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Existence and viability for fractional differential equations with initial conditions at inner points

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Abstract

This paper is concerned with nonlinear fractional differential equations with the Caputo derivative. Existence results are obtained for terminal value problems and initial value problems with initial conditions at inner points. It is also proved that the sufficient condition in order that a locally closed subset be a viable domain is the tangency condition. As a corollary, the existence of positive solutions is obtained.

Keywords: Fractional derivative, differential equation, initial value problem, viability, tangency condition.
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1. Introduction

This paper is mainly concerned with the existence and viability results for the nonlinear fractional differential equation

$$^cD_a^\alpha y(x) = f(x, y(x)), \qquad x \geq x_0 \in (a, b), \tag{1.1}$$

with the initial value condition at an inner point (IVP for short)

$$y(x_0) = y_0, \tag{1.2}$$

where $0 < \alpha \leq 1$, $^cD_a^\alpha$ is the Caputo fractional derivative, $f : [a, b] \times \mathbb{R}^m \to \mathbb{R}^m$ is a given function satisfying some assumptions that will be specified later.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, biology, economics, control theory,

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signal and image processing, etc. which involve fractional order derivatives. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. Consequently, the subject of fractional differential equations is gaining much importance and attention; see [8, 9, 10, 13, 15, 20, 23, 24]. There are a large number of papers dealing with the existence or properties of solutions to fractional differential equations. For an extensive collection of such results, we refer the reader to the monograph [13] by Kilbas et al.. Very recently, a new fractional derivative without singular kernel was introduced by Caputo and Fabrizio in [4] and the properties of such fractional derivative are discussed in [16]. Some applications to nonlinear Fisher's reaction-diffusion equations and heat transfer model are studied in [1] and [2] respectively.

The viability problem was initialed in 1940s [18] and is still one of the active directions of differential equations, see [3, 5, 6, 11, 17, 19, 21, 22]. As for the fractional version, J. Ciotie and A. Răscanu first showed in [7] some viability results for multidimensional time-dependent stochastic differential equations driven by a fractional Brownian motion. They proved a type of the Nagumo theorem on the viability inspired by the work of Nualart and Răscanu [20]. Later in [12], E. Girejko et al. proved a sufficient condition for the viability of nonlinear fractional differential equations with the Caputo fractional derivative. A brief reviews of the main contributions in this area can be found in [12].

In the mentioned papers for viability results of the fractional version, the authors considered the case that the initial conditions are at the endpoints of the definition interval of the Caputo fractional derivative (Definition 2.2). However, the fractional derivative is in fact an interval function, which depends on the starting-point of the definition interval. And this is the most significant difference from the classical integer order derivative.

Let us investigate the fractional differential equations

$$ {}^cD_0^\alpha y_1(x) = x^2, \qquad x \geq 1 \tag{1.3}$$

and

$$ {}^cD_1^\alpha y_2(x) = x^2, \qquad x \geq 1 \tag{1.4}$$

with $0 < \alpha < 1$ and the same initial value condition

$$y_1(1) = y_2(1) = \frac{2}{\Gamma(3+\alpha)}.$$

A direct computation deduces that the solutions to the above initial value problems are

$$y_1(x) = \frac{2x^{2+\alpha}}{\Gamma(3+\alpha)}$$

and

$$y_2(x) = \frac{2(x-1)^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{2(x-1)^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{(x-1)^{\alpha}}{\Gamma(1+\alpha)} + \frac{2}{\Gamma(3+\alpha)}$$

respectively. By a numerical method we can find that $y_1(x) \neq y_2(x)$ for $x > 1$. This example shows that ${}^cD_0^\alpha$ and ${}^cD_1^\alpha$ are two different 'fractional derivatives' and equations (1.3) and (1.4) are two different equations.

Recall that a subset $D \subset \mathbb{R}^m$ is said to be the viable domain of the differential equation

$$y'(x) = f(x, y(x)), \qquad x \in (a, b), \tag{1.5}$$

if for every $(x_0, y_0) \in (a, b) \times D$, there is a $T > 0$ with $x_0 + T < b$ such that equation (1.5) admits a solution $y(\cdot) : [x_0, x_0+T] \to D$ satisfying $y(x_0) = y_0$. What should be noticed here is that the initial point $x_0 \in (a, b)$ is arbitrary while the equation is fixed. To generalize the viability problem to the fractional case, one should retain the fact that the equations are fixed and independent of the initial value.

Motivated by the above comment, in this paper, we study the existence and viability of solutions to the nonlinear Caputo fractional differential equation modeled as (1.1), with the initial conditions at inner points of the definition interval of the fractional derivative. Inspired by [5] and [11], we only suppose that the function f on the right hand side of the equation is of Caratheodory type. To the best of my knowledge, there is no result on the fractional viability with the initial values at inner points. In this case, the equivalent integral equation is a Volterra-Fredholm equation. The technical difficulty comes from the Fredholm part, which is in fact a delayed problem, that the approximate solutions should be constructed different from the Volterra case.

The paper is organized as follows. In Section 2 we introduce the definitions of fractional integrals and derivatives, and some basic results that will be used for viability. In Section 3 we give the existence results for terminal value problems and initial value problems under several assumptions. In Section 4 we study a sufficient condition for the viability to problem (1.1).

2. Preliminaries and lemmas

In this section we collect some definitions and results needed in our further investigations.

Let $C([a,b];\mathbb{R}^m)$ be the Banach space of all continuous functions $u:[a,b]\to\mathbb{R}^m$ with the norm $\|u\|_\infty = \sup\{\|u(x)\| : x\in[a,b]\}$ and $L^1((a,b);\mathbb{R}^m)$ the Banach space of all measurable functions $u:[a,b]\to\mathbb{R}^m$ which are Lebesgue integrable, equipped with the norm $\|u\|_1 = \int_a^b \|u(x)\|dx$. Here the vector norm $\|u\| = \sqrt{u_1^2+u_2^2+\cdots+u_m^2}$ for $u\in\mathbb{R}^m$.

