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Lightcone dual surfaces and hyperbolic dual surfaces of spacelike curves in de Sitter 3-space

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Abstract

In this paper, we consider the spacelike curves in de Sitter space and we investigate the singularities of lightcone dual surfaces and hyperbolic dual surfaces of these spacelike curves in the framework of the theory of Legendrian dualities between pseudo-spheres in Minkowski space. We classify the singularities of these subjects and reveal the relationships between their singularities and geometric invariants of spacelike curves under the action of the Lorentz group. As application and illustration of the main results, an example is given.

Keywords: De Sitter space, Legendrian dualities, lightcone dual surfaces.
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1. Introduction

In the theory of relativity, the future lightcone of an event is the boundary of its causal future in Minkowski space-time [11]. Up to now, different types of surfaces and curves in the future lightcone such as spacelike surfaces and null curves have been studied [7, 8, 9, 10]. In fact, any simply connected two dimensional Riemannian manifold can be isometrically immersed in a lightcone in Minkowski four space and a famous global geometry property is that a compact spacelike surface in a lightcone is diffeomorphic to a two dimensional sphere S^2. Moreover, from the relations between the conformal transformation group and the Lorentzian group of $\mathbb{R}^n_1$, and the submanifolds of the Riemannian sphere S^n and the submanifolds of the lightcone LC^{n+1}, we know that it is important to study submanifolds of the lightcone [8].

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In this paper, we investigate generic singularities of lightcone dual surfaces, which are spacelike surfaces in the lightcone (cf., Proposition 2.1), and hyperbolic dual surfaces generated by spacelike curves in de Sitter space. Our findings indicate that there are two kinds of spacelike dual surfaces of spacelike curves. One is the dual of the spacelike curve of the de Sitter 3-space embedded in the lightcone and another is the dual of the spacelike curve of the de Sitter 3-space embedded in the hyperbolic space (cf., Proposition 3.1). By definition, these two kinds of dual surfaces are different. The main results are Theorems 5.3 and 6.3. These results give a classification of the singularities of lightcone dual surfaces and hyperbolic dual surfaces for generic spacelike curves in de Sitter 3-space.

The rest of this paper is organized as follows. Firstly, we introduce some basic concepts. In Section 3, we investigate the relationships among the hyperbolic dual surfaces, the lightcone dual surfaces and the spacelike curves by Legendrian dualities [4]. Then, we introduce two different families of functions on spacelike curves $\boldsymbol{\gamma}$ that will be useful to study the singularities of the lightcone dual surfaces and the hyperbolic dual surfaces in Section 4. Afterwards, some general results on the singularity theory are used for families of function germs and the main results (Theorem 5.3 and Theorem 6.3) are proved in Section 5 and Section 6. As application and illustration of the main results, we give an example in Section 7.

All maps considered here are of class C^∞ unless otherwise stated.

2. The basic concepts

In this section, we will use some basic concepts and results in [2, 11]. Let $\mathbb{R}^4$ be a four-dimensional vector space, for any two vectors $\mathbf{x} = (x_0, x_1, x_2, x_3)$, $\mathbf{y} = (y_0, y_1, y_2, y_3)$ in $\mathbb{R}^4$, their pseudo scalar product is defined by $\langle \mathbf{x}, \mathbf{y}\rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$. The pair $(\mathbb{R}^4, \langle , \rangle)$ is called Minkowski space-time. We denote it as $\mathbb{R}^4_1$.

For any vectors $\mathbf{x} = (x_0, x_1, x_2, x_3)$, $\mathbf{y} = (y_0, y_1, y_2, y_3)$ and $\mathbf{z} = (z_0, z_1, z_2, z_3)$ in $\mathbb{R}^4_1$, their pseudo vector product is defined by

$$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \begin{vmatrix} -\mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{vmatrix},$$

where $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the canonical basis of $\mathbb{R}^4_1$. We remark that $\langle \mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}, \mathbf{w}\rangle = det(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w})$. A non-zero vector $\mathbf{x}$ in $\mathbb{R}^4_1$ is called spacelike, lightlike or timelike if $\langle \mathbf{x}, \mathbf{x}\rangle > 0$, $\langle \mathbf{x}, \mathbf{x}\rangle = 0$, $\langle \mathbf{x}, \mathbf{x}\rangle < 0$, respectively. The norm of $\mathbf{x} \in \mathbb{R}^4_1$ is defined by $\parallel \mathbf{x} \parallel = \sqrt{|\langle \mathbf{x}, \mathbf{x}\rangle|}$.

We define *de Sitter three-space* by

$$S^3_1 = \{\mathbf{x} \in \mathbb{R}^4_1 \mid \langle \mathbf{x}, \mathbf{x}\rangle = 1\}.$$

Let $\boldsymbol{\gamma} : I \to S^3_1 \subset \mathbb{R}^4_1$ be a smooth regular curve in S^3_1 (i.e., $\dot{\boldsymbol{\gamma}}(t) \neq 0$ for any $t \in I$), where I is an open interval. The curve $\boldsymbol{\gamma}$ is called a *spacelike curve*, if its velocity is $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle > 0$ for any $t \in I$. The arc-length of a spacelike curve $\boldsymbol{\gamma}(t)$, measured from $\boldsymbol{\gamma}(t_0)(t_0 \in I)$ is $s(t) = \int_{t_0}^{t} \parallel \dot{\boldsymbol{\gamma}}(t) \parallel dt$. Then the parameter s is determined such that $\parallel \boldsymbol{\gamma}'(s) \parallel = 1$, where $\boldsymbol{\gamma}'(s) = \frac{d\boldsymbol{\gamma}}{ds}$. So we say that a spacelike curve $\boldsymbol{\gamma}$ is parameterized by arc-length if it satisfies that $\parallel \boldsymbol{\gamma}'(s) \parallel = 1$. Throughout the remainder in this paper we denote the parameter s of $\boldsymbol{\gamma}$ as the arc-length parameter. Employing the usual terminology, the spacelike unit vector fields $\mathbf{t}(s) = \boldsymbol{\gamma}'(s)$ is call the tangent vector of $\boldsymbol{\gamma}$ at s. Under the assumption that $\langle \mathbf{t}'(s), \mathbf{t}'(s)\rangle \neq 1$, one can construct a unit vector $\mathbf{n}(s) = \frac{\mathbf{t}'(s)+\boldsymbol{\gamma}(s)}{\|\mathbf{t}'(s)+\boldsymbol{\gamma}(s)\|}$. Moreover, define $\mathbf{e}(s) = \boldsymbol{\gamma}(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}(s)$, then we can define a pseudo orthonormal frame $\{\boldsymbol{\gamma}(s), \mathbf{t}(s), \mathbf{n}(s), \mathbf{e}(s)\}$ of $\mathbb{R}^4_1$ along $\boldsymbol{\gamma}(s)$. By the standard arguments, we can show the following Frenet-Serret type formula:

$$\begin{pmatrix} \boldsymbol{\gamma}'(s) \\ \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{e}'(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & k_g(s) & 0 \\ 0 & k_g(s)\delta(s) & 0 & \tau_g(s) \\ 0 & 0 & \tau_g(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma}(s) \\ \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{e}(s) \end{pmatrix}.$$

Here, $\delta(s) = -\langle \mathbf{n}(s), \mathbf{n}(s)\rangle$, $k_g(s) = \parallel \mathbf{t}^{'}(s) + \boldsymbol{\gamma}(s) \parallel$ and $\tau_g(s) = \frac{1}{k_g^2(s)} det(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}^{'}(s), \boldsymbol{\gamma}^{''}(s), \boldsymbol{\gamma}^{'''}(s))$.

