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Scrambled sets of shift operators

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Abstract

In this paper, some characterizations about orbit invariants, p-scrambled points and scrambled sets are obtained. Applying these results solves a conjecture and two problems given in [X. Fu, Y. You, Nonlinear Anal., **71** (2009), 2141–2152].

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1. Introduction and preliminaries

A *topological dynamical system* (briefly, *dynamical system*), is denoted by a pair (X, f), where X is a complete metric space without isolated points and $f : X \longrightarrow X$ is continuous. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. For a dynamical system (X, f), the set of *fixed points* and *periodic points* of f are denoted by $\mathrm{Fix}(f)$ and $\mathrm{Per}(f)$, respectively. The *positive orbit* of x is the set $\mathrm{orb}_f^+(x) = \{f^n(x) : n \in \mathbb{Z}^+\}$.

The complexity of a dynamical system is a central topic of research since the term of chaos was introduced by Li and Yorke [5] in 1975, known as Li-Yorke chaos today. In their study, Li and Yorke suggested considering 'divergent pairs' (x, y), which are *proximal* but not *asymptotic*, i.e.,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0, \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

In this context, a subset $D \subset X$ containing at least two points is called a *scrambled set* of (X, f) or simply of f, if for any pair of distinct points $x, y \in D$, (x, y) is proximal but not asymptotic. If a scrambled set D of f is also uncountable, it is called a *Li-Yorke chaotic set* for f, and f is said to be *chaotic in the sense of*

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Li-Yorke. As is well known, sensitivity is widely understood as a key ingredient of chaos and was popularized by the meteorologist Lorenz thought the so-called 'butterfly effect'. More recent results on sensitivity can be found in [4, 10, 11, 13].

A generalization of Li-Yorke chaos is proposed by Schweizer and Smítal in [8], which is equivalent to having a positive topological entropy and some other concepts of chaos when restricted to a compact interval [8] or a hyperbolic symbolic space [6]. It is remarkable that this equivalence does not transfer to higher dimensions, e.g. positive topological entropy does not imply distributional chaos in the case of triangular maps in the unit square [9] (the same happens when the dimension is zero [7]).

For any pair $(x, y) \in X \times X$ and for any $n \in \mathbb{N}$, the *distributional function* $F^n_{x,y} : \mathbb{R} \longrightarrow [0, 1]$ is defined by

$$F^n_{x,y}(t, f) = \frac{1}{n} \left|\left\{i : d(f^i(x), f^i(y)) < t, 1 \leq i \leq n\right\}\right|,$$

where $|A|$ denotes the cardinality of the set A. The *lower* and *upper distributional functions* generated by f, x and y are defined as

$$F_{x,y}(t, f) = \liminf_{n\to\infty} F^n_{x,y}(t, f),$$

and

$$F^*_{x,y}(t, f) = \limsup_{n\to\infty} F^n_{x,y}(t, f),$$

respectively. Both functions $F_{x,y}$ and $F^*_{x,y}$ are non-decreasing and $F_{x,y} \leq F^*_{x,y}$.

A dynamical system (X, f) is *distributionally ε-chaotic* for a given $\varepsilon > 0$ if there exists an uncountable subset $S \subset X$ such that for any pair of distinct points $x, y \in S$, one has $F^*_{x,y}(t, f) = 1$ for all $t > 0$ and $F_{x,y}(\varepsilon, f) = 0$. The set S is a *distributionally ε-chaotic set* and the pair (x, y) a *distributionally ε-chaotic pair*. If (X, f) is distributionally ε-chaotic for any given $0 < \varepsilon < \operatorname{diam} X$, then (X, f) is said to exhibit *maximal distributional chaos*.

Definition 1.1 ([2, 3])**.** For an integer $p > 0$, a point $x \in X$ is *p-scrambled* if the pair $(x, f^p(x))$ is proximal but not asymptotic, i.e.,

$$\liminf_{n\to\infty} d(f^n(x), f^n(f^p(x))) = 0, \quad \limsup_{n\to\infty} d(f^n(x), f^n(f^p(x))) > 0.$$

In [3], Fu and You proved that if for all $p \in \mathbb{N}$, x is p-scrambled, then the $\mathrm{orb}_f(x)$ is a scrambled set. In section 2 below, more results on p-scrambled points will be given.