Definition 2.1 ([9]). Let $\alpha>0$ be a fixed number. The Riemann-Liouville fractional integral of order $\alpha>0$ of the function $h:[a,b]\to\mathbb{R}^m$ is defined by

$$I_a^\alpha h(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}h(t)\mathrm{d}t, \quad x\in[a,b],$$

where $\Gamma(\cdot)$ denotes the Gamma function, i.e., $\Gamma(z)=\int_0^\infty \mathrm{e}^{-t}t^{z-1}\mathrm{d}t$.

It has been shown that the fractional integral operator I_a^α transforms the space $L^1((a,b);\mathbb{R}^m)$ into $C([a,b];\mathbb{R}^m)$ and some other properties of I_a^α are referred to [9].

Definition 2.2 ([9]). Let $h:[a,b]\to\mathbb{R}^m, \alpha>0$ and $m=[\alpha]+1$. The Caputo fractional derivative of order α of h at the point x is defined by

$${}^cD_a^\alpha h(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-t)^{m-\alpha-1}h^{(m)}(t)\mathrm{d}t, \quad x\in[a,b].$$

${}^cD_a^\alpha$ is also called the Caputo fractional differential operator.

A function $f:[a,b]\times\mathbb{R}^m\to\mathbb{R}^m$ is said to satisfy the Caratheodory condition if $f(\cdot,y):[a,b]\to\mathbb{R}^m$ is measurable for every $y\in\mathbb{R}^m$ and $f(x,\cdot):\mathbb{R}^m\to\mathbb{R}^m$ is continuous for almost every $x\in[a,b]$. A Caratheodory type function has the following Scorza Dragoni property ([14]). We denote by m the Lebesgue measure on $\mathbb{R}$.

Theorem 2.3. *Let X,Y be separable metric spaces, $I=(a,b)$ and $f:I\times X\to Y$ a function satisfying the Caratheodory condition. Then for each $\varepsilon>0$, there exists a compact subset $K\subset I$ such that $m(I\backslash K)<\varepsilon$ and the restriction of f to $K\times X$ is continuous.*

Similar to the case for ordinary differential equations, we define the viability of the subset for fractional differential equations.

Definition 2.4. Let $D\subset\mathbb{R}^m$ be nonempty and $f:(a,b)\times D\to\mathbb{R}^m$ be of the Caratheodory type. We say that D is a viable domain of the fractional differential equation (1.1) if for a.e. $x_0\in(a,b)$ and every $y_0\in D$, there is a $T>0$ with $x_0+T<b$, such that (1.1) has at least one solution $y:[x_0,x_0+T]\to D$ satisfying the initial condition (1.2).

Let $D \subset \mathbb{R}^m$ be a locally closed subset, i.e., for each $\zeta \in D$, there is a $\rho > 0$ such that $B(\zeta, \rho) \cap D$ is closed ([22]) and $f : (a, b) \times \mathbb{R}^m \to \mathbb{R}^m$ a function of the Caratheodory type (as usual, here $B(\zeta, \rho) = \{x \in \mathbb{R}^m : \|x - \zeta\| \leq \rho\}$, the closed ball centered at ζ with radius ρ). The condition often in consideration for viable domains is the following tangency condition ([5, 18]).

Definition 2.5. We say that f satisfies the tangency condition with respect to the set D if

$$\liminf_{h\downarrow 0} \frac{1}{h} d(y_0 + hf(x_0, y_0), D) = 0 \tag{2.1}$$

for a.e. $x_0 \in (a, b)$ and all $y_0 \in D$, where $d(\eta, D)$ denote the distance from the point $\eta \in \mathbb{R}^m$ to the subset $D \subset \mathbb{R}^m$.

The following property is useful for fractional cases. The proof is similar to the one in [12], so we omit it.

Lemma 2.6. *Let $D \subset \mathbb{R}^m$ be a locally closed subset and $\eta \in \mathbb{R}^m$. Then the equality* (2.1) *holds if and only if for every $\varepsilon > 0$, there exist $h \in (0, \varepsilon)$ and $p_h \in B(0, \varepsilon)$ with the property*

$$x_0 + \frac{h^\alpha}{\Gamma(\alpha + 1)} f(x_0, y_0) + p_h \in D.$$

Since we only consider the case that $0 < \alpha < 1$, the equality (2.1) is equivalent to

$$\liminf_{h\downarrow 0} \frac{1}{h^\alpha} d(y_0 + h^\alpha f(x_0, y_0), D) = 0.$$

Using this expression as well as Theorem 2.3 and mimicking the process of the proof of [5, Theorem 2.3], one can obtain the following result, which is also a variant of the Lebesgue derivative type.

Theorem 2.7. *Let $D \subset \mathbb{R}^m$ be nonempty and $f : (a, b) \times D \to \mathbb{R}^m$ be of Caratheodory type. Then there exists a negligible subset $Z \subset (a, b)$ such that for every $x \in (a, b)\backslash Z$, one has*

$$\liminf_{h\downarrow 0} \frac{1}{h^\alpha} \int_x^{x+h} (x + h - t)^{\alpha - 1} f(t, u(t))dt = f(x, u(x)) \tag{2.2}$$

for all continuous functions $u : (a, b) \times D \to \mathbb{R}^m$.

