Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2577–2589
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Singularity analysis of pseudo null hypersurfaces and pseudo hyperbolic hypersurfaces

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Communicated by R. Saadati

Abstract

This paper introduces the notions of pseudo null curves in Minkowski 4-space. Meanwhile, some geometrical characterizations and the singularities of pseudo null hypersurfaces and pseudo hyperbolic hypersurfaces, which are generated by pseudo null curves, are considered in this paper.

Keywords: Minkowski space, singularity, pseudo null hypersurfaces, pseudo hyperbolic hypersurfaces.
2010 MSC: 58K27, 53B50, 53A35.

1. Introduction

Minkowski space is a real vector space with a symmetric bilinear form. And Minkowski space form with the positive curvature is called de Sitter space. We know that de Sitter 3-space is a vacuum solution of the Einstein equation and an important cosmological model for physical universe [2, 4, 7]. The Euclidean rectifying curves have many interesting geometric properties. In Minkowski 3-space, the rectifying curves have similar geometric properties as in the Euclidean 3-space. However, in Minkowski 4-space, it is more interesting for the rectifying curves using the null frame vector. The pseudo null hypersurfaces and pseudo hyperbolic hypersurfaces are two of those rectifying surfaces. The authors have considered the geometrical properties of spacelike curves and spacelike surfaces in Minkowski space [5]; Bilici obtained the geometrical properties of involutes of spacelike curves in Minkowski 3-space [1]. In particular, when the Frenet frame along a spacelike or a timelike curve contains a null vectors, such curve is said to be a pseudo null curve. The Frenet equations of a pseudo null curve, lying fully in $\mathbb{R}^4_1$, are given in [6, 11]. However, most of papers and books are studying the geometrical properties of spacelike curves without any null Fernet frames in Minkowski 4-space. In this paper, we consider the pseudo null hypersurfaces and pseudo hyperbolic

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Received 2015-06-19

hypersurfaces, which are generated by pseudo null curves on de Sitter 3-space, as the most elementary case for the study of the singularities of pseudo hyperbolic hypersurfaces in non-flat Lorentzian space forms.

On the other hand, singularity theory, which is a direct descendant of differential calculus, is certain to have a great deal of interest to say about geometry, equation, physic, astronomy and other disciplines [3, 5, 8, 9, 10]. In general, the current theory always does not allow for singularities. However, it is unavoidable in some real life circumstances. Thus, we apparently need to understand the ontology of singularities if we want to research the nature of space and time in the actual universe. At present, the studying of singularities is mainly concentrated in general surfaces [3, 5, 8, 9]. The authors considered a classification of the singularities of lightlike surfaces with co-dimensional two for generic space-like curves on de Sitter 3-space and a geometric characterization of the singularities [5], which motivate us to investigate the differential geometry of pseudo null curves. The most interesting case is the contact of pseudo null curves with hyperbolic 3-space. We consider the geometric characterizations of pseudo null curves and the singularities of pseudo null hypersurfaces and pseudo hyperbolic hypersurfaces in this paper.

The remainder of this paper is organized as follows: Section 2 reviews some basic notions about the Minkowski space and gives the main results about the classifications of singularities (Theorem 2.1 and Theorem 2.2). Section 3 considers some height functions on pseudo null curves. Also, the versal properties of those height functions are used to prove Theorem 2.1 and Theorem 2.2 in Section 4. Section 5 gives the generic properties of pseudo null curves on de Sitter 3-space to introduce the stability of singularity. In the last section of this paper, we supply an example to explain the singular of pseudo null hypersurfaces and pseudo hyperbolic hypersurfaces of pseudo null curves, respectively.

We shall assume that all the maps and manifolds in this paper are C^∞, unless the contrary is explicitly stated.

2. Preliminaries and the main result

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) | x_i \in \mathbb{R}\ (i = 1, 2, 3, 4)\ \}$ be a 4-dimensional vector space. For any two vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ and $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$, the *symmetric bilinear form* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4.$$

$(\mathbb{R}^4, \langle , \rangle)$ is called *4-dimensional Minkowski space* and written by $\mathbb{R}^4_1$.

A vector $\boldsymbol{x}$ in $\mathbb{R}^4_1$ is called a *spacelike vector*, a *null vector* or a *timelike vector* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ is positive, zero or negative, respectively. The *norm* of $\boldsymbol{x} \in \mathbb{R}^4_1$ is defined by $\|\boldsymbol{x}\| = (\mathrm{sign}(\boldsymbol{x})\langle \boldsymbol{x}, \boldsymbol{x} \rangle)^{1/2}$, where $\mathrm{sign}(\boldsymbol{x})$ denotes the signature of $\boldsymbol{x}$ which is given by $\mathrm{sign}(\boldsymbol{x})$=1, 0 or -1 when $\boldsymbol{x}$ is spacelike, lightlike or timelike vector. For any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^4_1$, we say $\boldsymbol{x}$ is *pseudo-perpendicular* to $\boldsymbol{y}$ if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$. For vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ and $\boldsymbol{z} = (z_1, z_2, z_3, z_4)$ in $\mathbb{R}^4_1$, we define a vector $\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z}$ by

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & -\mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ is the canonical base of $\mathbb{R}^4_1$. One can easily show that

$$\langle \boldsymbol{a}, \boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} \rangle = \det(\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}).$$

In $\mathbb{R}^4_1$, we introduce some typical manifolds,

$$\begin{aligned} \textit{de Sitter 3-space} \quad & \mathbb{S}^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}, \\ \textit{hyperbolic 3-space} \quad & \mathbb{H}^3 = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\}, \\ \textit{lightcone} \quad & \mathbb{LC}^* = \{\boldsymbol{x} \in \mathbb{R}^4_1 \setminus \{\mathbf{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}, \end{aligned}$$

we call $\mathbb{S}^3_1, \mathbb{H}^3, \mathbb{LC}^*$ the pseudo spheres in Minkowski 4-space. For a vector $\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in \mathbb{LC}^*$, if $v_4 = 1$, vector $\boldsymbol{v}$ is denoted by $\widetilde{\boldsymbol{v}}$.

Let $\boldsymbol{\gamma} : I \to \mathbb{R}^4_1$ by $\boldsymbol{\gamma}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be a regular curve in $\mathbb{R}^4_1$ (*i.e.*, $\dot{\boldsymbol{\gamma}}(t) \neq \mathbf{0}$ for any $t \in I$), where I is an open interval. For any $t \in I$, a curve $\boldsymbol{\gamma}$ is called *spacelike curve*, *null (lightlike) curve* or *timelike curve* if $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle > 0, \langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle = 0$ or $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t)\rangle < 0$, respectively. The arc-length of a non null curve $\boldsymbol{\gamma}(t)$ measured from $\boldsymbol{\gamma}(t_0)$ ($t_0 \in I$) is

$$s(t) = \int_{t_0}^{t} \|\dot{\boldsymbol{\gamma}}(t)\| dt.$$

The parameter s is determined as $\|\boldsymbol{\gamma}'(s)\| = 1$ for a non null curve, where $\boldsymbol{\gamma}'(s) = (d\boldsymbol{\gamma}/ds)(s)$. For a curve $\boldsymbol{\gamma}(s)$ on de Sitter 3-space, we call curve $\boldsymbol{\gamma}(s)$ be *pseudo null curve* if its tangent vector $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ is spacelike vector, the normal vector $\boldsymbol{n}_1(s)$ and the binormal vector $\boldsymbol{n}_2(s)$ are null vectors. The pseudo-orthonormal frame of pseudo null curve $\boldsymbol{\gamma}(s)$ is $\{\boldsymbol{\gamma}(s),\ \boldsymbol{t}(s),\ \boldsymbol{n}_1(s),\ \boldsymbol{n}_2(s)\}$ satisfying

$$\begin{aligned}
&\langle \boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle = \langle \boldsymbol{t}(s), \boldsymbol{t}(s)\rangle = \langle \boldsymbol{n}_1(s), \boldsymbol{n}_2(s)\rangle = 1,\\
&\langle \boldsymbol{\gamma}(s), \boldsymbol{t}(s)\rangle = \langle \boldsymbol{\gamma}(s), \boldsymbol{n}_1(s)\rangle = \langle \boldsymbol{\gamma}(s), \boldsymbol{n}_2(s)\rangle = 0,\\
&\langle \boldsymbol{t}(s), \boldsymbol{n}_1(s)\rangle = \langle \boldsymbol{t}(s), \boldsymbol{n}_2(s)\rangle = 0,\\
&\langle \boldsymbol{n}_1(s), \boldsymbol{n}_1(s)\rangle = \langle \boldsymbol{n}_2(s), \boldsymbol{n}_2(s)\rangle = 0.
\end{aligned}$$