We define *hyperbolic three-space* by

$$H_+^3 = \{\mathbf{x} \in \mathbb{R}_1^4 \mid \langle \mathbf{x}, \mathbf{x}\rangle = -1, x_0 \geq 1\}.$$

In addition, we define *the future lightcone at the origin* by

$$LC_+^3 = \{\mathbf{x} \in \mathbb{R}_1^4 \mid \langle \mathbf{x}, \mathbf{x}\rangle = 0,\ x_0 > 0\}.$$

For the case that $\delta(s) = -1$, we define *the first lightcone dual surface* of the spacelike curve by

$$\mathbf{FLS}_\gamma : I \times \mathbb{R} \to LC_+^3,\ \mathbf{FLS}_\gamma(s, \theta) = \boldsymbol{\gamma}(s) + \sinh\theta \mathbf{n}(s) + \cosh\theta \mathbf{e}(s).$$

Under this assumption, we also define *the first hyperbolic dual surface* of the spacelike curve by

$$\mathbf{FHD}_\gamma : I \times \mathbb{R} \to H_+^3,\ \mathbf{FHD}_\gamma(s, \theta) = \sinh\theta \mathbf{n}(s) + \cosh\theta \mathbf{e}(s).$$

For the case that $\delta(s) = 1$, we define *the second lightcone dual surface* of the spacelike curve by

$$\mathbf{SLS}_\gamma : I \times \mathbb{R} \to LC_+^3,\ \mathbf{SLS}_\gamma(s, \theta) = \boldsymbol{\gamma}(s) + \cosh\theta \mathbf{n}(s) + \sinh\theta \mathbf{e}(s).$$

We also define *the second hyperbolic dual surface* of the spacelike curve by

$$\mathbf{SHD}_\gamma : I \times \mathbb{R} \to H_+^3,\ \mathbf{SHD}_\gamma(s, \theta) = \cosh\theta \mathbf{n}(s) + \sinh\theta \mathbf{e}(s).$$

In this paper, we consider the singularities of these surfaces. Then we have the following proposition.

Proposition 2.1. *Let $\boldsymbol{\gamma} : I \to S_1^3$ be a unit speed spacelike curve with $k_g(s) \neq 0$ and $\tau_g(s) \neq 0$, then we have the following claims.*

(1) $\mathbf{FLS}_\gamma(s, \theta)$ *is a spacelike surface in the lightcone and (s, θ) is a singular point of $FLS_\gamma(s, \theta)$ if and only if* $\sinh\theta = \frac{1}{k_g(s)}$. *i.e.*

$$\mathbf{FLS}_\gamma(s, \theta(s)) = \boldsymbol{\gamma}(s) + \frac{1}{k_g(s)}\mathbf{n}(s) + \frac{\sqrt{1 + k_g^2(s)}}{k_g(s)}\mathbf{e}(s).$$

(2) $\mathbf{FHD}_\gamma(s, \theta)$ *is a spacelike surface and (s, θ) is a singular point of* $\mathbf{FHD}_\gamma(s, \theta)$ *if and only if* $\sinh\theta = 0$, *i.e.*

$$\mathbf{FHD}_\gamma(s, \theta(s)) = \mathbf{e}(s).$$

(3) $\mathbf{SLS}_\gamma(s, \theta)$ *is a regular spacelike surface.*

(4) $\mathbf{SHD}_\gamma(s, \theta)$ *is a regular spacelike surface.*

Proof. By some calculations, we can get that

$$\left\langle \frac{\partial \mathbf{FLS}_\gamma}{\partial s}(s, \theta), \frac{\partial \mathbf{FLS}_\gamma}{\partial s}(s, \theta) \right\rangle = (1 - k_g(s)\sinh\theta)^2 + \tau_g^2(s) > 0$$

and

$$\left\langle \frac{\partial \mathbf{FLS}_\gamma}{\partial \theta}(s, \theta), \frac{\partial \mathbf{FLS}_\gamma}{\partial \theta}(s, \theta) \right\rangle = (k_g(s)\sinh\theta)^2 + \tau_g^2(s) > 0.$$

This means that they are spacelike vectors. By the definitions of spacelike surfaces in the lightcone and first lightcone dual surface, this means that $\mathbf{FLS}_\gamma(s, \theta)$ is a spacelike surface lightcone. On the other hand, $\frac{\partial \mathbf{FLS}_\gamma}{\partial s}(s,\theta)$ and $\frac{\partial \mathbf{FLS}_\gamma}{\partial \theta}(s,\theta)$ are linearly dependent if and only if $1-k_g(s)\sinh\theta=0$. This means that $\sinh\theta=\frac{1}{k_g(s)}$. The assertion (1) follows.

(2) By the similar reason, we have the assertion (2).

(3) By some calculations, we can get that

$$\left\langle \frac{\partial \mathbf{SLS}_\gamma}{\partial s}(s,\theta), \frac{\partial \mathbf{SLS}_\gamma}{\partial s}(s,\theta)\right\rangle = (1+k_g(s)\cosh\theta)^2+\tau_g^2(s)>0$$

and

$$\left\langle \frac{\partial \mathbf{SLS}_\gamma}{\partial \theta}(s,\theta), \frac{\partial \mathbf{SLS}_\gamma}{\partial \theta}(s,\theta)\right\rangle = (k_g(s)\cosh\theta)^2+\tau_g^2(s)>0.$$

This means that they are spacelike vectors. By the definitions of spacelike surfaces in the lightcone and second lightcone dual surface, this means that $\mathbf{SLS}_\gamma(s,\theta)$ is a spacelike surface in the lightcone. On the other hand, $\frac{\partial \mathbf{SLS}_\gamma}{\partial s}(s,\theta)$ and $\frac{\partial \mathbf{SLS}_\gamma}{\partial \theta}(s,\theta)$ are linearly dependent if and only if $1+k_g(s)\cosh\theta=0$. This means that $\cosh\theta=-\frac{1}{k_g(s)}$. Note that it is impossible for the reason that $k_g(s)=\parallel \mathbf{t}^{'}(s)+\gamma(s)\parallel>0$ and $\cosh\theta>0$. This means that $\frac{\partial \mathbf{SLS}_\gamma}{\partial s}(s,\theta)$ and $\frac{\partial \mathbf{SLS}_\gamma}{\partial \theta}(s,\theta)$ are linearly independent. The assertion (3) follows.

(4) By the similar reason, we have the assertion (4). □

By Proposition 2.1, both the second lightcone dual surface and the second hyperbolic dual surface are regular surfaces, this means that they are locally diffeomorphic to a plane $\mathbb{R}^2$. For our purpose, we know that only the first lightcone dual surface and the first hyperbolic dual surface have special interested in singularity theory (i.e. we only care for the case that $\delta(s)=-1$). To describe their singularities, we find the lightcone invariants of $\boldsymbol{\gamma}$ as follows:

$$\rho(s)=k_g^2(s)\tau_g^2(s)[1+k_g^2(s)]-[k_g^{'}(s)]^2$$

and

$$\sigma(s)=k_g^{''}(s)+k_g(s)\tau_g^2(s)-(2k_g^{'}(s)\tau_g(s)+k_g(s)\tau_g^{'}(s))\sqrt{1+k_g^2(s)}.$$

3. Legendrian dualities among lightcone dual surfaces, hyperbolic dual surfaces and spacelike curves

We introduce the Legendrian dualities between pseudo-spheres in Minkowski space-time which has been proved to be a basic tool for the study of surfaces in pseudo-spheres in Minkowski space [3, 4, 5, 6, 12]. We define one-forms $\langle d\mathbf{v},\omega\rangle=-\omega_0 dv_0+\sum_{i=1}^3\omega_i dv_i$ and $\langle \mathbf{v},d\omega\rangle=-v_0 d\omega_0+\sum_{i=1}^3 v_i d\omega_i$ on $\mathbb{R}^4_1\times\mathbb{R}^4_1$ and consider the following three double fibrations.