Definition 1.2 ([3])**.** A map γ is said to be *orbit invariant* on $D \subset X$ under f, if

(1) $\gamma : \cup_{n=0}^{+\infty} f^n(D) \longrightarrow (0, 1)$ is a function;

(2) $\gamma|_D$ is injective;

(3) $\gamma(f^n(x)) = \gamma(x)$, $\forall x \in D$, $\forall n \geq 0$,

i.e., γ has the same value on an orbit, but different values on different orbits.

Let $A = \{0, 1, \ldots, N - 1\}$ for some integer $N \geq 2$ with a discrete metric d, and denote by $\Sigma(N)$ the space consisting of one-sided sequences in A. So, $x \in \Sigma(N)$ may be denoted by $x = x_1x_2\cdots$, $x_i \in A$, $\forall i \in \mathbb{N}$. Let $\Sigma(N)$ be endowed with the product topology. Then, $\Sigma(N)$ is metrizable, and a metric on $\Sigma(N)$ can be chosen to be

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i}, \quad \forall x = x_1x_2\cdots,\ y = y_1y_2\cdots \in \Sigma(N).$$

Define $\sigma : \Sigma(N) \longrightarrow \Sigma(N)$ by

$$\sigma(x_1x_2\cdots) = x_2x_3\cdots,$$

called the *shift* on $\Sigma(N)$, which is continuous. Also, $(X, \sigma|_X)$ is called a *shift space* or *subshift*, where X is a closed and invariant subset of $\Sigma(N)$.

To characterize the scrambled sets of σ, Fu and You [3] proved the following result.

Theorem 1.3 ([3])**.** *Let $D \subset \Sigma(2)$ with $|D| \geq 2$. If there exists an orbit invariant on D under σ, and $\eta(x,y) = 1$ for all $x, y \in D$, then D is a scrambled set of σ, and moreover $\exists p \in \mathrm{Per}(\sigma)$, such that $\forall m, n \in \mathbb{Z}^+$, $m \neq n$, $\sigma^m(D) \cap \sigma^n(D) \subset \mathrm{orb}_\sigma^+(p)$, where*

$$\eta(x,y) = \limsup_{n\to\infty} \frac{1}{n} \left|\{k : x_k = y_k,\ 1 \leq k \leq n\}\right|.$$

At the same time, the following conjecture was made in [3]:

Conjecture 1.4 ([3])**.** The condition $\eta(x,y) = 1$ for $x, y \in D$ in Theorem 1.3 may be replaced by a weaker condition that $\eta(x,y) \geq \delta$ for some (or any given) positive constant $\delta < 1$.

At the end of [3], Fu and You also posed the following two open problems on the scrambled sets of σ:

Question 1.5 ([3])**.** Is it possible to formulate some necessary and sufficient conditions for D to be a scrambled set of σ?

Question 1.6 ([3])**.** Does there exists a subshift $(X, \sigma|_X)$ with the whole space X being a scrambled set of $\sigma|_X$?

In this paper, we further investigate the structures of the Li-Yorke chaotic sets and the distributionally chaotic sets generated by shift and weighted-shift operators. In Section 2, we prove Conjecture 1.4 and answer Question 1.5 and Question 1.6 (see Example 2.6, Theorem 2.3, and Theorem 2.5, respectively).

2. Scrambled (chaotic) sets and orbit invariants

In this section, we further study the structure of Li-Yorke chaotic sets of the shift operator. Note that some results of the Li-Yorke chaotic sets for the shift operator is already obtained by Fu et al. in [2, 3]. Our new contribution here is to characterize Li-Yorke chaotic sets by orbit invariants, Furstenberg families and p-scrambled points. Applying these concepts and relevant results, Conjecture 1.4 and Question 1.5 are both solved.

Theorem 2.1. *For a dynamical system (X, f), there exists an orbit invariant map on $D \subset X$ if and only if $|D| \leq |(0,1)|$ and $\mathrm{orb}_f^+(x) \cap \mathrm{orb}_f^+(y) = \emptyset$, $\forall x, y \in D$, $x \neq y$.*

Proof. Sufficiency. Because $|D| \leq |(0,1)|$, there exists an injection $\gamma^* : D \longrightarrow (0,1)$. Define

$$\gamma : \cup_{n=0}^{+\infty} f^n(D) \longrightarrow (0,1)$$

by

$$\gamma(\mathrm{orb}_f^+(x)) = \gamma^*(x), \quad x \in D.$$

As $\mathrm{orb}_f^+(x) \cap \mathrm{orb}_f^+(y) = \emptyset$ holds for any $x, y \in D$ with $x \neq y$, γ is well defined and it can be verified that γ is an orbit invariant map on D.