The Frenet type formulas as following [6],

$$\begin{cases}
\boldsymbol{\gamma}'(s) = \boldsymbol{t}(s),\\
\boldsymbol{t}'(s) = -\boldsymbol{\gamma}(s) + \boldsymbol{n}_1(s) + \kappa(s)\boldsymbol{n}_2(s),\\
\boldsymbol{n}_1'(s) = -\kappa(s)(s)\boldsymbol{t}(s) + \tau(s)\boldsymbol{n}_1(s),\\
\boldsymbol{n}_2'(s) = -\boldsymbol{t}(s) - \tau(s)\boldsymbol{n}_2(s),
\end{cases} \tag{2.1}$$

where $\kappa(s) = \langle \boldsymbol{t}'(s), \boldsymbol{n}_1(s)\rangle$ and $\tau(s) = \langle \boldsymbol{n}_1'(s), \boldsymbol{n}_2(s)\rangle$.

Let $\boldsymbol{\gamma} : I \to \mathbb{S}^3_1$ be a pseudo null curve, we define two maps, one is $\mathbb{L} : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^4_1$ by

$$\mathbb{L}(s, \mu, \eta) = \boldsymbol{\gamma}(s) + \mu\boldsymbol{n}_1(s) + \eta\boldsymbol{n}_2(s).$$

We call the image of $\mathbb{L}$ the *pseudo null hypersurface* associated to the pseudo null curve $\boldsymbol{\gamma}(s)$; the other is $\widehat{\mathbb{L}} : I \times \mathbb{R} \to \mathbb{H}^3$ by

$$\widehat{\mathbb{L}}(s, \mu) = \boldsymbol{\gamma}(s) + \mu\boldsymbol{n}_1(s) - \frac{2}{\mu}\boldsymbol{n}_2(s).$$

We call the image of $\widehat{\mathbb{L}}$ the *pseudo hyperbolic hypersurface* on hyperbolic 3-space. And it is obvious that the pseudo hyperbolic hypersurface is a special pseudo null hypersurface when $\mu\eta = -2$.

Let $F : \mathbb{S}^3_1 \longrightarrow \mathbb{R}$ be a submersion and $f : I \longrightarrow \mathbb{S}^3_1$ be a pseudo null curve. We say that f and $F^{-1}(0)$ have *k-point* contact at $t = t_0$ if the function $g(t) = (F \circ f)(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$ and $g^{(k)}(t_0) \neq 0$. We also say that f and $F^{-1}(0)$ have at least *k-point* contact at $t = t_0$ if the function $g(t) = (F \circ f)(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$. The main results as following:

Theorem 2.1. *Let $\boldsymbol{\gamma}(s)$ be a pseudo null curve on de Sitter 3-space with $\kappa'(s) - 2\tau(s)\kappa(s) \neq 0$ for $\forall s \in I$. $\boldsymbol{v}_0 = \mathbb{L}(s_0, \mu_0, \eta_0)$ and osculating sphere $LC(\boldsymbol{v}_0) = \{\boldsymbol{u} \in \mathbb{S}^3_1 \mid \langle \boldsymbol{u}, \boldsymbol{v}_0\rangle = 1\}$, we have the following,*

1. *$\boldsymbol{\gamma}(s)$ and $LC(\boldsymbol{v}_0)$ have at least 2-point contact for s_0.*
2. *$\boldsymbol{\gamma}(s)$ and $LC(\boldsymbol{v}_0)$ have at least 3-point contact for s_0 if and only if*

$$\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \frac{\tau(s_0)}{\kappa'(s_0) - 2\tau(s_0)\kappa(s_0)}\boldsymbol{n}_1(s_0) + \frac{\kappa'(s_0) - 3\tau(s_0)\kappa(s_0)}{\kappa'(s_0) - 2\tau(s_0)\kappa(s_0)}\boldsymbol{n}_2(s_0)$$

and $\sigma(s) = \tau(s)\kappa''(s) + (2\kappa(s) + \tau'(s) - \tau^2(s))\kappa'(s) - 2\kappa(s)\tau(s)\tau'(s) - 2\kappa(s)\tau^3(s) \neq 0$. Under this condition, the germ of image $\mathbb{L}$ at $\mathbb{L}(s_0, \mu_0, \eta_0)$ is diffeomorphic to the cuspidal edge $C \times \mathbb{R}$. (Fig. 1)

3. *$\boldsymbol{\gamma}(s)$ and $LC(\boldsymbol{v}_0)$ have at least 4-point contact for s_0 if and only if*

$$\boldsymbol{v}_0 = \boldsymbol{\gamma}(s_0) + \frac{\tau(s_0)}{\kappa'(s_0) - 2\tau(s_0)\kappa(s_0)}\boldsymbol{n}_1(s_0) + \frac{\kappa'(s_0) - 3\tau(s_0)\kappa(s_0)}{\kappa'(s_0) - 2\tau(s_0)\kappa(s_0)}\boldsymbol{n}_2(s_0)$$

and $\sigma(s) = 0, \sigma'(s) \neq 0$, where $\sigma(s)$ is a new geometry invariant, which can effect the singularities of the hypersurfaces. Under this condition, the germ of image $\mathbb{L}$ at $\mathbb{L}(s_0, \mu_0, \eta_0)$ is diffeomorphic to the swallowtail SW. (Fig. 2).

When the condition $\kappa'(s) - 2\tau(s)\kappa(s) = 0$, we have the following result.

Theorem 2.2. *Let $\boldsymbol{\gamma}(s)$ be a pseudo null curve on de Sitter 3-space, $\widehat{\boldsymbol{v}}_0 = \widehat{\mathbb{L}}(s_0, \mu_0) \in \mathbb{H}^3$ and hyperbolic osculating sphere $\widehat{LC}(\widehat{\boldsymbol{v}}_0) = \{\boldsymbol{u} \in \mathbb{S}^3_1 \mid \langle \boldsymbol{u}, \widehat{\boldsymbol{v}}_0\rangle = 1\}$ we have the following,*

1. *$\boldsymbol{\gamma}(s)$ and $\widehat{LC}(\widehat{\boldsymbol{v}}_0)$ have at least 2-point contact for s_0.*

2. *$\boldsymbol{\gamma}(s)$ and $\widehat{LC}(\widehat{\boldsymbol{v}}_0)$ have at least 3-point contact for s_0 if and only if*

$$\widehat{\boldsymbol{v}}_0 = \boldsymbol{\gamma}(s_0) + \frac{1 \pm \sqrt{1 + 8\kappa(s_0)}}{2\kappa(s_0)}\boldsymbol{n}_1(s_0) + \frac{1 \mp \sqrt{1 + 8\kappa(s)}}{2}\boldsymbol{n}_2(s_0)$$

and $\widehat{\sigma}(s) = \kappa'(s) - 2\kappa(s)\tau(s) = 0, \widehat{\sigma}'(s) \neq 0, \kappa(s) \neq 0$. Under this condition, the germ of image $\widehat{\mathbb{L}}$ at $\widehat{\mathbb{L}}(s_0, \mu_0)$ is diffeomorphic to the cuspidal edge $C \times \mathbb{R}$. (Fig. 1)