(1) (a) $H^3_+\times S^3_1\supset\Delta_1=\{(\mathbf{v},\mathbf{w})\mid\langle\mathbf{v},\mathbf{w}\rangle=0\}$,
(b) $\pi_{11}:\Delta_1\longrightarrow H^3_+, \pi_{12}:\Delta_1\longrightarrow S^3_1$,
(c) $\theta_{11}=\langle d\mathbf{v},\mathbf{w}\rangle|_{\Delta_1}, \theta_{12}=\langle \mathbf{v},d\mathbf{w}\rangle|_{\Delta_1}$;

(2) (a) $H^3_+\times LC^3_+\supset\Delta_2=\{(\mathbf{v},\mathbf{w})\mid\langle\mathbf{v},\mathbf{w}\rangle=-1\}$,
(b) $\pi_{21}:\Delta_2\longrightarrow H^3_+, \pi_{22}:\Delta_2\longrightarrow LC^3_+$,
(c) $\theta_{21}=\langle d\mathbf{v},\mathbf{w}\rangle|_{\Delta_2}, \theta_{22}=\langle \mathbf{v},d\mathbf{w}\rangle|_{\Delta_2}$;

(3) (a) $LC^3_+\times S^3_1\supset\Delta_3=\{(\mathbf{v},\mathbf{w})\mid\langle\mathbf{v},\mathbf{w}\rangle=1\}$,
(b) $\pi_{31}:\Delta_3\longrightarrow LC^3_+, \pi_{32}:\Delta_3\longrightarrow S^3_1$,
(c) $\theta_{31}=\langle d\mathbf{v},\mathbf{w}\rangle|_{\Delta_3}, \theta_{32}=\langle \mathbf{v},d\mathbf{w}\rangle|_{\Delta_3}$.

Here $\pi_{i1}(\mathbf{v},\mathbf{w})=\mathbf{v}, \pi_{i2}(\mathbf{v},\mathbf{w})=\mathbf{w}$. We remark that $\theta_{i1}^{-1}(0)$ and $\theta_{i2}^{-1}(0)$ define the same tangent hyperplane field over Δ_i which is denoted by K_i $(i=1,2,3)$. The basic duality theorem is that each (Δ_i,K_i) is a contact manifold and both of $\pi_{ij}(j=1,2)$ are Legendrian fibrations. If there exists an isotropic mapping $i : L \longrightarrow \Delta_i$, which means that $i^*\theta_{i1}=0$, we say that $\pi_{i1}(i(L))$ and $\pi_{i2}(i(L))$ are Δ_i-dual to each other $(i=1,2,3)$. It is easy to see that the condition $i^*\theta_{i1}=0$ is equivalent to $i^*\theta_{i2}=0$. Then we have the following proposition on the relationships among the lightcone dual surfaces, the hyperbolic dual surfaces, and the spacelike curves with the help of the above Legendrian dualities.

Proposition 3.1. *Let $\boldsymbol{\gamma} : I \to S_1^3$ be a unit speed spacelike curve with $k_g(s) \neq 0$, then we have the following claims.*

(A) *For the case that $\delta(s)=-1$, we have the followings.*
  (1) *$\boldsymbol{\gamma}(s)$ and $\mathbf{FHD}_\gamma(s,\theta)$ are Δ_1-dual to each other.*
  (2) *$\boldsymbol{\gamma}(s)$ and $\mathbf{FLS}_\gamma(s,\theta)$ are Δ_3-dual to each other.*
  (3) *$\mathbf{FLS}_\gamma(s,\theta)$ and $\mathbf{FHD}_\gamma(s,\theta)$ are Δ_2-dual to each other.*

(B) *For the case that $\delta(s)=1$, we have the followings.*
  (1) *$\boldsymbol{\gamma}(s)$ and $\mathbf{SHD}_\gamma(s,\theta)$ are Δ_1-dual to each other.*
  (2) *$\boldsymbol{\gamma}(s)$ and $\mathbf{SLS}_\gamma(s,\theta)$ are Δ_3-dual to each other.*
  (3) *$\mathbf{SLS}_\gamma(s,\theta)$ and $\mathbf{SHD}_\gamma(s,\theta)$ are Δ_2-dual to each other.*

Proof. (A) Consider the mapping
$$\mathfrak{L}_1(s,\theta)=(\mathbf{FHD}_\gamma(s,\theta),\boldsymbol{\gamma}(s)).$$
Then we have $\langle \mathbf{FHD}_\gamma(s,\theta),\boldsymbol{\gamma}(s)\rangle=0$ and
$$\begin{aligned}\mathfrak{L}_1^*\theta_{12}(s,\theta)&=\langle \mathbf{FHD}_\gamma(s,\theta),\boldsymbol{\gamma}'(s)\rangle\\&=\langle \sinh\theta\mathbf{n}(s)+\cosh\theta\mathbf{e}(s),\mathbf{t}(s)\rangle\\&=0.\end{aligned}$$
The assertion (1) holds.

Consider the mapping $\mathfrak{L}_3(s,\theta)=(\mathbf{FLS}_\gamma(s,\theta),\boldsymbol{\gamma}(s))$. Then we have
$$\langle \mathbf{FLS}_\gamma(s,\theta),\boldsymbol{\gamma}(s)\rangle=1$$
and $\mathfrak{L}_3^*\theta_{32}(s,\theta)=0$. The assertion (2) holds.

To prove the claim (3), we consider the mapping
$$\mathfrak{L}_2(s,\theta)=(\mathbf{FHD}_\gamma(s,\theta),\mathbf{FLS}_\gamma(s,\theta)).$$
We also define a mapping
$$\Psi_{12}:\Delta_1\to\Delta_2 \text{ by } \Psi_{12}(\mathbf{v},\mathbf{w})=(\mathbf{v},\mathbf{v}+\mathbf{w}),$$
and the converse mapping
$$\Psi_{21}:\Delta_2\to\Delta_1 \text{ by } \Psi_{21}(\mathbf{v},\mathbf{w})=(\mathbf{v},\mathbf{w}-\mathbf{v}).$$
Moreover, we have that
$$\begin{aligned}\Psi_{12}^*\theta_{22}(\mathbf{v},\mathbf{w})&=\langle \mathbf{v},d\mathbf{v}+d\mathbf{w}\rangle\\&=\langle \mathbf{v},d\mathbf{w}\rangle\\&=\theta_{12}.\end{aligned}$$
Thus Ψ_{12} is a contact diffeomorphism from Δ_1 to Δ_2. By definition we have
$$\Psi_{12}\circ\mathfrak{L}_1(s,\theta)=\mathfrak{L}_2(s,\theta),$$
so that the image of $\mathfrak{L}_2(s,\theta)$ is a Legendrian submanifold in Δ_2. Then we have the assertion (3) follows.

(B) Using the same computation as the proof of (A), we can get (B). □

4. Lightcone height functions and timelike height functions

In order to study the singularities of the lightcone dual surfaces and the hyperbolic dual surfaces of spacelike curves in S_1^3, we introduce two very useful different families of functions on spacelike curves in de Sitter 3-space. Let $\boldsymbol{\gamma} : I \to S_1^3$ be a unit speed spacelike curve, we define two families of functions as follows:

$$H^L : I \times LC_+^3 \longrightarrow \mathbb{R},\ H^L(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v} \rangle - 1,$$

$$H^T : I \times H_+^3 \longrightarrow \mathbb{R},\ H^T(s, \mathbf{v}) = \langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle.$$

We call H^L a *lightcone height function* of the curve $\boldsymbol{\gamma}$. For any fixed $\mathbf{v} \in LC_+^3$, we denote $(h_v^L)(s) = H^L(s, \mathbf{v})$.

We call H^T a *timelike height function* of the curve $\boldsymbol{\gamma}$. For any fixed $\mathbf{v} \in H_+^3$, we denote $(h_v^T)(s) = H^T(s, \mathbf{v})$.

By making tedious calculations, we have the following propositions which contain some geometric invariants $\rho(s)$, $\sigma(s)$ and $\tau_g(s)$.