3. Existence results

In this section, we study the initial value problem for nonlinear fractional differential equations with initial conditions at inner points. More precisely, we will prove a Peano type theorem of the fractional version. Since the fractional derivative of a function y at an inner point $x \in (a, b)$ is determined by the values of y on the interval $[a, x]$, we begin with the so called terminal value problem

$$\begin{cases} {}^cD_a^\alpha y(x) = f(x, y(x)), \quad & a \leq x \leq x_0 \in (a, b), \\ y(x_0) = y_0. \end{cases} \tag{3.1}$$

As indicated in [9], Problem (3.1) is equivalent to the integral equation

$$y(x) = y_0 + \int_a^x \frac{(x - t)^{\alpha - 1}}{\Gamma(\alpha)} f(t, y(t))dt - \int_a^{x_0} \frac{(x_0 - t)^{\alpha - 1}}{\Gamma(\alpha)} f(t, y(t))dt. \tag{3.2}$$

Theorem 3.1. *Let $0 < \alpha < 1$ and $G = [a,b] \times \mathbb{R}^m$. Let $f : G \to \mathbb{R}^m$ be continuous and fulfill a Lipschitz condition with respect to the second variable with a Lipschitz constant L, i.e.*

$$\|f(x,y_2) - f(x,y_1)\| \le L\|y_2 - y_1\|, \quad (x,y_1),(x,y_2) \in G.$$

Then for $(x_0,y_0) \in G$ with $x_0 < a + \left(\frac{\Gamma(\alpha+1)}{2L}\right)^{1/\alpha}$, there exists a unique solution $y \in C([a,x_0];\mathbb{R}^m)$ to the terminal value problem (3.1).

Proof. We define a mapping $T : C([a,x_0],\mathbb{R}) \to C([a,x_0],\mathbb{R})$ by

$$(Ty)(x) = y_0 + \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t,y(t))dt - \int_a^{x_0} \frac{(x_0-t)^{\alpha-1}}{\Gamma(\alpha)} f(t,y(t))dt$$

for $y \in C([a,x_0],\mathbb{R})$ and $x \in [a,x_0]$. Then for any $y_1, y_2 \in C([a,x_0],\mathbb{R})$ and $x \in [a,x_0]$, we have

$$\begin{aligned}
\|(Ty_2)(x) - (Ty_1)(x)\| \le & \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|f(t,y_2(t)) - f(t,y_1(t))\|dt \\
& + \int_a^{x_0} \frac{(x_0-t)^{\alpha-1}}{\Gamma(\alpha)} \|f(t,y_2(t)) - f(t,y_1(t))\|dt \\
\le & L \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \|y_2(t) - y_1(t)\|dt \\
& + L \int_a^{x_0} \frac{(x_0-t)^{\alpha-1}}{\Gamma(\alpha)} \|y_2(t) - y_1(t)\|dt \\
\le & \frac{2L(x_0-a)^\alpha}{\Gamma(\alpha+1)} \|y_2 - y_1\|_\infty.
\end{aligned}$$

And hence

$$\|Ty_2 - Ty_1\|_\infty \le K\|y_2 - y_1\|_\infty$$

with $K = \frac{2L(x_0-a)^\alpha}{\Gamma(\alpha+1)}$. Since $x_0 < a + \left(\frac{\Gamma(\alpha+1)}{2L}\right)^{1/\alpha}$, we get that $\frac{2L(x_0-a)^\alpha}{\Gamma(\alpha+1)} < 1$. Thus an application of Banach's fixed point theorem yields the existence and uniqueness of solution to our integral equation (3.2). □

Remark 3.2. The condition $x_0 < a + \left(\frac{\Gamma(\alpha+1)}{2L}\right)^{1/\alpha}$ means that the point x_0 cannot be far away from a. However, the following example shows that we cannot expect that there exists a solution to (3.1) for each $x_0 \in (a,b]$.

Example 3.3. Consider the differential equation with the Caputo fractional derivative

$${}^cD_0^{1/2}y(x) = \frac{2\sqrt{x}}{\sqrt{\pi c}} y^2(x),$$

where $c > 0$ is a constant. A direct computation shows that it admits a solution

$$y(x) = \frac{1}{\sqrt{c-x}},$$

whose existence interval is $[0,c)$.

However, from the proof of Theorem 3.1 we can see that if the Lipschitz constant L is small enough, then x_0 can be extended to the whole interval. Thus we have the following result.

Theorem 3.4. *Let $0 < \alpha < 1$ and $G = [a,b] \times \mathbb{R}^m$. Let $f : G \to \mathbb{R}^m$ be continuous and fulfill a Lipschitz condition with respect to the second variable with a Lipschitz constant L. If $L < \frac{\Gamma(\alpha+1)}{2(b-a))^\alpha}$, then for every $(x_0,y_0) \in G$, there exists a unique solution $y \in C[a,x_0]$ to the terminal value problem* (3.1).

Now we prove an existence result to the initial value problem at inner points (1.1)-(1.2) based on the generalized Banach contraction principle.

Theorem 3.5. *Let $0 < \alpha < 1$ and $G = [a, b] \times \mathbb{R}^m$. Let $f : G \to \mathbb{R}^m$ be continuous and fulfill a Lipschitz condition with respect to the second variable with a Lipschitz constant L. Suppose $(x_0, y_0) \in G$ with $x_0 < a + \left(\frac{\Gamma(\alpha+1)}{2L}\right)^{1/\alpha}$. Then for $h > 0$, there exists a unique solution on $[a, x_0 + h]$ to IVP (1.1)-(1.2) .*

Proof. In view of [9], a function $y \in C([a, x_0 + h]; \mathbb{R}^m)$ is a solution to (1.1)-(1.2) if and only if y satisfies

$$y(x) = y_0 + \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t, y(t))dt - \int_a^{x_0} \frac{(x_0-t)^{\alpha-1}}{\Gamma(\alpha)} f(t, y(t))dt \tag{3.3}$$

for $t \in [a, x_0 + h]$.