3. *$\boldsymbol{\gamma}(s)$ and $\widehat{LC}(\widehat{\boldsymbol{v}}_0)$ have at least 4-point contact for s_0 if and only if*

$$\widehat{\boldsymbol{v}}_0 = \boldsymbol{\gamma}(s_0) + \frac{1 \pm \sqrt{1 + 8\kappa(s_0)}}{2\kappa(s_0)}\boldsymbol{n}_1(s_0) + \frac{1 \mp \sqrt{1 + 8\kappa(s_0)}}{2}\boldsymbol{n}_2(s_0)$$

and $\widehat{\sigma}(s) = \widehat{\sigma}'(s) = 0$, $\widehat{\widehat{\sigma}}(s) = \kappa'''(s) - 12\kappa(s)\tau(s)\tau'(s) + 4\kappa(s)\tau^3(s) - \kappa(s)\tau''(s) - \kappa(s)\tau(s)\tau''(s) \neq 0, \kappa(s) \neq 0$. Under this condition, the germ of image $\widehat{\mathbb{L}}$ at $\widehat{\mathbb{L}}(s_0, \mu_0)$ is diffeomorphic to the swallowtail SW. (Fig. 2)

Remark 2.3. When $\boldsymbol{\gamma}(s)$ be a pseudo null curve on de Sitter 3-space with the curvature $\kappa(s) = 0$, by the same methods in [6], we know the curve $\boldsymbol{\gamma}(s)$ is a plane curve, and $\boldsymbol{\gamma}(s)$ has only one order contact with hyperbolic 3-space.

Here $C \times \mathbb{R} = \{(x_1, x_2, x_3) \mid x_1 = u, x_2 = \pm v^{1/2}, x_3 = v^{1/3}\}$ is the cuspidal edge and $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$ is the swallowtail.

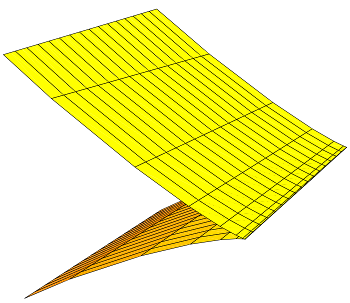

Figure 1: cuspidaledge

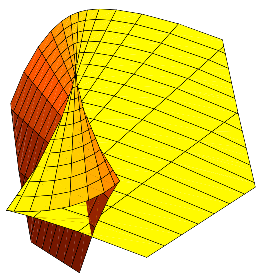

Figure 2: swallowtail

3. Pseudo null height function and pseudo hyperbolic height function

The section is to construct a germ of family of functions $F : I \times \mathbb{R}^4_1 \to \mathbb{R}$ with parameter space $\mathbb{R}^4_1$, such that the germ at s_0 of the subset in question is the bifurcation set or discriminant of the family, and to show that the family of functions is a versal deformation of the germ at s_0 of the function $h(s)$ defined by $h_v(s) = H(s, \boldsymbol{v})$.

Let $\boldsymbol{\gamma}(s)$ be a pseudo null curve on de Sitter 3-space, we define a function $H : I \times \mathbb{R}^4_1 \to \mathbb{R}$ by

$$H(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s), \boldsymbol{v} \rangle - 1,$$

which is called *pseudo null height function* of $\boldsymbol{\gamma}(s)$. Denoted $h_v(s) = H(s, \boldsymbol{v})$ for any fixed vector $\boldsymbol{v} \in \mathbb{R}^4_1$. Also, we can define three functions to describe the contact relationship between pseudo null curves and pseudo spheres: *pseudo de Sitter height function* $\widetilde{H} : I \times \mathbb{S}^3_1 \to \mathbb{R}$ by

$$\widetilde{H}(s, \widetilde{\boldsymbol{v}}) = \langle \boldsymbol{\gamma}(s), \widetilde{\boldsymbol{v}} \rangle - 1,$$

pseudo lightcone height function $\overline{H} : I \times \mathbb{LC}^* \to \mathbb{R}$ by

$$\overline{H}(s, \overline{\boldsymbol{v}}) = \langle \boldsymbol{\gamma}(s), \overline{\boldsymbol{v}} \rangle - 1,$$

pseudo hyperbolic height function $\widehat{H} : I \times \mathbb{H}^3 \to \mathbb{R}$ by

$$\widehat{H}(s, \widehat{\boldsymbol{v}}) = \langle \boldsymbol{\gamma}(s), \widehat{\boldsymbol{v}} \rangle - 1.$$

Proposition 3.1. *Suppose $\boldsymbol{\gamma}(s)$ be a pseudo null curve on de Sitter 3-space with $\kappa'(s) - 2\tau(s)\kappa(s) \neq 0$ for $\forall s \in I$ and $\boldsymbol{v} \in \mathbb{R}^4_1$. Then,*

1. $h_v(s) = h'_v(s) = 0$ if and only if there exist two real numbers μ, η such that $\boldsymbol{v} = \boldsymbol{\gamma}(s) + \mu \boldsymbol{n}_1(s) + \eta \boldsymbol{n}_2(s)$.

2. $h_v(s) = h'_v(s) = h''_v(s) = 0$ if and only if $\boldsymbol{v} = \boldsymbol{\gamma}(s) + \mu \boldsymbol{n}_1(s) + (1 - \kappa(s)\mu)\boldsymbol{n}_2(s)$.

3. $h_v(s) = h'_v(s) = h''_v(s) = h^{(3)}_v(s) = 0$ if and only if

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) + \frac{\tau(s)}{\kappa'(s) - 2\tau(s)\kappa(s)} \boldsymbol{n}_1(s) + \frac{\kappa'(s) - 3\tau(s)\kappa(s)}{\kappa'(s) - 2\tau(s)\kappa(s)} \boldsymbol{n}_2(s)$$

and $\sigma(s) = \tau(s)\kappa''(s) + (2\kappa(s) + \tau'(s) - \tau^2(s))\kappa'(s) - 2\kappa(s)\tau(s)\tau'(s) - 2\kappa(s)\tau^3(s) \neq 0$.

4. $h_v(s) = h'_v(s) = h''_v(s) = h^{(3)}_v(s) = h^{(4)}_v(s) = 0$ if and only if

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) + \frac{\tau(s)}{\kappa'(s) - 2\tau(s)\kappa(s)} \boldsymbol{n}_1(s) + \frac{\kappa'(s) - 3\tau(s)\kappa(s)}{\kappa'(s) - 2\tau(s)\kappa(s)} \boldsymbol{n}_2(s)$$

and $\sigma(s) = 0, \sigma'(s) \neq 0$.

Proof.

1. Supposing there are four real numbers $\lambda, \omega, \mu, \eta$ satisfying $\boldsymbol{v} = \lambda \boldsymbol{\gamma}(s) + \omega \boldsymbol{t}(s) + \mu \boldsymbol{n}_1(s) + \eta \boldsymbol{n}_2(s) \in \mathbb{R}^4_1$, we obtain $\omega = 0$ and $\lambda = 1$ by $h_v(s) = h'_v(s) = 0$. Therefore, the assertion 1 follows.