Proposition 4.1. *Let $\boldsymbol{\gamma} : I \to S_1^3$ be a unit speed spacelike curve with $k_g(s) \neq 0$ and $\delta(s) = -1$, then we have the following claims.*

(1) $(h_v^L)(s) = 0$ *if and only if there are real numbers $\lambda, \mu, \nu \in \mathbb{R}$ such that $\mathbf{v} = \boldsymbol{\gamma}(s) + \lambda \mathbf{t}(s) + \mu \mathbf{n}(s) + \nu \mathbf{e}(s)$ and $\lambda^2 + \mu^2 - \nu^2 = -1$.*

(2) $(h_v^L)(s) = (h_v^L)'(s) = 0$ *if and only if there is a real number $\theta \in \mathbb{R}$ such that $\mathbf{v} = \boldsymbol{\gamma}(s) + \sinh\theta \mathbf{n}(s) + \cosh\theta \mathbf{e}(s)$.*

(3) $(h_v^L)(s) = (h_v^L)'(s) = (h_v^L)''(s) = 0$ *if and only if*

$$\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_g(s)}\mathbf{n}(s) + \frac{\sqrt{1 + k_g^2(s)}}{k_g(s)}\mathbf{e}(s).$$

(4) $(h_v^L)(s) = (h_v^L)'(s) = (h_v^L)''(s) = (h_v^L)^{(3)}(s) = 0$ *if and only if*

$$\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_g(s)}\mathbf{n}(s) + \frac{\sqrt{1 + k_g^2(s)}}{k_g(s)}\mathbf{e}(s)$$

and $\rho(s) = 0$.

(5) $(h_v^L)(s) = (h_v^L)'(s) = (h_v^L)''(s) = (h_v^L)^{(3)}(s) = (h_v^L)^{(4)}(s) = 0$ *if and only if*

$$\mathbf{v} = \boldsymbol{\gamma}(s) + \frac{1}{k_g(s)}\mathbf{n}(s) + \frac{\sqrt{1 + k_g^2(s)}}{k_g(s)}\mathbf{e}(s)$$

and $\rho(s) = \sigma(s) = 0$.

Proof. By definition and the Frenet-Serret type formulae, we have

(a) $(h_v^L)'(s) = \langle \mathbf{t}(s), \mathbf{v} \rangle$,

(b) $(h_v^L)''(s) = \langle -\boldsymbol{\gamma}(s) + k_g(s)\mathbf{n}(s), \mathbf{v} \rangle$,

(c) $(h_{\mathbf{v}}^L)^{(3)}(s) = \langle (\delta(s)k_g^2(s) - 1)\mathbf{t}(s) + k_g'(s)\mathbf{n}(s) + k_g(s)\tau_g(s)\mathbf{e}(s), \mathbf{v} \rangle$,

(d) $(h_v^L)^{(4)}(s) = \langle 3\delta(s)k_g(s)k_g'(s)\mathbf{t}(s) + (\delta(s)k_g^3(s) - k_g(s) + k_g''(s) + k_g(s)\tau_g^2(s))\mathbf{n}(s) + (2k_g'(s)\tau_g(s) + k_g(s)\tau_g'(s))\mathbf{e}(s) - (\delta(s)k_g^2(s) - 1)\boldsymbol{\gamma}(s), \mathbf{v} \rangle$.

By the conditions that $(h_v^L)(s) = 0$, $\mathbf{v} \in LC_+^3$ and $\delta(s) = -1$, we have that there are real numbers λ, μ, ν such that $\mathbf{v} = \boldsymbol{\gamma}(s) + \lambda\mathbf{t}(s) + \mu\mathbf{n}(s) + \nu\mathbf{e}(s)$ and $\lambda^2 + \mu^2 - \nu^2 = -1$. The converse direction also holds. By the above formula (a), we have $(h_v^L)(s) = (h_v^L)'(s) = 0$ if and only if $\lambda = 0$. This means that $\mu^2 - \nu^2 = -1$. Let $\mu = sinh\theta, \nu = cosh\theta$, we have $\mathbf{v} = \boldsymbol{\gamma}(s) + \sinh\theta\mathbf{n}(s) + \cosh\theta\mathbf{e}(s)$. By the above formula (b), the assertion (3) holds. By the similar arguments to the above cases we can show that the assertion (4) and assertion (5) holds. □

Proposition 4.2. *Let $\boldsymbol{\gamma} : I \to S_1^3$ be a unit speed spacelike curve with $\delta(s) = -1$ and $k_g(s) \neq 0$, then we have the following claims.*

(1) $(h_v^T)(s) = (h_v^T)'(s) = 0$ *if and only if there are real numbers θ such that* $\mathbf{v} = \sinh\theta\mathbf{n}(s) + \cosh\theta\mathbf{e}(s)$.

(2) $(h_v^T)(s) = (h_v^T)'(s) = (h_v^T)''(s) = 0$ *if and only if* $\mathbf{v} = \mathbf{e}(s)$.

(3) $(h_v^T)(s) = (h_v^T)'(s) = (h_v^T)''(s) = (h_v^T)^{(3)}(s) = 0$ *if and only if* $\mathbf{v} = \mathbf{e}(s)$ *and* $\tau_g(s) = 0$

(4) $(h_v^T)(s) = (h_v^T)'(s) = (h_v^T)''(s) = (h_v^T)^{(3)}(s) = (h_v^T)^{(4)}(s) = 0$ *if and only if* $\mathbf{v} = \mathbf{e}(s)$ *and* $\tau_g(s) = \tau_g'(s) = 0$.

Proof. By the calculations of fourth order derivatives of the timelike height function $(h_v^T)(s)$, we can show the assertions similar way to the proof of Proposition 4.1. □

5. Singularities of lightcone dual surfaces and hyperbolic dual surfaces

In this section we use some general results on the singularity theory for families of function germs to classify the singularities of the lightcone dual surfaces and the hyperbolic dual surfaces. Detailed descriptions can be found in the book [1]. Let function germ $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$, where $f(s) = F(s, \mathbf{x}_0)$. We say that $f(s)$ has A_k-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that $f(s)$ has $A_{\geq k}$-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let $F(s, \mathbf{x})$ be an unfolding of $f(s)$ and $f(s)$ has A_k-singularity $(k \geq 1)$ at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}(s, \mathbf{x})$ at s_0 by

$$j^{(k-1)}\left(\frac{\partial F}{\partial x_i}(s, \mathbf{x}_0)\right)(s_0) = \sum_{j=1}^{k-1} a_{ji}(s - s_0)^j, \ i = 1, \cdots, r.$$

Then $F(s, \mathbf{x})$ is called an $\mathcal{R}$-versal unfolding if the $k \times r$ matrix of coefficients (a_{0i}, a_{ji}) has rank k $(k \leq r)$, where $a_{0i} = \frac{\partial F}{\partial x_i}(s_0, \mathbf{x}_0)$. We now introduce an important set concerning the unfolding. We define the following set

$$\mathfrak{D}_F^l = \left\{\mathbf{x} \in \mathbb{R}^r \mid \exists s \in \mathbb{R}, F(s, \mathbf{x}) = \frac{\partial F}{\partial s}(s, \mathbf{x}) = \cdots = \frac{\partial^l F}{\partial s^l}(s, \mathbf{x}) = 0\right\},$$

which is called *a discriminant set of order l*. Then $\mathfrak{D}_F^1 = \mathfrak{D}_F$ and $\mathfrak{D}_F^2$ is the set of singular points of $\mathfrak{D}_F$.

We have the following classification result (cf.,[1]).

Theorem 5.1. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \longrightarrow \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has the A_k singularity at s_0. Suppose that $F(s, \mathbf{x})$ is an $\mathcal{R}$-versal unfolding, then we have the following claims.*

(a) *If $k = 1$, then $\mathfrak{D}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$ and $\mathfrak{D}_F^2 = \varnothing$.*

(b) *If $k = 2$, then $\mathfrak{D}_F$ is locally diffeomorphic to $C(2,3) \times \mathbb{R}^{r-2}$, $\mathfrak{D}_F^2$ is diffeomorphic to $\{0\} \times \mathbb{R}^{r-2}$ and $\mathfrak{D}_F^3 = \varnothing$.*

(c) *If $k = 3$, then $\mathfrak{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$, $\mathfrak{D}_F^2$ is diffeomorphic to $C(2,3,4) \times \mathbb{R}^{r-3}$, $\mathfrak{D}_F^3$ is diffeomorphic to $\{0\} \times \mathbb{R}^{r-3}$ and $\mathfrak{D}_F^4 = \varnothing$.*

Here, we respectively call $C(2,3)\times\mathbb{R}=\{(x_1,x_2)\mid x_1^2=x_2^3\}\times\mathbb{R}$ *a cuspidal edge,*

$$SW=\{(x_1,x_2,x_3)\mid x_1=3u^4+u^2v, x_2=4u^3+2uv, x_3=v\}$$

a swallowtail, $C(2,3,4)=\{(t^2,t^3,t^4)\in\mathbb{R}^3\mid t\in\mathbb{R}\}$ *a* $(2,3,4)$*-cusp* (*cf.*, Fig. 1).