By Theorem 3.1, there exists a unique function $y^* \in C([a, x_0]; \mathbb{R}^m)$ satisfying

$$y^*(x) = y_0 + \int_a^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t, y^*(t))dt - \int_a^{x_0} \frac{(x_0-t)^{\alpha-1}}{\Gamma(\alpha)} f(t, y^*(t))dt. \tag{3.4}$$

Extend y^* to $[a, x_0 + h]$, also denoted by y^*, by

$$\begin{cases} y^*(x) = y^*(x), & x \in [a, x_0], \\ y^*(x) = y_0, & x \in [x_0, x_0 + h]. \end{cases} \tag{3.5}$$

For $z \in C([x_0, x_0 + h]; \mathbb{R}^m)$ with $z(x_0) = 0$, we extend z to $[a, x_0 + h]$, still denoted by $\tilde{z}$, by

$$\begin{cases} \tilde{z}(x) = 0, & x \in [a, x_0], \\ \tilde{z}(x) = z(x), & x \in [x_0, x_0 + h]. \end{cases} \tag{3.6}$$

It is easily seen that a function $y \in C([a, x_0 + h]; \mathbb{R}^m)$ satisfies (3.3) if and only if there is a function $z \in C([x_0, x_0 + h]; \mathbb{R}^m)$ with $z(x_0) = 0$ such that $y = y^* + \tilde{z}$ on $[a, x_0 + h]$. Moreover, y and y^* agree on $[a, x_0]$ and for $x \in [x_0, x_0 + h]$, we have

$$\begin{aligned} \tilde{z}(x) =& \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t, y^*(t) + \tilde{z}(t))dt - \frac{1}{\Gamma(\alpha)} \int_a^{x_0} (x_0-t)^{\alpha-1} f(t, y^*(t) + \tilde{z}(t))dt \\ =& \frac{1}{\Gamma(\alpha)} \int_a^{x_0} \left[(x-t)^{\alpha-1} - (x_0-t)^{\alpha-1}\right] f(t, y^*(t) + \tilde{z}(t))dt + \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-t)^{\alpha-1} f(t, y^*(t) + \tilde{z}(t))dt. \end{aligned}$$

Due to (3.5) and (3.6) this equation can be rewritten as

$$\begin{aligned} z(x) =& \frac{1}{\Gamma(\alpha)} \int_a^{x_0} \left[(x-t)^{\alpha-1} - (x_0-t)^{\alpha-1}\right] f(t, y^*(t))dt + \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-t)^{\alpha-1} f(t, z(t) + y_0)dt \\ =& \ g(x) + \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-t)^{\alpha-1} f(t, z(t) + y_0)dt \end{aligned} \tag{3.7}$$

for $x \in [x_0, x_0 + h]$, where $g(x) = \frac{1}{\Gamma(\alpha)} \int_a^{x_0} \left[(x-t)^{\alpha-1} - (x_0-t)^{\alpha-1}\right] f(t, y^*(t))dt$ with $g(x_0) = 0$. Since y^* is uniquely determined on $[a, x_0]$, g is a known function. Let $W = \{z \in C([x_0, x_0 + h]; \mathbb{R}^m) : z(x_0) = 0\}$ endowed with the supremum norm $\|z\|_\infty = \sup_{x \in [x_0, x_0+h]} \|z(x)\|$. Then W becomes a Banach space. Define an operator $T : W \to W$ by

$$(Tz)(x) = g(x) + \frac{1}{\Gamma(\alpha)} \int_{x_0}^x (x-t)^{\alpha-1} f(t, z(t) + y_0)dt$$

for $z \in W$ and all $x \in [x_0, x_0 + h]$. Obviously if z is a fixed point of T, then $y = y^* + \tilde{z}$ is a solution to (1.1)-(1.2) and vise versa. Below we prove that T has a unique fixed point in W by the generalized Banach contraction principle.

We first note that T is well-defined due to the continuity of the function g and the fact that $g(x_0) = 0$. Next we prove that for any $z_1, z_2 \in W$,

$$\|T^n z_2 - T^n z_1\|_\infty \leq \frac{(Lh^\alpha)^n}{\Gamma(n\alpha + 1)} \|z_2 - z_1\|_\infty$$

for every $n \in \mathbb{N}$. In fact, take arbitrary $z_1, z_2 \in W$. Then for every $x \in [x_0, x_0 + h]$, we have

$$\begin{aligned}\big\|Tz_2(x) - Tz_1(x)\big\| \leq& \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \big\|f(t, z_2(t) + y_0) - f(t, z_1(t) + y_0)\big\| dt \\ \leq& \frac{L}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \big\|z_2(t) - z_1(t)\big\| dt \\ =& L I_{x_0}^{\alpha} \|z_2(\cdot) - z_1(\cdot)\|(x).\end{aligned}$$

Then we have

$$\begin{aligned}\big\|T^2 z_2(x) - T^2 z_1(x)\big\| \leq& \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \big\|f(t, Tz_2(t) + y_0) - f(t, Tz_1(t) + y_0)\big\| dt \\ \leq& \frac{L}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \big\|Tz_2(t) - Tz_1(t)\big\| dt \\ =& \frac{L^2}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} \big(I_{x_0}^{\alpha} \|z_2 - z_1\|(t)\big) dt \\ =& L^2 I_{x_0}^{2\alpha} \|z_2(\cdot) - z_1(\cdot)\|(x).\end{aligned}$$

By induction, we deduce that for $n \in \mathbb{N}$ and every $x \in [x_0, x_0 + h]$,

$$\begin{aligned}\big\|T^n z_2(x) - T^n z_1(x)\big\| \leq L^n I_{x_0}^{n\alpha} \big\|z_2(\cdot) - z_1(\cdot)\big\|(x) =& \frac{L^n}{\Gamma(n\alpha)} \int_{x_0}^{x} (x-t)^{n\alpha-1} \big\|z_2(t) - z_1(t)\big\| dt \\ \leq& \frac{L^n}{\Gamma(n\alpha)} \int_{x_0}^{x} (x-t)^{n\alpha-1} dt \big\|z_2 - z_1\big\|_\infty \\ =& \frac{L^n}{\Gamma(n\alpha + 1)} (x - x_0)^{n\alpha} \big\|z_2 - z_1\big\|_\infty \\ \leq& \frac{L^n}{\Gamma(n\alpha + 1)} h^{n\alpha} \big\|z_2 - z_1\big\|_\infty.\end{aligned}$$