Necessity. Let $\alpha : \cup_{n=0}^{+\infty} f^n(D) \longrightarrow (0,1)$ be an orbit invariant map on D. It is clear that $|D| = |\alpha(D)| \leq |(0,1)|$.

It remains to show that $\mathrm{orb}_f^+(x) \cap \mathrm{orb}_f^+(y) = \emptyset$, $\forall x, y \in D$, $x \neq y$.

Suppose that there exist $x, y \in D$ with $x \neq y$ such that $\mathrm{orb}_f^+(x) \cap \mathrm{orb}_f^+(y) \neq \emptyset$. This implies that there exist $m, n \in \mathbb{Z}^+$ such that $f^m(x) = f^n(y)$. Thus

$$\alpha(x) = \alpha(f^m(x)) = \alpha(f^n(y)) = \alpha(y), \tag{2.1}$$

which contradicts that α is orbit invariant. □

Theorem 2.2. *Let D be a scrambled set of f with $|D| \leq |(0,1)|$ and $\mathrm{Per}(f) \neq \emptyset$. Then, the following statements are equivalent:*

(1) There exists an orbit invariant map on D.

(2) $f^m(D) \cap f^n(D) \subset \mathrm{Per}(f)$, $\forall m, n \in \mathbb{Z}^+$, $m \neq n$.

(3) There exists a $p \in \mathrm{Per}(f)$ *such that* $\forall m, n \in \mathbb{Z}^+$, $m \neq n$, $f^m(D) \cap f^n(D) \subset \mathrm{orb}_f^+(p)$.

(4) $\mathrm{orb}_f^+(x) \cap \mathrm{orb}_f^+(y) = \emptyset$, $\forall x, y \in D$, $x \neq y$.

Proof. Applying Theorem 2.1 and [2, Theorem 2.2, Lemma 2.3], we have (2) $\Longleftrightarrow$ (3) $\Longrightarrow$ (1) $\Longleftrightarrow$ (4), so it suffices to show that (4) $\Longrightarrow$ (2).

Given any fixed $m, n \in \mathbb{Z}^+$ with $m < n$, for any $x^* \in f^m(D) \cap f^n(D)$, there exist $x, y \in D$ such that $f^m(x) = x^* = f^n(y)$. Combining this with hypothesis (4), it follows that $x = y$, so that

$$f^m(x) = x^* = f^{n-m}(f^m(x)) = f^{n-m}(x^*) \in \mathrm{Per}(f). \tag{2.2}$$

□

In [3], Fu and You thought that a possible route to solve Question 1.5 is to re-define the function $\eta(\cdot,\cdot)$. Theorem 2.3 shows that this works. For this, let us first recall some notations [1]. For the set of positive integers $\mathbb{N}$, denote by $\mathcal{P} = \mathcal{P}(\mathbb{N})$ the collection of all subsets of $\mathbb{N}$. A subset F of $\mathcal{P}$ is called a *Furstenberg family* (briefly, a family), if it is hereditary upward, i.e., $F_1 \subset F_2$ and $F_1 \in F$ imply $F_2 \in F$. Let $\mathscr{F}_{inf}$ be the family of all infinite subsets of $\mathbb{N}$. A subset F of $\mathbb{N}$ is called *thick* if it contains arbitrarily long runs of positive integers, i.e., for any $n \in \mathbb{N}$, there exists some $a_n \in \mathbb{N}$ such that $\{a_n + 1, \dots, a_n + n\} \subset F$. The families of all thick sets of $\mathbb{N}$ is denoted by $\mathscr{F}_t$. For $A \subset \mathbb{Z}^+$, define

$$\overline{d}(A) = \limsup_{n\to+\infty} \frac{1}{n} \left|A \cap [0, n-1]\right| \text{ and } \underline{d}(A) = \liminf_{n\to+\infty} \frac{1}{n} \left|A \cap [0, n-1]\right|.$$

Then, $\overline{d}(A)$ and $\underline{d}(A)$ are *the upper density* and *the lower density* of A, respectively.