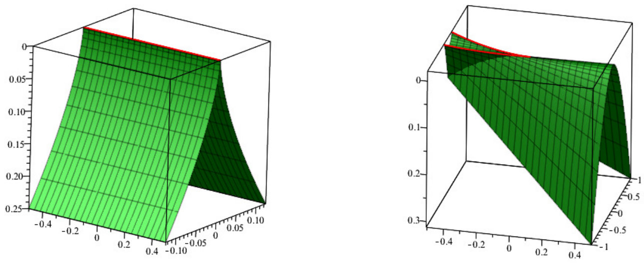

Figure 1: Cuspidal edge and Swallowtail.

By Proposition 4.1, Proposition 4.2 and the definitions of the discriminant set, we have

$$\begin{aligned}\mathfrak{D}_{H^L}&=\{\mathbf{v}=\boldsymbol{\gamma}(s)+\sinh\theta\mathbf{n}(s)+\cosh\theta\mathbf{e}(s)\mid s\in I,\theta\in\mathbb{R}\},\\ \mathfrak{D}_{H^T}&=\{\mathbf{v}=\sinh\theta\mathbf{n}(s)+\cosh\theta\mathbf{e}(s)\mid s\in I,\theta\in\mathbb{R}\}.\end{aligned}$$

These are the first lightcone dual surface and the second hyperbolic dual surface of $\boldsymbol{\gamma}(s)$ respectively. We have the following key propositions on $H^L(s,\mathbf{v})$ and $H^T(s,\mathbf{v})$.

Proposition 5.2. *Under the conditions of Propositions* 4.1 *and* 4.2, *we have the following claims.*

(1) *If* $h^L_{\mathbf{v}_0}(s)$ *has* A_k*-singularity at* s_0 $(k=1,2,3)$, *then* H^L *is an* $\mathcal{R}$*-versal unfolding of* $h^L_{\mathbf{v}_0}$.
(2) *If* $h^T_{v_0}(s)$ *has* A_k*-singularity at* s_0 $(k=1,2,3)$, *then* H^T *is an* $\mathcal{R}$*-versal unfolding of* $h^T_{\mathbf{v}_0}$.

Proof. (1) We consider the pseudo orthonormal basis $\mathbf{e}_0=\gamma(s),\mathbf{e}_1=\mathbf{t}(s),\mathbf{e}_2=\mathbf{n}(s),\mathbf{e}_3=\mathbf{e}(s)$ instead of the canonical basis of $\mathbb{R}^4_1$. Then, we denote that $\boldsymbol{\gamma}(s)=(x_0(s),x_1(s),x_2(s),x_3(s))$ and

$$\mathbf{v}=\left(v_0,v_1,v_2,\pm\sqrt{v_0^2-v_1^2-v_2^2}\right).$$

Under this notation, we have

$$H^L(s,\mathbf{v})=-x_0(s)v_0+x_1(s)v_1+x_2(s)v_2\pm x_3(s)\sqrt{v_0^2-v_1^2-v_2^2}-1.$$

For a fixed $\mathbf{v}_0=\left(v_0,v_1,v_2,\pm\sqrt{v_0^2-v_1^2-v_2^2}\right)$, the two-jet of $\frac{\partial H^L}{\partial v_i}(s,\mathbf{v}_0)(i=1,2,3)$ at s_0 is given by

$$j^2\left(\frac{\partial H^L}{\partial v_i}(s,\mathbf{v}_0)\right)(s_0)=\frac{\partial}{\partial s}\frac{\partial H^L}{\partial v_i}(s-s_0)+\frac{1}{2}\frac{\partial^2}{\partial s^2}\frac{\partial H^L}{\partial v_i}(s-s_0)^2.$$

It is enough to show that the rank of the following matrix A is three.

$$\begin{pmatrix}-x_0(s_0)\pm\frac{v_0x_3(s_0)}{\sqrt{v_0^2-v_1^2-v_2^2}} & x_1(s_0)\mp\frac{v_1x_3(s_0)}{\sqrt{v_0^2-v_1^2-v_2^2}} & x_2(s_0)\mp\frac{v_2x_3(s_0)}{\sqrt{v_0^2-v_1^2-v_2^2}}\\ -x_0'(s_0)\pm\frac{v_0x_3'(s_0)}{\sqrt{v_0^2-v_1^2-v_2^2}} & x_1'(s_0)\mp\frac{v_1x_3'(s_0)}{\sqrt{v_0^2-v_1^2-v_2^2}} & x_2'(s_0)\mp\frac{v_2x_3'(s_0)}{\sqrt{v_0^2-v_1^2-v_2^2}}\\ -x_0''(s_0)\pm\frac{v_0x_3''(s_0)}{\sqrt{v_0^2-v_1^2-v_2^2}} & x_1''(s_0)\mp\frac{v_1x_3''(s_0)}{\sqrt{v_0^2-v_1^2-v_2^2}} & x_2''(s_0)\mp\frac{v_2x_3''(s_0)}{\sqrt{v_0^2-v_1^2-v_2^2}}\end{pmatrix}.$$

We denote that $\mathbf{a} = \begin{pmatrix} x_0(s_0) \\ x_0'(s_0) \\ x_0''(s_0) \end{pmatrix}$, $\mathbf{b}_i = \begin{pmatrix} x_i(s_0) \\ x_i'(s_0) \\ x_i''(s_0) \end{pmatrix}$, $i = 1, 2, 3$.

$$\begin{aligned} detA &= \pm\frac{v_0 det(\mathbf{b}_3, \mathbf{b}_1, \mathbf{b}_2)}{\sqrt{v_0^2 - v_1^2 - v_2^2}} \pm \frac{v_1 det(\mathbf{a}, \mathbf{b}_3, \mathbf{b}_2)}{\sqrt{v_0^2 - v_1^2 - v_2^2}} \pm \frac{v_2 det(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_3)}{\sqrt{v_0^2 - v_1^2 - v_2^2}} - \frac{v_3}{v_3} det(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2) \\ &= \frac{v_0 det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)}{\pm\sqrt{v_0^2 - v_1^2 - v_2^2}} - \frac{v_1 det(\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3)}{\pm\sqrt{v_0^2 - v_1^2 - v_2^2}} - \frac{v_2 det(\mathbf{b}_1, \mathbf{a}, \mathbf{b}_3)}{\pm\sqrt{v_0^2 - v_1^2 - v_2^2}} - \frac{v_3}{v_3} det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{a}) \\ &= \frac{v_0}{v_3} det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) - \frac{v_1}{v_3} det(\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3) - \frac{v_2}{v_3} det(\mathbf{b}_1, \mathbf{a}, \mathbf{b}_3) - \frac{v_3}{v_3} det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{a}). \end{aligned}$$

Since

$$\boldsymbol{\gamma}(s_0) \wedge \boldsymbol{\gamma}'(s_0) \wedge \boldsymbol{\gamma}''(s_0) = (-det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3), -det(\mathbf{a}, \mathbf{b}_2, \mathbf{b}_3), -det(\mathbf{b}_1, \mathbf{a}, \mathbf{b}_3), -det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{a}))$$

and $\mathbf{v}_0 \in \mathfrak{D}_{H^L}$ is a singular point, we have $\mathbf{v}_0 = \boldsymbol{\gamma}(s_0) + \frac{1}{k_g(s_0)}\mathbf{n}(s_0) + \frac{\sqrt{1+k_g^2(s_0)}}{k_g(s_0)}\mathbf{e}(s_0)$. Therefore we have

$$\begin{aligned} detA &= \left\langle \left(\frac{v_0}{v_3}, \frac{v_1}{v_3}, \frac{v_2}{v_3}, \frac{v_3}{v_3}\right), \boldsymbol{\gamma}(s_0) \wedge \boldsymbol{\gamma}'(s_0) \wedge \boldsymbol{\gamma}''(s_0) \right\rangle \\ &= \frac{1}{v_3}\left\langle \gamma(s_0) + \frac{1}{k_g(s_0)}\mathbf{n}(s_0) + \frac{\sqrt{1+k_g^2(s_0)}}{k_g(s_0)}\mathbf{e}(s_0), k_g(s_0)\mathbf{e}(s_0) \right\rangle \\ &= \frac{1}{v_3}\sqrt{1+k_g^2(s_0)} \neq 0. \end{aligned}$$

Thus, we have $rankA = 3$. This completes the proof of claim (1).