Take supremum on both side we obtain that

$$\big\|T^n z_2 - T^n z_1\big\|_\infty \leq \frac{(Lh^\alpha)^n}{\Gamma(n\alpha + 1)} \big\|z_2 - z_1\big\|_\infty$$

for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{(Lh^\alpha)^n}{\Gamma(n\alpha+1)} = 0$, we can take a natural number n_0 large enough such that $\frac{(Lh^\alpha)^{n_0}}{\Gamma(n_0\alpha+1)} < \frac{1}{2}$. Hence

$$\|T^{n_0} z_2 - T^{n_0} z_1\|_\infty \leq \frac{1}{2} \|z_2 - z_1\|_\infty.$$

By the generalized Banach contraction principle, T has a unique fixed point z in W and this completes the proof. □

Remark 3.6. Although x_0 is bounded with respect to the Lipschitzian constant L and the fractional order α, the number $h > 0$ is unrestricted. That is to say, we actually get a global existence result for IVP (1.1) with initial conditions (1.2) at inner points. It is interesting to compare with [8] and [13], where the authors considered the Riemann-Liouville type fractional, with x_0 being unbounded and $h > 0$ being bounded.

Remark 3.7. The existence and uniqueness of solutions to the fractional differential equations with initial conditions at inner points (1.1)-(1.2) was studied in [13] (Theorem 3.20 for Riemann-Liouville fractional derivative and Theorem 2.27 for Caputo version). However, by a careful check of the proof one can find that the existence results are discussed in the interval $(x_0 - h, x_0 + h)$. From the definition of the solutions to problem (1.1)-(1.2), (see also Theorem 6.18 in [9]), this is not appropriate.

Next we want to study the case that f satisfies the Caratheodory condition. For simplicity, we limit to the case that f is locally bounded. We list the hypotheses.

(H_1) $f : [a, b] \times C([a, b]; \mathbb{R}^m) \to C([a, b]; \mathbb{R}^m)$ satisfies the Caratheodory condition.

(H_2) For every $r > 0$, there is a constant $M_r > 0$, such that $\|f(x, y)\| \le M_r$ for a.e. $x \in [a, b]$ and $y \in \mathbb{R}^m$ with $\|y\| \le r$.

We prove a local existence result.

Theorem 3.8. *Let $0 < \alpha < 1$ and $G = [a, b] \times \mathbb{R}^m$. Assume that the hypotheses $(H_1) - (H_2)$ hold and suppose $(x_0, y_0) \in G$. Further assume that there exists a real number $r > 0$ solving the inequality*

$$\frac{(x_0 - a)^\alpha}{\Gamma(\alpha + 1)} \frac{M_r}{r} < \frac{1}{2}. \tag{3.8}$$

Then there exists an $h > 0$ such that the IVP (1.1)-(1.2) has at least a solution $y \in C([x_0, x_0 + h]; \mathbb{R}^m)$.

Proof. On account of the hypothesis (3.8), we can find constants $r_0 > 0$ and $h > 0$ with

$$\|y_0\| + \frac{2(x_0 + h - a)^\alpha M_{r_0}}{\Gamma(\alpha + 1)} < r_0. \tag{3.9}$$

Define an operator $T : C([a, x_0 + h]; \mathbb{R}^m) \to C([a, x_0 + h]; \mathbb{R}^m)$ by

$$Ty(x) = y_0 + \frac{1}{\Gamma(\alpha)}\Big(\int_a^x (x - t)^{\alpha-1} f(t, y(t))dt - \int_a^{x_0} (x_0 - t)^{\alpha-1} f(t, y(t))dt\Big)$$

for $y \in C([a, x_0 + h]; \mathbb{R}^m)$ and $x \in [a, x_0 + h]$. It then follows from the hypotheses $(H_1) - (H_2)$ as well as the Lebesgue dominated convergence theorem that T is well-defined, i.e., Ty is continuous on $[a, x_0 + h]$ for every $y \in C([a, x_0 + h]; \mathbb{R}^m)$ and that T is continuous. Further, let $B_{r_0} = \{y \in C([a, x_0 + h]; \mathbb{R}^m) : \|y\|_\infty \le r_0\}$. Then B_{r_0} is a bounded closed subset of $C([a, x_0 + h]; \mathbb{R}^m)$. For every $y \in B_{r_0}$ and $x \in [a, x_0 + h]$, we have

$$\begin{aligned}\|Ty(x)\| \le& \|y_0\| + \frac{1}{\Gamma(\alpha)}\Big(\int_a^x (x - t)^{\alpha-1} \|f(t, y(t))\|dt + \int_a^{x_0} (x_0 - t)^{\alpha-1} \|f(t, y(t))\|dt\Big)\\ \le& \|y_0\| + \frac{M_{r_0}}{\Gamma(\alpha)}\Big(\int_a^x (x - t)^{\alpha-1} dt + \int_a^{x_0} (x_0 - t)^{\alpha-1} dt\Big)\\ \le& \|y_0\| + \frac{2M_{r_0}(x_0 + h - a)^\alpha}{\Gamma(\alpha + 1)} \le r_0\end{aligned}$$

due to (H_2) and (3.9), which implies that $TB_{r_0} \subset B_{r_0}$.