Similarly, define the *upper Banach density* and the *lower Banach density* of A as

$$\mathrm{BD}^*(A) = \limsup_{|I|\to+\infty} \frac{|A \cap I|}{|I|} \text{ and } \mathrm{BD}_*(A) = \liminf_{|I|\to+\infty} \frac{|A \cap I|}{|I|},$$

where I is over all non-empty finite intervals of $\mathbb{Z}^+$. It is well known that $A \subset \mathbb{Z}^+$ is thick if and only if $\mathrm{BD}^*(A) = 1$. Moreover, define $DIF(x, y) = \{k \in \mathbb{N} : x_k \neq y_k\}$ and $IDE(x, y) = \{k \in \mathbb{N} : x_k = y_k\}$, $\forall x, y \in \Sigma(N)$. Clearly, $\eta(x, y) = \overline{d}(IDE(x, y))$.

Theorem 2.3. *$D \subset \Sigma(N)$ is a scrambled set of σ if and only if for any $x, y \in D$ with $x \neq y$, $DIF(x, y) \in \mathscr{F}_{inf}$ and $IDE(x, y) \in \mathscr{F}_t$, i.e., $\mathrm{BD}^*(IDE(x, y)) = 1$.*

Proof. Sufficiency. Given any $x, y \in D$ with $x \neq y$, since $DIF(x, y) \in \mathscr{F}_{inf}$, we may assume that $DIF(x, y) = \{n_k\}_{k\in\mathbb{N}}$. Then, for any $k \in \mathbb{N}$, we have $\rho(\sigma^{n_k-1}(x), \sigma^{n_k-1}(y)) \geq \frac{1}{2}$. This implies that

$$\limsup_{n\to\infty} \rho(\sigma^n(x), \sigma^n(y)) \geq \limsup_{k\to\infty} \rho(\sigma^{n_k-1}(x), \sigma^{n_k-1}(y)) \geq \frac{1}{2}. \tag{2.3}$$

As $IDE(x, y) \in \mathscr{F}_t$, for any $m \in \mathbb{N}$, there exists some $a_m \in \mathbb{N}$ such that $\{a_m + 1, \dots, a_m + m\} \in IDE(x, y)$. Hence,

$$d(\sigma^{a_m}(x), \sigma^{a_m}(y)) \leq \sum_{i=m+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^m}. \tag{2.4}$$

Thus

$$\liminf_{n\to\infty} \rho(\sigma^n(x), \sigma^n(y)) \leq \liminf_{m\to\infty} \rho(\sigma^{a_m}(x), \sigma^{a_m}(y)) = 0. \tag{2.5}$$

Since x and y are arbitrary, it follows that D is a scrambled set of σ.

Necessity. For any fixed $x, y \in D$ with $x \neq y$, it is clear that $DIF(x,y) \in \mathscr{F}_{inf}$ as

$$\limsup_{n\to\infty} \rho(\sigma^n(x), \sigma^n(y)) > 0.$$

Since $\liminf_{n\to\infty} \rho(\sigma^n(x), \sigma^n(y)) = 0$, it follows that for any $k \in \mathbb{N}$, there exists some $b_k \in \mathbb{N}$ such that $\rho(\sigma^{b_k}(x), \sigma^{b_k}(y)) \leq \frac{1}{2^{k+1}}$. This implies that the first k symbols of $\sigma^{b_k}(x)$ and $\sigma^{b_k}(y)$ coincide correspondingly for any $k \in \mathbb{N}$. So

$$\bigcup_{k=1}^{\infty} \{b_k+1, \ldots, b_k+k\} \subset IDE(x,y) \text{ and } \bigcup_{k=1}^{\infty} \{b_k+1, \ldots, b_k+k\} \in \mathscr{F}_t. \tag{2.6}$$

Therefore, $IDE(x,y) \in \mathscr{F}_t$ which is equivalent to $\mathrm{BD}^*(IDE(x,y)) = 1$. □

Theorem 2.4. *Assume that $N \geq 2$ and $x \in \Sigma(N)$. Then, the following statements are equivalent:*

(1) x is 1-scrambled.

(2) x is p-scrambled for all $p \in \mathbb{N}$.

(3) $\mathrm{orb}_\sigma^+(x)$ is a scrambled set of σ.

Proof. It is clear that (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1). Combining this with [3, Lemma 3.2] which shows that (2) $\Longrightarrow$ (3), it suffices to show that (1) $\Longrightarrow$ (2).