(2) Using the same computation as the proof of (1), we can get (2). □

Theorem 5.3. *Let $\boldsymbol{\gamma} : I \to S_1^3$ be a unit speed spacelike curve with $k_g(s) \neq 0$ and $\delta(s) = -1$.*

(A) *For the first lightcone dual surface $\mathbf{FLS}_\gamma(s, \theta)$ of $\boldsymbol{\gamma}$, we have the following claims.*

(1) *The germ of the image of the first lightcone dual surface $\mathbf{FLS}_\gamma(s, \theta)$ is locally diffeomorphic to cuspidal edge $C(2, 3) \times \mathbb{R}$ at s_0 if*

$$\mathbf{v}_0 = \boldsymbol{\gamma}(s_0) + \frac{1}{k_g(s_0)}\mathbf{n}(s_0) + \frac{\sqrt{1+k_g^2(s_0)}}{k_g(s_0)}\mathbf{e}(s_0)$$

and $\rho(s_0) \neq 0$. In this case the critical value set of the first lightcone dual surface

$$\mathbf{FLS}_\gamma(s_0, \theta(s_0)) = \boldsymbol{\gamma}(s_0) + \frac{1}{k_g(s_0)}\mathbf{n}(s_0) + \frac{\sqrt{1+k_g^2(s_0)}}{k_g(s_0)}\mathbf{e}(s_0)$$

is locally diffeomorphic to a line.

(2) *The germ of the image of the first lightcone dual surface $\mathbf{FLS}_\gamma(s, \theta)$ is locally diffeomorphic to swallowtail SW at s_0 if*

$$\mathbf{v}_0 = \boldsymbol{\gamma}(s_0) + \frac{1}{k_g(s_0)}\mathbf{n}(s_0) + \frac{\sqrt{1+k_g^2(s_0)}}{k_g(s_0)}\mathbf{e}(s_0),$$

$\rho(s_0) = 0$ and $\sigma(s_0) \neq 0$. In this case the critical value set of the first lightcone dual surface

$$\mathbf{FLS}_\gamma(s_0, \theta(s_0)) = \boldsymbol{\gamma}(s_0) + \frac{1}{k_g(s_0)}\mathbf{n}(s_0) + \frac{\sqrt{1+k_g^2(s_0)}}{k_g(s_0)}\mathbf{e}(s_0)$$

is locally diffeomorphic to the $(2, 3, 4)$-cusp.

(B) *For the first hyperbolic dual surface* $\mathbf{FHD}_\gamma(s,\theta)$ *of* $\boldsymbol{\gamma}$, *we have the following claims.*

(1) *The germ of the image of the first hyperbolic dual surface* $\mathbf{FHD}_\gamma(s,\theta)$ *is locally diffeomorphic to cuspidal edge* $C(2,3)\times\mathbb{R}$ *if* $\mathbf{v}_0=\mathbf{e}(s_0)$ *and* $\tau_g(s_0)\neq 0$. *In this case the critical value set of the first hyperbolic dual surface* $\mathbf{FHD}_\gamma(s_0,\theta(s_0))=\mathbf{e}(s_0)$ *is locally diffeomorphic to a line.*

(2) *The germ of the image of the first hyperbolic dual surface* $\mathbf{FHD}_\gamma(s,\theta)$ *is locally diffeomorphic to swallowtail* SW *at* s_0 *if* $\mathbf{v}_0=\mathbf{e}(s_0)$, $\tau_g(s_0)=0$ *and* $\tau'_g(s_0)\neq 0$. *In this case the critical value set of the first hyperbolic dual surface* $\mathbf{FHD}_\gamma(s_0,\theta(s_0))=\mathbf{e}(s_0)$ *is locally diffeomorphic to the* $(2,3,4)$*-cusp.*

Proof. (A) First, we consider the assertion (A). By Proposition 4.1, the discriminant set of $H^L(s,\mathbf{v})$ is

$$\mathfrak{D}_{H^L}=\{\mathbf{v}=\boldsymbol{\gamma}(s)+\sinh\theta\mathbf{n}(s)+\cosh\theta\mathbf{e}(s)\mid s\in I,\theta\in\mathbb{R}\}.$$

This means that the discriminant set of the lightcone height function is the image of the first lightcone dual surface of $\boldsymbol{\gamma}(s)$. By Proposition 5.2, H^L is an $\mathcal{R}$-versal unfolding of of $h^L_{\mathbf{v}_0}$ at s_0 if $h^L_{\mathbf{v}_0}$ has A_k-singularity for $k=1,2,3$. By Proposition 4.1, $h^L_{v_0}$ has the A_2-singularity at s_0 if

$$\mathbf{v}_0=\boldsymbol{\gamma}(s_0)+\frac{1}{k_g(s_0)}\mathbf{n}(s_0)+\frac{\sqrt{1+k_g^2(s_0)}}{k_g(s_0)}\mathbf{e}(s_0)$$

and $\rho(s_0)\neq 0$. In this case, by Theorem 5.1, the germ of the image of the first lightcone dual surface $\mathbf{FLS}_\gamma(s,\theta)$ is locally diffeomorphic to the cuspidal edge $C(2,3)\times\mathbb{R}$. Moreover, $h^L_{v_0}$ has A_3-singularity at s_0 for

$$\mathbf{v}_0=\boldsymbol{\gamma}(s_0)+\frac{1}{k_g(s_0)}\mathbf{n}(s_0)+\frac{\sqrt{1+k_g^2(s_0)}}{k_g(s_0)}\mathbf{e}(s_0),$$

$\rho(s_0)=0$ and $\sigma(s_0)\neq 0$. In this case, the germ of the image of the first lightcone dual surface $\mathbf{FLS}_\gamma(s,\theta)$ is locally diffeomorphic to swallowtail SW . By Theorem 5.1, the critical value set of the first lightcone dual surface is locally diffeomorphic to the line and the $(2,3,4)$-cusp respectively. This completes the proof of claim (A).

(B) By the similar arguments to the above, if we consider the timelike height function H^T, we can show the assertion (B) of Theorem 5.3. This completes the proof. □

6. Generic properties

In this section we consider generic properties of spacelike curves in S^3_1. The main tool is a transversality theorem. Let $Emb_{SD}(I,S^3_1)$ be the space of spacelike embeddings $\boldsymbol{\gamma}:I\to S^3_1$ with $\langle\mathbf{t}',\mathbf{t}'\rangle\neq 1$ equipped with Whitney C^∞-topology. We also consider the function $\mathcal{H}^L:S^3_1\times LC^3_+\to\mathbb{R}$ defined by $\mathcal{H}^L(\mathbf{u},\mathbf{v})=\langle\mathbf{u},\mathbf{v}\rangle-1$. We claim that $\mathcal{H}^L_v$ is a submersion for any $\mathbf{v}\in LC^3_+$, where $\mathcal{H}^L_{\mathbf{v}}(\mathbf{u})=\mathcal{H}^L(\mathbf{u},\mathbf{v})$. For any $\boldsymbol{\gamma}\in Emb_{SD}(I,S^3_1)$, we have $H^L=\mathcal{H}^L\circ(\boldsymbol{\gamma}\times id_{LC^3_+})$. We also have the ℓ-jet extension $j^\ell_1H^L:I\times LC^3_+\to J^\ell(I,\mathbb{R})$ defined by $j^\ell_1H^L(s,\mathbf{v})=j^\ell h^L_{\mathbf{v}}(s)$. We consider the trivialization $J^\ell(I,\mathbb{R})\equiv I\times\mathbb{R}\times J^\ell(1,1)$. For any submanifold $Q\subset J^\ell(1,1)$, we denote that $\widetilde{Q}=I\times\{0\}\times Q$. Then we have the following proposition as a corollary of Lemma 6 in Wassermann [13].