Now we show that T is completely continuous. To this end, we first prove that T maps bounded subsets in $C([a, x_0 + h]; \mathbb{R}^m)$ into bounded subsets. It suffices to show that TB_r is bounded for every $B_r = \{y \in C([a, x_0 + h]; \mathbb{R}^m) : \|y\|_\infty \le r\}$ with fixed $r > 0$. Let $y \in B_r$. Then by (H_2) we have for every $x \in [a, x_0 + h]$,

$$\begin{aligned}\|Ty(x)\| \le& \|y_0\| + \frac{1}{\Gamma(\alpha)}\Big(\int_a^x (x - t)^{\alpha-1} \|f(t, y(t))\|dt + \int_a^{x_0} (x_0 - t)^{\alpha-1} \|f(t, y(t))\|dt\Big)\\ \le& \|y_0\| + \frac{M_r}{\Gamma(\alpha)}\Big(\int_a^x (x - t)^{\alpha-1} dt + \int_a^{x_0} (x_0 - t)^{\alpha-1} dt\Big)\\ \le& \|y_0\| + \frac{2M_r b^\alpha}{\Gamma(\alpha + 1)}.\end{aligned}$$

It follows that $\|Ty\|_\infty \le \|y_0\| + \frac{2M_r b^\alpha}{\Gamma(\alpha+1)}$ which is independent of $y \in B_r$. Hence TB_r is bounded.

Next we prove that T maps bounded subsets into equicontinuous subsets. Let $y \in B_r$ be arbitrary and $x_1, x_2 \in [a, x_0 + h]$ with $x_1 < x_2$. Then we have

$$\begin{aligned}\|Ty(x_2) - Ty(x_1)\| \le & \frac{1}{\Gamma(\alpha)}\Big(\int_a^{x_1} \big|(x_2 - t)^{\alpha-1} - (x_1 - t)^{\alpha-1}\big| \|f(t, y(t))\| dt \\ & + \int_{x_1}^{x_2} (x_2 - t)^{\alpha-1} \|f(t, y(t))\| dt\Big) \\ \le & \frac{M_r}{\Gamma(\alpha)}\Big(\int_a^{x_1} \big[(x_1 - t)^{\alpha-1} - (x_2 - t)^{\alpha-1}\big] dt + \int_{x_1}^{x_2} (x_2 - t)^{\alpha-1} dt\Big) \\ \le & \frac{M_r}{\Gamma(\alpha+1)} \big[(x_2 - a)^\alpha - (x_1 - a)^\alpha\big],\end{aligned}$$

which converges to 0 as $x_2 - x_1 \to 0$ and the convergence is independent of $y \in B_r$. Thus TB_r is equicontinuous.

We have shown that T maps bounded subsets in $C([a, x_0+h]; \mathbb{R}^m)$ to bounded and equicontinuous subsets. By Arzela-Ascoli's theorem, we conclude that T is a completely continuous operator. An application of the Schauder fixed point theorem shows that there exists at least a fixed point y of T in B_{r_0}, which is the solution to (1.1)-(1.2) on $[x_0, x_0 + h]$ and the proof is completed. □

4. Viability

In this section we discuss the viability of solutions for the nonlinear fractional differential equation (1.1). The main result is the following theorem.

Theorem 4.1. *Let $D \subset \mathbb{R}^m$ be a locally closed subset, $0 < \alpha < 1$ and assume that the hypotheses $(H_1)-(H_2)$ hold. Further, assume that there is a number $r_0 > 0$ satisfying $\frac{(b-a)^\alpha M_{r_0}}{\Gamma(\alpha+1) r_0} < \frac{1}{2}$. If f satisfies the tangency condition (2.1) for every $y_0 \in D$, then D is the viable domain of the fractional differential equation (1.1).*

To prove Theorem 4.1, we need the following lemma. Let $x_0 \in (a, b)$ and $y_0 \in D$ be arbitrary. Take $r > 0$ such that $B(y_0, r) \cap D$ is closed. From Theorem 3.8, there is a $\bar{y} \in C([a, b]; \mathbb{R})$ satisfying

$$\bar{y}(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t, \bar{y}(t)) dt \tag{4.1}$$

for $x \in [a, x_0]$ and $\bar{y}(x_0) = y_0$, i.e. $\bar{y}(x_0) = \frac{1}{\Gamma(\alpha)} \int_a^{x_0} (x_0 - t)^{\alpha-1} f(t, \bar{y}(t)) dt$. Let $r_1 = \sup_{x\in[a,x_0]} \|\bar{y}(x)\|$. Then by (H_2), there exists an M_{r_1} with $\|f(x, \bar{y}(x))\| \le M_{r_1}$ for all $x \in [a, x_0]$. Since for $x > x_0$,

$$\begin{aligned}\frac{1}{\Gamma(\alpha)} \Big\| \int_a^{x_0} \Big((x - t)^{\alpha-1} - (x_0 - t)^{\alpha-1}\Big) f(t, \bar{y}(t)) \Big\| dt \le & \frac{M_{r_1}}{\Gamma(\alpha)} \int_a^{x_0} \Big|(x - t)^{\alpha-1} - (x_0 - t)^{\alpha-1}\Big| dt \\ = & \frac{M_{r_1}}{\Gamma(\alpha)} \Big[(x - a)^\alpha - (x_0 - a)^\alpha - (x - x_0)^\alpha\Big],\end{aligned}$$

we can choose $T > 0$ such that

$$\frac{1}{\Gamma(\alpha)} \Big\| \int_a^{x_0} \Big((x - t)^{\alpha-1} - (x_0 - t)^{\alpha-1}\Big) f(t, \bar{y}(t)) \Big\| dt \le \frac{r}{3} \tag{4.2}$$

for all $x \in [x_0, x_0 + T]$. Notice that we also have

$$\frac{1}{\Gamma(\alpha)} \Big\| \int_a^{x_0} \Big((x_2 - t)^{\alpha-1} - (x_1 - t)^{\alpha-1}\Big) f(t, \bar{y}(t)) \Big\| dt \le \frac{r}{3} \tag{4.3}$$