Denote $x = x_1x_2\cdots$. Theorem 2.3 and (1) together imply that $DIF(x,\sigma(x)) = \{k \in \mathbb{N} : x_k \neq x_{k+1}\} \in \mathscr{F}_{inf}$ and $IDE(x,\sigma(x)) = \{k \in \mathbb{N} : x_k = x_{k+1}\} \in \mathscr{F}_t$.

Claim 1. For any $p \geq 2$, $\limsup_{n\to\infty} \rho(\sigma^n(x), \sigma^n(\sigma^p(x))) > 0$.

Suppose that there exists some $p \geq 2$ such that $\limsup_{n\to\infty} \rho(\sigma^n(x), \sigma^n(\sigma^p(x))) = 0$. This implies that there exists a $K^* \in \mathbb{N}$ such that $\{k \in \mathbb{N} : k \geq K^*\} \subset IDE(x, \sigma^p(x)) = \{k \in \mathbb{N} : x_k = x_{k+p}\}$. Combining this with $DIF(x,\sigma(x)) \in \mathscr{F}_{inf}$, it can be verified that there exist $0 \leq j_1 < j_2 \leq p-1$ such that for any $n \in \mathbb{N}$,

$$x_{K^*+j_1+np} = x_{K^*+j_1} \neq x_{K^*+j_2} = x_{K^*+j_2+np}. \tag{2.7}$$

This means that $IDE(x,\sigma(x)) \notin \mathscr{F}_t$, which is a contradiction.

Claim 2. For any $p \geq 2$, $\liminf_{n\to\infty} \rho(\sigma^n(x), \sigma^n(\sigma^p(x))) = 0$.

Given any fixed $p \geq 2$, since $IDE(x,\sigma(x)) \in \mathscr{F}_t$, for any $k \in \mathbb{N}$ there exists $a_k \in \mathbb{N}$ such that $x_{a_k+1} = \cdots = x_{a_k+k}$. This implies that for any $k > p$ and any $a_k + 1 \leq i \leq a_k + k - p$, $\sigma^i(x) = \sigma^i(\sigma^p(x))$, i.e.,

$$\bigcup_{k=p+1}^{\infty} \{a_k+1, \ldots, a_k+k-p\} \subset IDE(x, \sigma^p(x)) \in \mathscr{F}_t. \tag{2.8}$$

Thus,

$$\liminf_{n\to\infty} \rho(\sigma^n(x), \sigma^n(\sigma^p(x))) = 0. \tag{2.9}$$

Combining Claim 1 with Claim 2 implies that x is p-scrambled for all $p \geq 2$, hence for all $p \in \mathbb{N}$. □

For the shift operator σ, Theorem 2.4 characterizes p-scrambled points. Now, we may further ask if Theorem 2.4 holds for a general dynamical system (X, f)? We conjecture that this is true. The following theorem gives a negative answer to Question 1.6. In contrast with Theorem 2.5, we [12] obtained that $(\Sigma(2), \sigma)$ contains an invariant distributionally ε-chaotic set for any $0 < \varepsilon < \mathrm{diam}\Sigma(2)$.

Theorem 2.5. *Dynamical system $(\Sigma(N), \sigma)$ does not contain invariant scrambled closed subsets.*

Proof. It suffices to check the case of $N = 2$, because the rest cases can be verified similarly. Suppose that D is an invariant scrambled closed subset of σ. Noting that for any fixed $x \in D$, $\liminf_{n\to\infty} d(\sigma^n(x),\sigma^n(\sigma(x))) = 0$, it follows that there exists an increased sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $\lim_{k\to\infty} d(\sigma^{n_k}(x), \sigma^{n_k}(\sigma(x))) = 0$. Without loss of generality, assume that $\lim_{k\to\infty} \sigma^{n_k}(x) = z \in D$. Then,

$$d(z, \sigma(z)) = \lim_{k\to\infty} d(\sigma^{n_k}(x), \sigma^{n_k}(\sigma(x))) = 0, \text{ i.e., } z \in D \cap \mathrm{Fix}(f).$$

For any $p, q \in D \cap \mathrm{Fix}(f)$, it can be verified that $d(p, q) = \liminf_{n\to\infty} d(\sigma^n(p), \sigma^n(q)) = 0$, implying that the set $D \cap \mathrm{Fix}(f)$ only contains an element. Without loss of generality, assume that $D \cap \mathrm{Fix}(f) = \{(0, 0, 0, \ldots)\}$.