2. The easy computation is that

$$h''_v(s) = \langle -\gamma(s) + \boldsymbol{n}_1(s) + \kappa(s)\boldsymbol{n}_2(s), \lambda\boldsymbol{\gamma}(s) + \omega \boldsymbol{t}(s) + \mu \boldsymbol{n}_1(s) + \eta \boldsymbol{n}_2(s) \rangle = 0, \tag{3.1}$$

we can obtain $-1 + \kappa(s)\mu + \eta = 0$. Hence, the assertion 2 holds.

3. Basing on the above assumption and using Equations (2.1), we have

$$\begin{aligned} h_v^{(3)}(s) &= \langle -\gamma'(s) + \boldsymbol{n}_1'(s) + \kappa'(s)\boldsymbol{n}_2(s) + \kappa(s)\boldsymbol{n}_2'(s), \boldsymbol{v}\rangle \\ &= \langle \tau(s)\boldsymbol{n}_1(s) + (\kappa'(s) - \kappa(s)\tau(s))\boldsymbol{n}_2(s)), \boldsymbol{v}\rangle \\ &= \tau(s)\eta + (\kappa'(s) - \kappa(s)\tau(s))\mu \\ &= 0. \end{aligned} \tag{3.2}$$

Substituting the formula $1 - \kappa(s)\mu + \eta = 0$ into Equation (3.2), we can obtain $\mu = \frac{\tau(s)}{\kappa'(s)-2\tau(s)\kappa(s)}$ and $\eta = \frac{\kappa'(s)-3\tau(s)\kappa(s)}{\kappa'(s)-2\tau(s)\kappa(s)}$.

4. By the Equations (2.1) and (3.2), we have

$$\begin{aligned} h_v^{(4)}(s) &= \langle (\tau'(s) + \tau''(s))\boldsymbol{n}_1(s) + (\kappa''(s) - 2\kappa'(s)\tau(s) - \kappa(s)\tau'(s) + \kappa(s)\tau^2(s))\boldsymbol{n}_2(s), \boldsymbol{v}\rangle, \\ &= \tau\kappa''(s) + (2\kappa(s) + \tau' - \tau^2(s))\kappa'(s) - 2\kappa(s)\tau\tau'(s) - 2\kappa(s)\tau^3, \\ &= 0. \end{aligned} \tag{3.3}$$

By assertion 3,

$$\sigma(s) = \tau(s)\kappa''(s) + (2\kappa(s) + \tau'(s) - \tau^2(s))\kappa'(s) - 2\kappa(s)\tau(s)\tau'(s) - 2\kappa(s)\tau^3(s) = 0$$

and $\sigma'(s) \neq 0$.

When $\kappa'(s) - 2\tau(s)\kappa(s) = 0, \kappa(s) \neq 0$, we can consider the other contact relation. □

Proposition 3.2. *Suppose $\boldsymbol{\gamma}(s)$ be a pseudo null curve on de Sitter 3-space and $\boldsymbol{v} \in \mathbb{H}^3$, for $\forall s \in I$. Then,*

1. *$\widehat{h}_v(s) = 0$ if and only if there exist three real numbers λ, μ, η such that $\boldsymbol{v} = \boldsymbol{\gamma}(s) + \lambda\boldsymbol{t}(s) + \mu\boldsymbol{n}_1(s) + \eta\boldsymbol{n}_2(s)$ and $1 + \lambda^2 + \mu\eta = -1$.*

2. *$\widehat{h}_v(s) = \widehat{h}'_v(s) = 0$ if and only if $\boldsymbol{v} = \boldsymbol{\gamma}(s) + \mu\boldsymbol{n}_1(s) + (1 - \kappa(s)\mu)\boldsymbol{n}_2(s)$ and $\mu\eta = -2$.*

3. *$\widehat{h}_v(s) = \widehat{h}'_v(s) = \widehat{h}''_v(s) = 0$ if and only if*

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) + \frac{1 \pm \sqrt{1 + 8\kappa(s)}}{2\kappa(s)}\boldsymbol{n}_1(s) + \frac{1 \mp \sqrt{1 + 8\kappa(s)}}{2}\boldsymbol{n}_2(s)$$

and $\widehat{\sigma}(s) = \kappa'(s) - 2\kappa(s)\tau(s) \neq 0$.

4. *$\widehat{h}_v(s) = \widehat{h}'_v(s) = \widehat{h}''_v(s) = \widehat{h}_v^{(3)}(s) = 0$ if and only if*

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) + \frac{1 \pm \sqrt{1 + 8\kappa(s)}}{2\kappa(s)}\boldsymbol{n}_1(s) + \frac{1 \mp \sqrt{1 + 8\kappa(s)}}{2}\boldsymbol{n}_2(s)$$

and $\widehat{\sigma}(s) = 0, \widehat{\sigma}'(s) \neq 0$.

5. *$\widehat{h}_v(s) = \widehat{h}'_v(s) = \widehat{h}''_v(s) = \widehat{h}_v^{(3)}(s) = \widehat{h}_v^{(4)}(s) = 0$ if and only if*

$$\boldsymbol{v} = \boldsymbol{\gamma}(s) + \frac{1 \pm \sqrt{1 + 8\kappa(s)}}{2\kappa(s)}\boldsymbol{n}_1(s) + \frac{1 \mp \sqrt{1 + 8\kappa(s)}}{2}\boldsymbol{n}_2(s)$$

and $\widehat{\sigma}(s) = \widehat{\sigma}'(s) = 0, \widehat{\widehat{\sigma}}(s) = \kappa'''(s) - 12\kappa(s)\tau(s)\tau'(s) + 4\kappa(s)\tau^3(s) - \kappa(s)\tau''(s) - \kappa(s)\tau(s)\tau''(s) \neq 0$.

The proof is the same as the proof of Proposition 3.1, we omit it here. Meanwhile, we can obtain the similar propositions for the pseudo de Sitter height function and pseudo lightcone height function.

4. Singularities of pseudo null hypersurfaces and pseudo hyperbolic hypersurfaces

In this section, we study the geometric properties of pseudo null hypersurfaces and pseudo hyperbolic hypersurfaces of pseudo null curves on de Sitter 3-space. Meanwhile, we use some general results on the singularity theory for families of function germs [3]. These properties will be stated following.

Proposition 4.1. *Suppose $\boldsymbol{\gamma}(s)$ be a pseudo null curve on de Sitter 3-space, the following assertions are established,*

1. *The singularities of $\mathbb{L}(s,\mu,\eta)$ and $\widehat{\mathbb{L}}(s,\mu)$ are the set $\{(s,\mu,\eta) \mid \eta = -1+\kappa(s)\mu, s \in I\}$ and $\{(s,\mu) \mid \mu = \frac{1\pm\sqrt{1+8\kappa(s)}}{2\kappa(s)}, \widehat{\sigma}(s) \neq 0\}$, respectively.*
2. *If $\boldsymbol{v}_0$ is a constant vector, then $\boldsymbol{\gamma}(s) \in LC(\boldsymbol{v}_0)$ and $\mu\kappa'(s) - \tau(s) = 0$.*
3. *If $\widehat{\boldsymbol{v}}_0 \in \mathbb{H}^3$ is a constant vector, then $\boldsymbol{\gamma}(s) \in \widehat{LC}(\widehat{\boldsymbol{v}}_0)$ and one can obtain $\tau(s)\kappa'(s)+\tau^2(s)\kappa(s)+2\kappa'^2(s) = 0$.*
4. *If $\overline{\boldsymbol{v}}_0 \in \mathbb{LC}^*$ is a constant vector, then $\boldsymbol{\gamma}(s) \in \overline{LC}(\overline{\boldsymbol{v}}_0) = \{\boldsymbol{u} \in \mathbb{S}^3_1 \mid \langle \boldsymbol{u}, \overline{\boldsymbol{v}}_0\rangle = 1\}$ and one can obtain $\tau(s)\kappa'(s) + \tau^2(s)\kappa(s) + \kappa'^2(s) = 0$.*
5. *If $\widetilde{\boldsymbol{v}}_0 \in \mathbb{S}^3_1$ is a constant vector, then $\boldsymbol{\gamma}(s) \in \widetilde{LC}(\widetilde{\boldsymbol{v}}_0) = \{\boldsymbol{u} \in \mathbb{S}^3_1 \mid \langle \boldsymbol{u}, \widetilde{\boldsymbol{v}}_0\rangle = 1\}$ and one can obtain $\kappa'(s) - \kappa(s)\tau(s) = 0$ or $\tau(s) = 0$.*

Proof.