for all $x_1, x_2 \in [x_0, x_0 + T]$. Moreover, we can choose T small enough such that

$$\frac{M_r T^\alpha}{\Gamma(\alpha+1)} \le \frac{r}{3}. \tag{4.4}$$

Lemma 4.2. *Suppose that the hypotheses of* Theorem 4.1 *hold and Z is the set given by* Theorem 2.7. *Then for each $(x_0, y_0) \in (a, b) \times D$ and $n \in \mathbb{N}$ and each open subset $L_n \in \mathbb{R}$ with $Z \in L_n$ and $m(L_n) < \frac{1}{n}$, there exist an $\bar{x} \in [x_0, x_0 + T]\backslash Z$, an increasing sequence $\{x_i^n\}_{i=1}^{\infty} \subset [x_0, x_0 + T]$ and an approximate solution y^n on $[x_0, x_0 + T]$ in the following sense:*

(i) $x_0^n = x_0, x_{i+1}^n - x_i^n = d_i^n < \frac{1}{n}$, $\lim_{i\to\infty} x_i^n = x_0 + T$;

(ii) $y^n(x_0) = y_0, y^n(x_i^n) = y_i^n \in D \cap B(y_0, r)$;

(iii) $h^n(t) = f(x_i^n, y^n(x_i^n))$ *in the case* $x_i^n \notin L_n$, *while* $h^n(t) = f(\bar{x}, y^n(x_i^n))$ *in the case* $x_i^n \in L_n$ *for* $t \in [x_i^n, x_{i+1}^n)$;

(iv) $y^n(x) = y_i^n + \frac{1}{\Gamma(\alpha)}\int_{x_i^n}^{x}(x-t)^{\alpha-1}h^n(t)dt + (x - x_i^n)p_i^n + \frac{1}{\Gamma(\alpha)}\int_a^{x_i^n}[(x-t)^{\alpha-1} - (x_i^n - t)^{\alpha-1}]h^n(t)dt$ *for* $x \in [x_i^n, x_{i+1}^n)$, *where* $y_i^n \in D$, $p_i^n \in \mathbb{R}^m$ *with* $\|p_i^n\| < \frac{1}{n}$.

Proof. Let $x_0 \in (a, b)$, $y_0 \in D$ and $n \in \mathbb{N}$ be given. We assume that the tangency condition (2.1) holds for every $x \in [x_0, x_0 + T]\backslash L_n$. Fix $\bar{x} \in [x_0, x_0 + T]\backslash L_n$. We construct y^n, h^n and x_i^n by induction. Let $h^n(t) = f(t, \bar{y}(t))$ for $t \in [a, x_0]$, where $\bar{y}$ is the function obtained by Theorem 3.8 and satisfying (4.1). Set $x_0^n = x_0$ and $y^n(x) = \bar{y}(x)$ for $x \in [a, x_0]$. So $y^n(x_0^n) = y_0$. To simplify notation, we drop n as a superscript for x_i, y_i, y, p_i etc.

Suppose that y and h are constructed on $[x_0, x_i]$. Then we define x_{i+1} in the following manner. If $x_i = x_0 + T$, set $x_{i+1} = x_0 + T$ and if $x_i < x_0 + T$, then we define x_{i+1} as in the following two cases.

In the case that $x_i \in L_n$, we set

$$\begin{aligned}\delta_i = \sup\Big\{&h \in (0, \frac{1}{n}]; x_i + h \le x_0 + T,\ [x_i, x_i + h] \subset L_n,\\ &d(y_i + \frac{h^\alpha}{\Gamma(\alpha+1)} f(\bar{x}, y_i); D) \le \frac{h}{6n}\Big\}.\end{aligned} \tag{4.5}$$

By Lemma 2.6, it is easily seen that $\delta_i > 0$. Choose a number $d_i \in (\frac{1}{2}\delta_i, \delta_i]$ such that

$$d(y_i + \frac{d_i^\alpha}{\Gamma(\alpha+1)} f(\bar{x}, y_i); D) \le \frac{d_i}{6n}. \tag{4.6}$$

Define $x_{i+1} = x_i + d_i$. By (4.6), there is a $y_{i+1} \in D$ such that

$$\|y_i + \frac{d_i^\alpha}{\Gamma(\alpha+1)} f(\bar{x}, y_i) - y_{i+1}\| \le \frac{d_i}{3n}.$$

Consequently, y_{i+1} can be written as

$$\begin{aligned}y_{i+1} = y_i &+ \frac{1}{\Gamma(\alpha)}\int_{x_i}^{x_{i+1}}(x_{i+1}-t)^{\alpha-1} f(\bar{x}, y_i)dt + (x_{i+1} - x_i)p_i\\ &+ \frac{1}{\Gamma(\alpha)}\int_a^{x_i}\big[(x_{i+1}-t)^{\alpha-1} - (x_i - t)^{\alpha-1}\big]h(t)dt.\end{aligned} \tag{4.7}$$

with $\|p_i\| \le \frac{1}{3n}$. Define h on $[x_i, x_{i+1})$ as $h(t) = f(\bar{x}, y_i)$.