Proposition 6.1. *Let* Q *be a submanifold of* $J^\ell(1,1)$. *Then the set*

$$T_Q=\{\boldsymbol{\gamma}\in Emb_{SD}(I,S^3_1)\mid j^\ell H^L \textit{is transversal to } Q\}$$

is a residual subset of $Emb_{SD}(I,S^3_1)$. *If* Q *is a closed subset, then* T_Q *is open.*

Let $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ which has an A_k-singularity at 0. It is well known that there exists a diffeomorphism germ $\phi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $f \circ \phi = \pm s^{k+1}$. This is the classification of A_k-singularities. For any $z = j^l f(0)$ in $J^\ell(1,1)$, we have the orbit $L^l(z)$ given by the action of the Lie group of l-jet diffeomorphism germs. If f has an A_k-singularity, then the codimension of the orbit is k. There is another characterization of $\mathcal{R}$-versal unfoldings as follows.

Proposition 6.2. *Let $F : (\mathbb{R} \times \mathbb{R}^r, \mathbf{0}) \to (\mathbb{R}, 0)$ be an r-parameter unfolding of $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ which has an A_k-singularity at 0. Then F is an $\mathcal{R}$-versal unfolding if and only if $j_1^l F$ is transversal to the orbit $L^l(\widetilde{j^l f(0)})$ for $l \geq k+1$. Here, $j_1^l F : (\mathbb{R} \times \mathbb{R}^r, \mathbf{0}) \to J^\ell(\mathbb{R}, \mathbb{R})$ is the l-jet extension of F given by $j_1^l F(s, \mathbf{x}) = j^l F_x(s)$.*

We can prove the following generic classification theorem.

Theorem 6.3. *Let $\boldsymbol{\gamma} : I \to S_1^3$ be a unit speed spacelike curve with $k_g(s) \neq 0$ and $\delta(s) = -1$. Then one have the following generic classification.*

(1) *There exists an open and dense subset $\mathcal{O}_1 \subset Emb_{SD}(I, S_1^3)$ such that for any $\boldsymbol{\gamma} \in \mathcal{O}_1$, the first lightcone dual surface of spacelike curve $\mathbf{FLS}_\gamma(s, \theta)$ of $\boldsymbol{\gamma}(s)$ is locally diffeomorphic to the cuspidal edge or swallowtail if the point is singular.*

(2) *There exists an open and dense subset $\mathcal{O}_2 \subset Emb_{SD}(I, S_1^3)$ such that for any $\boldsymbol{\gamma} \in \mathcal{O}_2$, the first hyperbolic dual surface $\mathbf{FHD}_\gamma(s, \theta)$ of $\boldsymbol{\gamma}(s)$ is locally diffeomorphic to the cuspidal edge or swallowtail if the point is singular.*

Proof. (1) For $l \geq 4$, we consider the decomposition of the jet space $J^\ell(1,1)$ into $L^l(1)$ orbits. We now define a semi-algebraic set by

$$\Sigma^l = \{z = j^l f(0) \in J^\ell(1,1) \mid f \text{ has an } A_{\geq 4} \text{ -singularity}\}.$$

Then the codimension of Σ^l is 4. Therefore, the codimension of $\widetilde{\Sigma_0^l} = I \times \{0\} \times \Sigma^l$ is 5. We have the orbit decomposition of $J^\ell(1,1) - \Sigma^l$ into

$$J^\ell(1,1) - \Sigma^l = L_0^l \cup L_1^l \cup L_2^l \cup L_3^l,$$

where L_k^l is the orbit through an A_k-singularity. Thus, the codimension of $\widetilde{L_k^L}$ is $k+1$. We consider the l-jet extension $j_1^\ell H^L$ of the lightcone height function H^L. By Proposition 6.1, there exists an open and dense subset $\mathcal{O}_1 \subset Emb_{SD}(I, S_1^3)$ such that $j_1^\ell H^L$ is transversal to L_k^l $(k = 0, 1, 2, 3)$ and the orbit decomposition of $\widetilde{\Sigma^l}$. This means that $j_1^\ell H^L(I \times LC_+^3) \cap \widetilde{\Sigma^l} = \emptyset$ and H^L is an $\mathcal{R}$-versal unfolding of h_v^L at any point $(s_0, \mathbf{v}_0)$.

By Theorem 5.1, the discriminant set of H^L (i.e., the first lightcone dual surface) is locally diffeomorphic to cuspidal edge or swallowtail if the point is singular.

(2) By the similar arguments to the above, if we consider the timelike height function H^T, we can show the assertion (2). This completes the proof. □

7. Example

In order to better illustrate our results, we give one example that consists of a lightcone dual surface and a hyperbolic dual surface. Furthermore, we depict these surfaces by computer.

Example 7.1. Let $\boldsymbol{\gamma}(s)$ be a unit speed spacelike curve on S_1^3 defined by

$$\boldsymbol{\gamma}(s) = \left(\frac{\sqrt{2}}{2}\sinh(\sqrt{2}s),\ \frac{\sqrt{2}}{2}\cosh(\sqrt{2}s),\ \frac{\sqrt{2}}{2}\sin(2s),\ \frac{\sqrt{2}}{2}\cos(2s) \right)$$

with respect to an arclength parameter s. It is easy to get the tangent vector $\mathbf{t}(s)$ which is given by

$$\mathbf{t}(s) = (\ \cosh(\sqrt{2}s),\ \sinh(\sqrt{2}s),\ \sqrt{2}\cos(2s),\ -\sqrt{2}\sin(2s)\ ).$$

Then we get that $\langle \mathbf{t}'(s), \mathbf{t}'(s)\rangle = 10 \neq 1$, $k_g(s) = 3$ and

$$\begin{aligned}
\tau_g(s) &= \frac{1}{k_g^2(s)} det(\boldsymbol{\gamma}(s), \boldsymbol{\gamma}^{'}(s), \boldsymbol{\gamma}^{''}(s), \boldsymbol{\gamma}^{'''}(s)) \\
&= \frac{1}{9}\begin{vmatrix}
\frac{\sqrt{2}}{2}sinh(\sqrt{2}s) & \frac{\sqrt{2}}{2}cosh(\sqrt{2}s) & \frac{\sqrt{2}}{2}sin(2s) & \frac{\sqrt{2}}{2}cos(2s) \\
cosh(\sqrt{2}s) & sinh(\sqrt{2}s) & \sqrt{2}cos(2s) & -\sqrt{2}sin(2s) \\
\sqrt{2}sinh(\sqrt{2}s) & \sqrt{2}cosh(\sqrt{2}s) & -2\sqrt{2}sin(2s) & -2\sqrt{2}cos(2s) \\
2cosh(\sqrt{2}s) & 2sinh(\sqrt{2}s) & -4\sqrt{2}cos(2s) & -4\sqrt{2}sin(2s)
\end{vmatrix} \\
&= 2\sqrt{2}.
\end{aligned}$$

Thus, we can get $\rho(s) = 720$ and $\sigma(s) = 24$. We obtain one of normal vector $\mathbf{n}(s)$ which is given by

$$\mathbf{n}(s) = \left(\frac{\sqrt{2}}{2}\sinh(\sqrt{2}s),\ \frac{\sqrt{2}}{2}\cosh(\sqrt{2}s), -\frac{\sqrt{2}}{2}\sin(2s), \frac{\sqrt{2}}{2}\cos(2s) \right).$$