1. Since $\mathbb{L}(s,\mu,\eta) = \boldsymbol{\gamma}(s) + \mu\boldsymbol{n}_1(s) + \eta\boldsymbol{n}_2(s)$, we have

$$\begin{aligned}\partial\mathbb{L}(s,\mu,\eta)/\partial s &= \boldsymbol{\gamma}'(s) + \mu\boldsymbol{n}_1'(s) + \eta\boldsymbol{n}_2'(s),\\ &= (1-\kappa(s)\mu-\eta)\boldsymbol{t}(s) + \mu\tau(s)\boldsymbol{n}_1(s) - \eta\tau(s)\boldsymbol{n}_2(s),\\ \partial\mathbb{L}(s,\mu,\eta)/\partial\mu &= \boldsymbol{n}_1(s),\\ \partial\mathbb{L}(s,\mu,\eta)/\partial\eta &= \boldsymbol{n}_2(s).\end{aligned} \tag{4.1}$$

The above three vectors are linearly dependent if and only if $\eta = -1+\kappa(s)\mu$. Hence, the assertion (1) is complete.

2. For two smooth functions $\nu, \eta : I \to \mathbb{R}$, we define a mapping $f_{\nu,\eta} : I \to \mathbb{R}^4_1$ by

$$f_{\nu,\eta}(s) = \boldsymbol{\gamma}(s) + \nu(s)\boldsymbol{n}_1(s) + \eta(s)\boldsymbol{n}_2(s).$$

Supposing $\boldsymbol{v}_0 = \mathbb{L}(s_0,\nu,\eta)$ is constant, we have

$$\begin{aligned}\frac{df_{\nu,\eta}(s)}{ds} &= \boldsymbol{\gamma}'(s) + \nu(s)\boldsymbol{n}_1'(s) + \nu'(s)\boldsymbol{n}_1 - (\kappa(s)\mu(s))'\boldsymbol{n}_2 + (1-\kappa(s)\mu(s))\boldsymbol{n}_2',\\ &= (\nu'(s)+\nu(s)\tau(s))\boldsymbol{n}_1(s) + [(\kappa(s)\mu(s))' - \tau(s) + \nu(s)\kappa(s)\tau(s)]\boldsymbol{n}_2(s),\\ &= 0.\end{aligned} \tag{4.2}$$

One obtains $\nu\kappa'(s) - \tau(s) = 0$ and $\langle\boldsymbol{\gamma}(s), \boldsymbol{v}_0\rangle = 1$. Hence, $\boldsymbol{\gamma}(s)$ is belonged to $LC(\boldsymbol{v}_0)$ and $\nu\kappa'(s) - \tau(s) = 0$.

3. If $\boldsymbol{v}_0 \in \mathbb{H}^3$ and $\mu = \frac{\tau(s)}{\kappa'(s)}$, one can obtain that the formula $\tau(s)\kappa'(s)+\tau^2(s)\kappa(s)+2\kappa'^2(s) = 0$ is established.

The other two cases are the same as assertion 3. Thus, we omit them here. □

Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be a function germ. We call F an r-parameter unfolding of f, where $f(s) = F_{x_0}(s, \boldsymbol{x}_0)$. We call $f(s)$ has A_k-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$ and $f^{(k+1)}(s_0) \neq 0$. We say that $f(s)$ has $A_{\geq k}$-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let F be an unfolding of f and $f(s)$ has A_k-singularity $(k \geq 1)$ at s_0. Denote the $(k-1)$-jet of the $\partial F/\partial x_i$ at s_0 by $j^{(k-1)}(\partial F/\partial x_i)(s, \boldsymbol{x}_0)(s_0) = \Sigma_{j=1}^{k-1} \alpha_{ji}(s - s_0)^j$ for $i = 1, 2, \ldots, r$. Then F is called a (p) versal unfolding if the $(k-1) \times r$ matrix of coefficients α_{ji} has rank $(k-1)(k-1 \leq r)$. Under the same as the above, F is called a versal unfolding if the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank $k(k \leq r)$, where $\alpha_{0i} = (\partial F/\partial x_i)(s_0, x_0)$. Let function germ $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an unfolding of f, we now introduce three important sets concerning the unfolding. The unfolding set of F is given by

$$\mathcal{Q}_F = \{(s, x) \in \mathbb{R} \times \mathbb{R}^r \mid \text{there exists } s, x \text{ with} \frac{\partial F}{\partial s}(s, x) = 0\}.$$

The discriminant set of F is given by

$$\mathcal{D}_F = \{x \in \mathbb{R}^r \mid \text{there exists } s \text{ with} F(s, x) = \frac{\partial F}{\partial s}(s, x) = 0\}.$$

The bifurcation set of F is the critical value set of the restriction to $\mathcal{Q}_F$ is given by

$$\mathcal{B}_F = \{x \in \mathbb{R}^r \mid \text{there exists } s \text{ with} \frac{\partial F}{\partial s}(s, x) = \frac{\partial^2 F}{\partial s^2}(s, x) = 0\}.$$

And the main theorems are following [3],

Theorem 4.2. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an r-parameter unfolding of $f(s)$ which has A_k-singularity at s_0.*

(1): Supposing F is a versal unfolding of f,

1. *If $k = 1$ then $\mathcal{D}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$,*
2. *If $k = 2$ then $\mathcal{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$,*
3. *If $k = 3$ then $\mathcal{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$, a point $\boldsymbol{x}_0 \in \mathbb{R}^r$ is called a fold point of a map germ $f : (\mathbb{R}^r, \boldsymbol{x}_0) \to (\mathbb{R}^r, f(\boldsymbol{x}_0))$ if there exist diffeomorphism germs $\phi : (\mathbb{R}^r, \boldsymbol{x}_0) \to (\mathbb{R}^r, 0)$ and $\psi : (\mathbb{R}^r, f(\boldsymbol{x}_0)) \to (\mathbb{R}^r, 0)$ such that $\psi \circ \phi(x_1, \ldots, x_r) = (x_1, \ldots, x_{r-1}, x^2)$.*

(2): Supposing that F is a(p) versal unfolding of f,

1. *If $k = 2$ then (s_0, x_0) is the fold point of $\pi \mid_{\mathcal{Q}_F}$ and $\mathcal{B}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$,*
2. *If $k = 3$ then $\mathcal{B}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*

By the Proposition 3.1, the discriminant sets of the height functions are given by

$$\mathcal{D}_{\widetilde{H}} = \{\boldsymbol{v} = \boldsymbol{\gamma}(s) + \lambda_i \boldsymbol{n}_i(s) \mid s, \lambda_i \in \mathbb{R}, i = 1, 2\},$$

$$\mathcal{D}_{\overline{H}} = \{\boldsymbol{v} = \boldsymbol{\gamma}(s) + \mu \boldsymbol{n}_1(s) - \frac{1}{\mu} \boldsymbol{n}_2(s) \mid s, \mu \in \mathbb{R}\},$$

or

$$\mathcal{D}_{\widehat{H}} = \{\boldsymbol{v} = \boldsymbol{\gamma}(s) + \mu \boldsymbol{n}_1(s) - \frac{2}{\mu} \boldsymbol{n}_2(s) \mid s, \mu \in \mathbb{R}\},$$

and the bifurcation set of H is given by

$$\mathcal{B}_H = \{\boldsymbol{v} = \boldsymbol{\gamma}(s) + \mu \boldsymbol{n}_1(s) - (1 - \kappa(s)\mu) \boldsymbol{n}_2(s) \mid s, \mu \in \mathbb{R}\}.$$

For those height functions $\widetilde{H}, \overline{H}, \widehat{H}, H$, we can consider the following theorems as [3].

