pq-1) \leq (\gamma+1+2\lambda)p + \min\{1,\delta\}pq + k + 2\mu = \Lambda_2.$$

Consequently, if

$$(Q+1)(pq-1) \leq \max\{\Lambda_1,\Lambda_2\},$$

then passing to the limit when $R \to \infty$ in (2.11), we obtain

$$\int_{\mathcal{H}} fu^p \, d\eta dt = 0 \implies u = 0,$$

which in turn leads to $v = 0$ via (2.10). Returning to inequalities (2.7) and (2.8) we obtain a contradiction. $\square$

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