Given any fixed $x = (x_1, x_2, x_3, \ldots) \in D \setminus \mathrm{Fix}(f)$, as $(x, \sigma(x))$ is proximal but not asymptotic, Theorem 2.3 implies that

$$\{i \in \mathbb{N} : x_i \neq x_{i+1}\} \in \mathscr{F}_{inf}, \tag{2.10}$$

and

$$\{i \in \mathbb{N} : x_i = x_{i+1}\} \in \mathscr{F}_t. \tag{2.11}$$

Applying (2.11) implies that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $x_{n_k} = x_{n_k+1} = \cdots = x_{n_k+k}$.

(a) If $\{k \in \mathbb{N} : x_{n_k} = x_{n_k+1} = \cdots = x_{n_k+k} = 1 \text{ for some } n_k \in \mathbb{N}\}$ is infinite, then we have $(1, 1, 1, \ldots) \in \omega(x, \sigma) \subset D$. This is a contradiction as $D \cap \mathrm{Fix}(f) = \{(0, 0, 0, \ldots)\}$.

(b) If $\{k \in \mathbb{N} : x_{n_k} = x_{n_k+1} = \cdots = x_{n_k+k} = 1 \text{ for some } n_k \in \mathbb{N}\}$ is finite, then there exists $K \in \mathbb{N}$ such that for any $k \geq K$, $x_{n_k} = x_{n_k+1} = \cdots = x_{n_k+k} = 0$. Combining this with (2.10) yields that $(1, 0, 0, \ldots) \in \omega(x, \sigma) \subset D$. This is impossible because $\lim_{n\to\infty} d(\sigma^n(0, 0, 0, \ldots), \sigma^n(1, 0, 0, \ldots)) = 0$.

□

Finally in this section, we show an example that Conjecture 1.4 mentioned earlier does not hold. Moreover, this example shows that [3, Corollary 3.6] is not true.

Example 2.6. Define

$$X_0 = \left\{x_1x_2\cdots \in \Sigma(2) : x_{4n+1} = 1, x_{4n+2} = x_{4n+3} = 0, \forall n \in \mathbb{Z}^+\right\},$$

$$X_1 = \left\{x_1x_2\cdots \in \Sigma(2) : x_{4n+4} = 1, x_{4n+1} = x_{4n+2} = 0, \forall n \in \mathbb{Z}^+\right\},$$

$$X_2 = \left\{x_1x_2\cdots \in \Sigma(2) : x_{4n+3} = 1, x_{4n+1} = x_{4n+4} = 0, \forall n \in \mathbb{Z}^+\right\},$$

and

$$X_3 = \left\{x_1x_2\cdots \in \Sigma(2) : x_{4n+2} = 1, x_{4n+3} = x_{4n+4} = 0, \forall n \in \mathbb{Z}^+\right\}.$$

It is easy to see that $\sigma(X_j) = X_{j+1 \pmod 4}$ and $X := \cup_{j=0}^{3} X_j$ is a closed invariant set under σ. For any $x = x_1x_2\cdots$, $y = y_1y_2\cdots \in X$, denote $x \sim y$, if $\mathrm{orb}_{\sigma}^+(x) \cap \mathrm{orb}_{\sigma}^+(y) \neq \emptyset$. It can be verified that '$\sim$' is an equivalence relation on X. For any $x \in X$, it is easy to see that the set $\{y \in X : y \sim x\}$ is countable and so the quotient set $X/\sim$ is uncountable. Taking a representative in each equivalence class of $X/\sim$, we get an uncountable set E. Without loss of generality, we may assume that $E \subset X_0$.

Fix a point $z \in E$ and define $D = (E \setminus \{z\}) \cup \{\sigma(z)\}$. Theorems 2.1 and Theorem 2.3 imply that the following three claims hold.

- **Claim 1.** There exists a surjective map $\gamma : \cup_{n=0}^{+\infty} \sigma^n(D) \longrightarrow (0, 1)$ such that $\gamma|_D$ is injective and $\gamma(\sigma^n(x)) = \gamma(x)$ for all $x \in D$ and for any $n \in \mathbb{Z}^+$.
- **Claim 2.** $\eta(x, y) \geq \frac{1}{4}$ for any $x, y \in D$.
- **Claim 3.** D is not a scrambled set of σ.

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