Theorem 4.3. *Let $H(s,\boldsymbol{v})$ be pseudo null height function of pseudo null curve $\boldsymbol{\gamma}(s)$ and $\boldsymbol{v}\in\mathcal{D}_H$. If h_v has A_k-singularity at s $(k=1,2,3)$, then h is a versal unfolding of h_v.*

Proof. We denote

$$\boldsymbol{\gamma}(s)=(x_1(s),x_2(s),x_3(s),x_4(s)) \text{ and } \boldsymbol{v}=(v_1,v_2,v_3,v_4)\in\mathbb{R}^4_1,$$

then

$$H(s,\boldsymbol{v})=\langle\boldsymbol{\gamma}(s),\boldsymbol{v}\rangle-1=x_1v_1+x_2v_2+x_3v_3-x_4v_4-1. \tag{4.3}$$

Thus,

$$(\partial H/\partial v_i)(s,v)=x_i(i=1,2,3),(\partial H/\partial v_4)(s,v)=-x_4,$$
$$\partial(\partial H/\partial v_i)/\partial s(s,v)=x_i'(i=1,2,3),\partial(\partial H/\partial v_4)/\partial s(s,v)=-x_4',$$
$$\partial^2(\partial H/\partial v_i)/\partial^2 s(s,v)=x_i''(i=1,2,3),\partial^2(\partial H/\partial v_4)/\partial^2 s(s,v)=-x_4''.$$

Let $(x_i's+(1/2)x_i''s^2)\mp(v_i/v_4)(x_4''s+(1/2)x_4''s^2)$ be the 2-jet of $\partial H/\partial v_i$ at s_0.

The condition for (p) versal can be checked as follows:

(1): By Proposition 3.1, h has the A_1-singularity at s_0 if and only if $\boldsymbol{v}=\boldsymbol{\gamma}(s)+\mu\boldsymbol{n}_1(s)+(1-\mu\kappa(s))\boldsymbol{n}_2(s)$, when h has A_1-singularity at s_0, we require the 1×4 matrix (x_1,x_2,x_3,x_4) to have rank 1, which always does since $\boldsymbol{\gamma}(s)$ is regular.

(2): It also follows from Proposition 3.1 that h has $A_{\geq 2}$-singularity at s_0 if and only if $\boldsymbol{v}=\boldsymbol{\gamma}(s)+\mu\boldsymbol{n}_1(s)+(1-\mu\kappa(s))\boldsymbol{n}_2(s)$ and $\sigma(s)=\tau(s)\kappa''(s)+(2\kappa(s)+\tau'(s)-\tau^2(s))\kappa'(s)-2\kappa(s)\tau(s)\tau'(s)-2\kappa(s)\tau^3(s)\neq 0$, when h has $A_{\geq 2}$-singularity at s_0, we require the 2×3 matrix

$$\begin{pmatrix} x_1 & x_1' \\ x_2 & x_2' \\ x_3 & x_3' \end{pmatrix}$$

to have rank 2, which follows from the proof of the case (3).

(3): By Proposition 3.1, h has the A_3-singularity at s_0 if and only if $\boldsymbol{v}=\boldsymbol{\gamma}(s)+\mu\boldsymbol{n}_1(s)+(1-\mu\kappa(s))\boldsymbol{n}_2(s)$ and $\sigma(s)=0$, $\sigma'(s)\neq 0$ and

$$\begin{aligned}\frac{\partial H}{\partial v_i}(s_0,v_0)+j^2(\frac{\partial H}{\partial v_i})(s_0,v_0) &= \frac{\partial H}{\partial v_i}(s_0,v_0)+\frac{\partial}{\partial s}(\frac{\partial H}{\partial v_i})(s_0,v_0)(s-s_0)+\frac{1}{2}\frac{\partial^2}{\partial^2 s}(\frac{\partial H}{\partial v_i})(s_0,v_0)(s-s_0)^2\\ &= \alpha_{0,i}+\alpha_{1,i}(s-s_0)+\frac{1}{2}\alpha_{2,i}(s-s_0)^2,\end{aligned} \tag{4.4}$$

when h has A_3-singularity at s_0, we require the 4×4 matrix

$$A=\begin{pmatrix}\alpha_{0,1} & \alpha_{0,2} & \alpha_{0,3} & \alpha_{0,4}\\ \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4}\\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4}\\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4}\end{pmatrix}=\begin{pmatrix} x_1 & x_2 & x_3 & -x_4\\ x_1' & x_2' & x_3' & -x_4'\\ x_1'' & x_2'' & x_3'' & -x_4''\\ x_1''' & x_2''' & x_3''' & -x_4'''\end{pmatrix}$$

to be nonsingular. Thus,

$$\begin{aligned} detA &= det(\boldsymbol{\gamma}(s_0),\boldsymbol{\gamma}'(s_0),\boldsymbol{\gamma}''(s_0),\boldsymbol{\gamma}'''(s_0)),\\ &= \langle\boldsymbol{\gamma}(s_0)\wedge\boldsymbol{\gamma}'(s_0)\wedge\boldsymbol{\gamma}''(s_0),\boldsymbol{\gamma}'''(s_0)\rangle,\\ &= \tau(s)-\kappa(s)\kappa'(s)+\kappa^2(s)\tau(s)\neq 0.\end{aligned} \tag{4.5}$$

Which implies that the rank of A is 4, which finishes the proof. □

Theorem 4.4. *Let $\widehat{H}(s,\widehat{\boldsymbol{v}})$ be pseudo hyperbolic height function of pseudo null curve $\boldsymbol{\gamma}(s)$ and $\widehat{\boldsymbol{v}}\in\mathcal{D}_{\widehat{H}}$. If $\widehat{h}_{\widehat{v}}$ has A_k-singularity at s $(k=1,2,3)$, then $\widehat{H}$ is a versal unfolding of $\widehat{h}_{\widehat{v}}$.*

Proof. We denote

$$\boldsymbol{\gamma}(s) = (x_1(s), x_2(s), x_3(s), x_4(s)) \text{ and } \widehat{\boldsymbol{v}} = (\widehat{v}_1, \widehat{v}_2, \widehat{v}_3, \widehat{v}_4) \in \mathbb{H}^3,$$

then

$$\widehat{H}(s, \widehat{\boldsymbol{v}}) = \langle \boldsymbol{\gamma}(s), \widehat{\boldsymbol{v}} \rangle - 1 = x_1\widehat{v}_1 + x_2\widehat{v}_2 + x_3\widehat{v}_3 - x_4\widehat{v}_4 - 1. \tag{4.6}$$