It is easy to get $\delta(s) = -1$. The other normal vector $\mathbf{e}(s)$ is given by

$$\begin{aligned}
\mathbf{e}(s) &= \boldsymbol{\gamma}(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}(s) \\
&= \begin{vmatrix}
-\mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\
\frac{\sqrt{2}}{2}\sinh(\sqrt{2}s) & \frac{\sqrt{2}}{2}\cosh(\sqrt{2}s) & \frac{\sqrt{2}}{2}\sin(2s) & \frac{\sqrt{2}}{2}\cos(2s) \\
\cosh(\sqrt{2}s) & \sinh(\sqrt{2}s) & \sqrt{2}\cos(2s) & -\sqrt{2}\sin(2s) \\
\frac{\sqrt{2}}{2}\sinh(\sqrt{2}s) & \frac{\sqrt{2}}{2}\cosh(\sqrt{2}s) & -\frac{\sqrt{2}}{2}\sin(2s) & -\frac{\sqrt{2}}{2}\cos(2s)
\end{vmatrix} \\
&= (\ \sqrt{2}\cosh(\sqrt{2}s),\ \sqrt{2}\sinh(\sqrt{2}s), \cos(2s),\ -\sin(2s)\ ).
\end{aligned}$$

Let $\sinh(\theta) = u,\ \cosh(\theta) = \sqrt{1+u^2}$. Thus, the first lightcone dual surface is given by

$$\mathbf{FLS}_\gamma(u,s) = \left(x_1(u,s), x_2(u,s), x_3(u,s), x_4(u,s)\right),$$

where

$$\begin{cases}
x_1(u,s) = \dfrac{\sqrt{2}}{2}(1+u)\sinh(\sqrt{2}s) + \sqrt{1+u^2}\sqrt{2}\cosh(\sqrt{2}s), \\
x_2(u,s) = \dfrac{\sqrt{2}}{2}(1+u)\cosh(\sqrt{2}s) + \sqrt{1+u^2}\sqrt{2}\sinh(\sqrt{2}s), \\
x_3(u,s) = \dfrac{\sqrt{2}}{2}(1-u)\sin(2s) + \sqrt{1+u^2}\cos(2s), \\
x_4(u,s) = \dfrac{\sqrt{2}}{2}(1-u)\cos(2s) - \sqrt{1+u^2}\sin(2s).
\end{cases}$$

The first hyperbolic dual surface is given by $\mathbf{FHD}_\gamma(u,s) = (y_1(u,s), y_2(u,s), y_3(u,s), y_4(u,s))$, where

$$\begin{cases}
y_1(u,s) = \dfrac{\sqrt{2}}{2}u\sinh(\sqrt{2}s) + \sqrt{1+u^2}\sqrt{2}\cosh(\sqrt{2}s), \\
y_2(u,s) = \dfrac{\sqrt{2}}{2}u\cosh(\sqrt{2}s) + \sqrt{1+u^2}\sqrt{2}\sinh(\sqrt{2}s), \\
y_3(u,s) = -\dfrac{\sqrt{2}}{2}u\sin(2s) + \sqrt{1+u^2}\cos(2s), \\
y_4(u,s) = -\dfrac{\sqrt{2}}{2}u\cos(2s) - \sqrt{1+u^2}\sin(2s).
\end{cases}$$

We obtain the vector parametric equations of the singular locus of the first lightcone dual surface and the first hyperbolic dual surface as follows:

$$\begin{aligned}\mathbf{SFLS}_\gamma(s) = \Big(&\frac{2\sqrt{2}}{3}\sinh(\sqrt{2}s) + \frac{2\sqrt{5}}{3}\cosh(\sqrt{2}s), \frac{2\sqrt{2}}{3}\cosh(\sqrt{2}s) + \frac{2\sqrt{5}}{3}\sinh(\sqrt{2}s),\\ &\frac{\sqrt{2}}{3}\sin(2s) + \frac{\sqrt{10}}{3}\cos(2s), \frac{\sqrt{2}}{3}\cos(2s) - \frac{\sqrt{10}}{3}\sin(2s)\Big),\end{aligned}$$

$$\begin{aligned}\mathbf{SFHD}_\gamma(s) = \Big(&\frac{\sqrt{2}}{6}\sinh(\sqrt{2}s) + \frac{2\sqrt{5}}{3}\cosh(\sqrt{2}s), \frac{\sqrt{2}}{6}\cosh(\sqrt{2}s) + \frac{2\sqrt{5}}{3}\sinh(\sqrt{2}s),\\ &-\frac{\sqrt{2}}{6}\sin(2s) + \frac{\sqrt{10}}{3}\cos(2s), -\frac{\sqrt{2}}{6}\cos(2s) - \frac{\sqrt{10}}{3}\sin(2s)\Big).\end{aligned}$$

We see that $\rho(s) = 720 \neq 0$ for arbitrary real numbers $s > 0$. Hence, we have that the first lightcone dual surface $\mathbf{FLS}_\gamma(s)$ is locally diffeomorphic to cuspidal edge at its singular points and the singular locus of the first lightcone dual surface $\mathbf{SFLS}_\gamma(s)$ is locally diffeomorphic to a line. We draw the pictures of the first lightcone dual surface and its singular locus by projecting them into three dimensional spaces, see Fig. 2. On the other hand, we see that $\tau_g(s) = 2\sqrt{2} \neq 0$ for arbitrary real numbers $s > 0$. Hence, we have that the first hyperbolic dual surface $\mathbf{FHD}_\gamma(s)$ is locally diffeomorphic to cuspidal edge at its singular points and the singular locus of the first hyperbolic dual surface $\mathbf{SFHD}_\gamma(s)$ is locally diffeomorphic to a line. We also draw the pictures of the first hyperbolic dual surface and its singular locus by projecting them into three dimensional spaces, see Fig. 3.

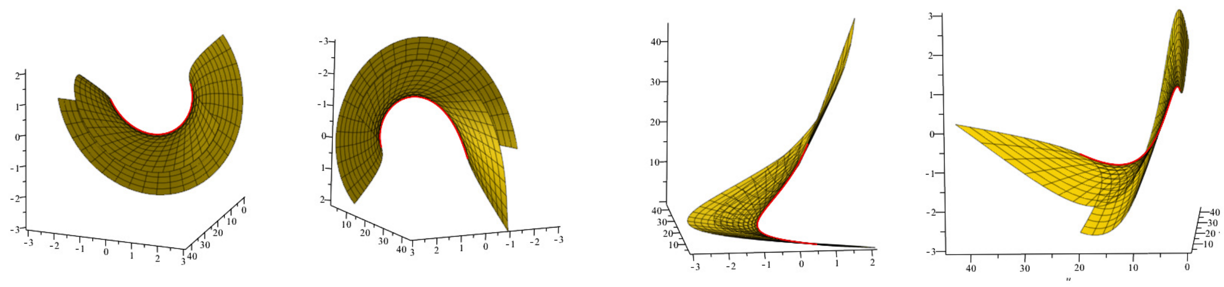

Figure 2: Projection of the first lightcone dual surface respectively on $x_2x_3x_4$-space,$x_1x_3x_4$-space,$x_1x_2x_4$-space,$x_1x_2x_3$-space.

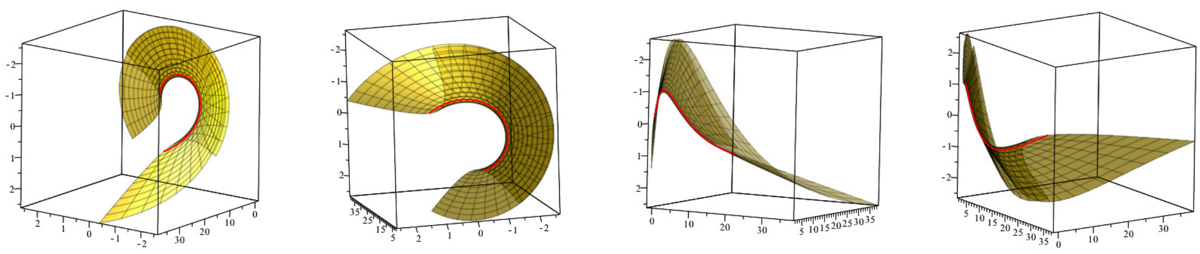

Figure 3: Projection of the first hyperbolic dual surface respectively on $y_2y_3y_4$-space,$y_1y_3y_4$-space,$y_1y_2y_4$-space,$y_1y_2y_3$-space.

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