Thus,

$$\begin{aligned}
&(\partial\widehat{H}/\partial\widehat{v}_i)(s, v) = x_i \mp (\widehat{v}_i/\widehat{v}_4)x_4,\\
&\partial(\partial\widehat{H}/\partial\widehat{v}_i)/\partial s(s, \widehat{\boldsymbol{v}}) = x'_i \mp (\widehat{v}_i/\widehat{v}_4)x'_4,\\
&\partial^2(\partial\widehat{H}/\partial\widehat{v}_i)/\partial^2 s(s, \widehat{\boldsymbol{v}}) = x''_i \mp (\widehat{v}_i/\widehat{v}_4)x''_4,
\end{aligned}$$

where $\widehat{v}_4 = \pm\sqrt{\widehat{v}_1^2 + \widehat{v}_2^2 + \widehat{v}_3^2 + 1}$ $(i = 1, 2, 3)$, so the 2-jet of $\partial\widehat{H}/\partial\widehat{v}_i$ at s_0 is

$$(x'_i s + (1/2)x''_i s^2) \mp (\widehat{v}_i/v_4)(x''_4 s + (1/2)x''_4 s^2).$$

The condition for (p) versal can be checked as follows:

(1): $\widehat{h}$ has the A_1-singularity at s_0 if and only if $\widehat{\boldsymbol{v}} = \boldsymbol{\gamma}(s) + \mu\boldsymbol{n}_1(s) - \frac{2}{\mu}\boldsymbol{n}_2(s)$, when $\widehat{h}$ has A_1-singularity at s_0, we require the 1×3 matrix $(x_1 \mp (\widehat{v}_1/\widehat{v}_4)x_4, x_2 \mp (\widehat{v}_2/\widehat{v}_4)x_4, x_3 \mp (\widehat{v}_3/\widehat{v}_4)x_4)$ to have rank 1, which it always does since $\boldsymbol{\gamma}(s)$ is regular.

(2): It also follows from Proposition 3.1 that $\widehat{h}$ has $A_{\geq 2}$-singularity at s_0 if and only if $\widehat{\boldsymbol{v}} = \boldsymbol{\gamma}(s) + \mu\boldsymbol{n}_1(s) - \frac{2}{\mu}\boldsymbol{n}_2(s)$ and $\widehat{\sigma}(s) = \tau(s)\kappa''(s) + (2\kappa(s) + \tau'(s) - \tau^2(s))\kappa'(s) - 2\kappa(s)\tau(s)\tau'(s) - 2\kappa(s)\tau^3(s) \neq 0$, when $\widehat{h}$ has $A_{\geq 2}$-singularity at s_0, we require the 2×3 matrix

$$\begin{pmatrix} x_1 \mp (\widehat{v}_1/\widehat{v}_4)x_4 & x'_1 \mp (\widehat{v}_1/\widehat{v}_4)x'_4 \\ x_2 \mp (\widehat{v}_2/\widehat{v}_4)x_4 & x'_2 \mp (\widehat{v}_2/\widehat{v}_4)x'_4 \\ x_3 \mp (\widehat{v}_3/\widehat{v}_4)x_4 & x'_3 \mp (\widehat{v}_3/\widehat{v}_4)x'_4 \end{pmatrix}$$

to have rank 2, which follows from the proof of the case (3).

(3): $\widehat{h}$ has the A_3-singularity at s_0 if and only if

$$\widehat{\boldsymbol{v}} = \boldsymbol{\gamma}(s) + \mu\boldsymbol{n}_1(s) - (2/\mu)\boldsymbol{n}_2(s)$$

and $\widehat{\sigma}(s) = \widehat{\sigma}'(s) = 0$, when $\widehat{h}$ has A_3-singularity at s_0, we require the 3×3 matrix

$$A = \begin{pmatrix} \alpha_{0,1} & \alpha_{0,2} & \alpha_{0,3} \\ \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \end{pmatrix}$$

to be nonsingular, where

$$\begin{aligned}
j^2(\partial\widehat{H}/\partial\widehat{v}_i)(s, \widehat{\boldsymbol{v}}_0)(s_0) &= \frac{\partial\widehat{H}}{\partial\widehat{v}_i}(s_0, \widehat{\boldsymbol{v}}_0) + \frac{\partial}{\partial s}(\frac{\partial\widehat{H}}{\partial\widehat{v}_i})(s_0, \widehat{\boldsymbol{v}}_0)(s - s_0) + \frac{1}{2}\frac{\partial^2}{\partial^2 s}(\frac{\partial\widehat{H}}{\partial\widehat{v}_i})(s_0, \widehat{\boldsymbol{v}}_0)(s - s_0)^2\\
&= \alpha_{0,i} + \alpha_{1,i}(s - s_0) + \frac{1}{2}\alpha_{2,i}(s - s_0)^2.
\end{aligned} \tag{4.7}$$

One denotes that

$$A(i, j, k) = det\begin{pmatrix} x_i(s) & x_j(s) & x_k(s) \\ x'_i(s) & x'_j(s) & x'_k(s) \\ x''_i(s) & x''_j(s) & x''_k(s) \end{pmatrix}.$$

We have

$$\begin{aligned} detA &= -(A(1,2,3) \mp \frac{\widehat{v}_1}{\widehat{v}_4} A(4,2,3) \mp \frac{\widehat{v}_2}{\widehat{v}_4} A(1,4,3) \mp \frac{\widehat{v}_3}{\widehat{v}_4} A(1,2,4)) \\ &= \pm(1/\widehat{v}_4)\langle \widehat{\boldsymbol{v}}, \boldsymbol{\gamma}(s) \wedge \boldsymbol{\gamma}'(s) \wedge \boldsymbol{\gamma}''(s) \rangle. \end{aligned} \tag{4.8}$$

Since $\widehat{\boldsymbol{v}} \in \mathcal{D}_{\widehat{H}}$ is a singular point, $\widehat{\boldsymbol{v}} = \boldsymbol{\gamma}(s) + \mu \boldsymbol{n}_1(s) - \frac{2}{\mu}\boldsymbol{n}_2(s)$ and

$$\begin{aligned} \boldsymbol{\gamma}(s) \wedge \boldsymbol{\gamma}'(s) \wedge \boldsymbol{\gamma}''(s) &= \boldsymbol{\gamma}(s) \wedge \boldsymbol{\gamma}'(s) \wedge (-\boldsymbol{\gamma}(s) + \boldsymbol{n}_1(s) + \kappa(s)\boldsymbol{n}_2(s)) \\ &= \boldsymbol{n}_2(s) - \kappa(s)\boldsymbol{n}_1(s). \end{aligned} \tag{4.9}$$

Therefore,

$$\begin{aligned} detA &= \pm(1/\widehat{v}_4)\langle \boldsymbol{\gamma}(s) + \mu \boldsymbol{n}_1(s) - \frac{2}{\mu}\boldsymbol{n}_2(s), \boldsymbol{n}_2(s) - \kappa(s)\boldsymbol{n}_1(s) \rangle \\ &= \pm \frac{1}{\widehat{v}_4} \frac{\kappa(s)(2\kappa'(s) - 2\tau(s)\kappa(s)) + \tau^2(s)}{\tau(s)(\kappa'(s) - 2\tau(s)\kappa(s))} \neq 0. \end{aligned} \tag{4.10}$$

This completes the proof. □

Proof of Theorem 2.1. Let $\boldsymbol{\gamma}(s)$ be a pseudo null curve on de Sitter 3-space with $\kappa'(s) - 2\tau(s)\kappa(s) \neq 0$. We define a function $\mathrm{G} : \mathbb{S}^3_1 \to \mathbb{R}$ by $\mathrm{G}(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle - 1$. Then we have $h_{v_0}(s) = \mathrm{G}(\boldsymbol{\gamma}(s), \boldsymbol{v}_0)$, since $LC(\boldsymbol{v}_0) = \mathrm{G}^{-1}(0)$ and 0 is a regular value of G. h_{v_0} has the A_k-singularity at s_0 if and only if $\boldsymbol{\gamma}(s)$ and $LC(\boldsymbol{v}_0)$ have $(k+1)$-point contact at s_0. By Proposition 3.1 and Theorems 4.2, 4.3, we get the results of Theorem 2.1.

Proof of Theorem 2.2. Let $\boldsymbol{\gamma}(s)$ be a pseudo null curve on de Sitter 3-space with $\kappa'(s) - 2\tau(s)\kappa(s) = 0$. We define a function $\widehat{\mathrm{G}} : \mathbb{S}^3_1 \to \mathbb{R}$ by $\widehat{\mathrm{G}}(\boldsymbol{u}) = \langle \boldsymbol{u}, \widehat{\boldsymbol{v}} \rangle - 1$. Then we have $\widehat{h}_{\widehat{v}_0}(s) = \widehat{\mathrm{G}}(\boldsymbol{\gamma}(s), \widehat{\boldsymbol{v}}_0)$, since $\widehat{LC}(\widehat{\boldsymbol{v}}_0) = \widehat{\mathrm{G}}^{-1}(0)$ and 0 is a regular value of $\widehat{\mathrm{G}}$. $\widehat{h}_{\widehat{v}_0}(s)$ has the A_k-singularity at s_0 if and only if $\boldsymbol{\gamma}(s)$ and $\widehat{LC}(\widehat{\boldsymbol{v}}_0)$ have $(k+1)$-point contact at s_0. By Proposition 3.2 and Theorems 4.2, 4.4, we get the results of Theorem 2.2.

For the other two types of height functions $\widetilde{H}, \overline{H}$, there are the same conclution as above, I omit them here. By well-known uniqueness theorems which follow easily from the definition of versality [5], the discriminant of the family is diffeomeomorphic to the discriminant of a standard versal deformation of a function having the same type of singularity. Therefore, the generic properties of pseudo null curves are the same as in [5].

5. Example

In this section, an example is given in order to verify the idea of Theorem 2.1 and Theorem 2.2.

Example 5.1. Let $\boldsymbol{\gamma}(s)$ be a pseudo null curve on $\mathbb{S}^3_1$ defined by

$$\boldsymbol{\gamma}(s) = \{\frac{\sqrt{3}}{3}s, \frac{1}{18}s^2 - 1, 2, \frac{1}{18}s^2 + 2\}$$

with respect to a distinguished parameter s, (Fig. 3) the Frenet frames as following

$$\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s) = \sqrt{3}\{\frac{\sqrt{3}}{3}, \frac{1}{9}s, 0, \frac{1}{9}s\},$$

$$\begin{aligned} \boldsymbol{n}_1(s) = \{252\sqrt{3} - 4\sqrt{3}s^2 + \wp, \quad & \tfrac{1}{s}(\varrho s - 3\sqrt{3}(252\sqrt{3} - 4\sqrt{3}s^2 + \wp), \\ & \tfrac{3}{2}\varrho - \tfrac{(18\sqrt{3}+\sqrt{3}s)252\sqrt{3}-4\sqrt{3}s^2+\wp}{12s}, \varrho\}, \end{aligned} \tag{5.1}$$

$$\boldsymbol{n}_2(s) = \{252\sqrt{3} - 4\sqrt{3}s^2 - \wp, \quad \frac{1}{s}(\varrho s - 3\sqrt{3}(252\sqrt{3} - 4\sqrt{3}s^2 - \wp), \quad \frac{3}{2}\varrho - \frac{(18\sqrt{3}+\sqrt{3}s)252\sqrt{3}-4\sqrt{3}s^2-\wp}{12s}, \varrho\}, \tag{5.2}$$

where $\wp = 36\sqrt{147 - 405s^2 - 21s^3 - 7s^4}$, $\varrho = s^4 + 84s^2 + 1620$.

Hence, the curvatures $\kappa(s) = \frac{252\sqrt{3}-4\sqrt{3}s^2+\wp}{s}$ and $\tau(s) = \frac{45\wp-13\sqrt{3}\wp'\varrho-63\varrho+36}{12s^3\wp}$. Thus, the pseudo null hypersurfaces (Fig. 3) (when $\eta = 1$) as

$$\begin{aligned}\mathbb{L}(s,\mu,\eta) =\{&\frac{\sqrt{3}}{3}s + \mu(252\sqrt{3} - 4\sqrt{3}s^2 + \wp) - \eta(252\sqrt{3} - 4\sqrt{3}s^2 - \wp),\\ &\frac{1}{18}s^2 - 1 + \mu(\frac{1}{s}(\varrho s - 3\sqrt{3}(252\sqrt{3} - 4\sqrt{3}s^2 + \wp)) + \eta(\frac{1}{s}(\varrho s - 3\sqrt{3}(252\sqrt{3} - 4\sqrt{3}s^2 - \wp)),\\ &2 + \mu(\frac{3}{2}\varrho - \frac{(18\sqrt{3}+\sqrt{3}s)252\sqrt{3} - 4\sqrt{3}s^2 + \wp}{12s}) + \eta(\frac{3}{2}\varrho - \frac{(18\sqrt{3}+\sqrt{3}s)252\sqrt{3} - 4\sqrt{3}s^2 - \wp}{12s}),\\ &\frac{1}{18}s^2 + 2 + (\mu+\eta)\varrho\},\end{aligned}$$

the pseudo hyperbolic hypersurfaces on hyperbolic 3-space (Fig. 4) as

$$\begin{aligned}\widehat{\mathbb{L}}(s,\mu) =\{&\frac{\sqrt{3}}{3}s + \mu(252\sqrt{3} - 4\sqrt{3}s^2 + \wp) - \frac{2}{\mu}(252\sqrt{3} - 4\sqrt{3}s^2 - \wp),\\ &\frac{1}{18}s^2 - 1 + \mu(\frac{1}{s}(\varrho s - 3\sqrt{3}(252\sqrt{3} - 4\sqrt{3}s^2 + \wp)) - \frac{2}{\mu}(\frac{1}{s}(\varrho s - 3\sqrt{3}(252\sqrt{3} - 4\sqrt{3}s^2 - \wp)),\\ &2 + \mu(\frac{3}{2}\varrho - \frac{(18\sqrt{3}+\sqrt{3}s)252\sqrt{3} - 4\sqrt{3}s^2 + \wp}{12s}) - \frac{2}{\mu}(\frac{3}{2}\varrho - \frac{(18\sqrt{3}+\sqrt{3}s)252\sqrt{3} - 4\sqrt{3}s^2 - \wp}{12s}),\\ &\frac{1}{18}s^2 + 2 + (\mu - \frac{2}{\mu})\varrho\}.\end{aligned}$$

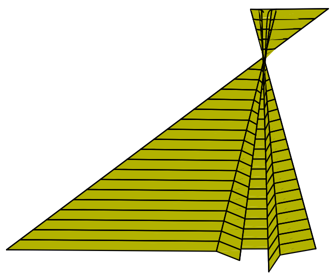

Figure 3: the pseudo null hypersurfaces when $\eta = 1$

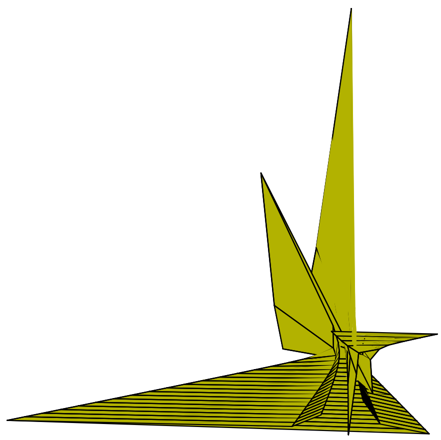

Figure 4: the pseudo hyperbolic hypersurfaces on hyperbolic 3-space

Acknowledgements

The authors express thank to the referee for his/her valuable suggestions. The author was supported by NSF of Shandong ZR2014AQ016, ZR2015AL003 and BRFCU 15CX02068A.

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