In the case that $x_i \notin L_n$, we set

$$\begin{aligned}\delta_i = \sup\Big\{&h \in (0, \frac{1}{n}]; x_i + h \le x_0 + T,\\ &d(y_i + \frac{h^\alpha}{\Gamma(\alpha+1)} f(x_i, y_i); D) \le \frac{h}{6n}\Big\}.\end{aligned} \tag{4.8}$$

By Lemma 2.6, we know that $\delta_i > 0$. Choose a number $d_i \in (\frac{1}{2}\delta_i, \delta_i]$ such that

$$d(y_i + \frac{d_i^\alpha}{\Gamma(\alpha+1)} f(x_i, y_i); D) \leq \frac{d_i}{6n}. \tag{4.9}$$

Define $x_{i+1} = x_i + d_i$. By (4.9), there is a $y_{i+1} \in D$ such that

$$\|y_i + \frac{d_i^\alpha}{\Gamma(\alpha+1)} f(x_i, y_i) - y_{i+1}\| \leq \frac{d_i}{3n}.$$

Consequently, y_{i+1} can be written as

$$\begin{aligned} y_{i+1} = y_i &+ \frac{1}{\Gamma(\alpha)} \int_{x_i}^{x_{i+1}} (x_{i+1} - t)^{\alpha-1} f(x_i, y_i) dt + (x_{i+1} - x_i) p_i \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^{x_i} \big[(x_{i+1} - t)^{\alpha-1} - (x_i - t)^{\alpha-1}\big] h(t) dt \end{aligned} \tag{4.10}$$

with $\|p_i\| \leq \frac{1}{3n}$. In this case, we define h on $[x_i, x_{i+1})$ as $h(t) = f(x_i, y_i)$. In both cases, we define y on $[x_i, x_{i+1}]$ as

$$\begin{aligned} y(x) = y_i &+ \frac{1}{\Gamma(\alpha)} \int_{x_i}^{x} (x - t)^{\alpha-1} h(t) dt + (x - x_i) p_i \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^{x_i} \big[(x - t)^{\alpha-1} - (x_i - t)^{\alpha-1}\big] h(t) dt. \end{aligned} \tag{4.11}$$

Let us define the step functions α^n and β^n as $\alpha^n(t) = x_i$ in the case $x_i \notin L_n$, $\alpha^n(t) = \bar{x}$ in the case $x_i \in L_n$ and $\beta^n(t) = x_i$ for $t \in [x_i, x_{i+1})$. Then h^n can be written as $h(t) = f(\alpha(t), y(\beta(t)))$. By the induction hypotheses, y can be written in the form

$$\begin{aligned} y(x) = y_0 &+ \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x - t)^{\alpha-1} h(t) dt + \sum_{k=0}^{i-1} (x_{k+1} - x_k) p_k + (x - x_i) p_i \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^{x_0} \big[(x - t)^{\alpha-1} - (x_0 - t)^{\alpha-1}\big] h(t) dt \end{aligned} \tag{4.12}$$

for $x \in [x_i, x_{i+1})$. We now check that $y^n(x) \in B(y_0, r)$ for sufficiently large n. We first notice that $\|p_k\| \leq \frac{1}{3n}$ for $k = 0, 1, \cdots, i$ and $\|h(t)\| \leq M_r$ by (H_2). Therefore, from (4.10), (4.3) and (4.4) we have

$$\begin{aligned} \|y(x) - y_0\| \leq& \frac{1}{\Gamma(\alpha)} \Big\| \int_{x_0}^{x} (x - t)^{\alpha-1} h(t) dt \Big\| + \Big\| \sum_{k=0}^{i-1} (x_{k+1} - x_k) p_k + (x - x_i) p_i \Big\| \\ &+ \frac{1}{\Gamma(\alpha)} \Big\| \int_a^{x_0} \big[(x - t)^{\alpha-1} - (x_0 - t)^{\alpha-1}\big] h(t) dt \Big\| \\ \leq& \frac{M_r (x - x_0)^\alpha}{\Gamma(\alpha+1)} + (x - x_0) \frac{1}{3n} + \frac{r}{3} \\ \leq& \frac{M_r T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{3n} + \frac{r}{3} \\ \leq& \frac{r}{3} + \frac{T}{3n} + \frac{r}{3} < r \end{aligned} \tag{4.13}$$

for sufficiently large n. This implies that $y^n(x) \in B(y_0, r)$ for all $x \in [x_0, x_{i+1}]$. Thus, the properties (ii), (iii) and (iv) are verified.

To prove property (i), we first note that $\lim_{i\to\infty} x_i$ exists since $\{x_i\}_{i=1}^\infty$ is increasing and $x_i \leq x_0 + T$ for all $i = 1, 2, \cdots$. Suppose that $\lim_{i\to\infty} x_i = x^*$, then $x^* \leq x_0 + T$. We have to prove that $x^* = x_0 + T$.

For this end, we first verify that $\lim_{i\to\infty} y_i$ exists. In fact, let $j > i$. Using (4.10) for $x = x_i$ and $x = x_j$ respectively, we derive that

$$\begin{aligned}\|y_j - y_i\| \le& \frac{1}{\Gamma(\alpha)}\Big\| \int_{x_0}^{x_i} \Big[(x_j - t)^{\alpha-1} - (x_i - t)^{\alpha-1}\Big] h(t)dt \\ &+ \int_{x_i}^{x_j} (x_j - t)^{\alpha-1} h(t)dt\Big\| + \Big\| \sum_{k=0}^{i-1} (x_{k+1} - x_k)p_k + (x - x_i)p_i\Big\| \\ &+ \frac{1}{\Gamma(\alpha)}\Big\| \int_{a}^{x_0} \big[(x_j - t)^{\alpha-1} - (x_i - t)^{\alpha-1}\big] h(t)dt\Big\| \\ \le& \frac{M_r}{\Gamma(\alpha+1)}\Big[(x_j - x_0)^{\alpha} - (x_i - x_0)^{\alpha} + 2(x_j - x_i)^{\alpha}\Big] + \frac{x_j - x_i}{n} \\ &+ \frac{M_r}{\Gamma(\alpha+1)}\Big[(x_j - a)^{\alpha} - (x_i - a)^{\alpha} + (x_j - x_0)^{\alpha} - (x_i - x_0)^{\alpha}\Big].\end{aligned} \tag{4.14}$$

From the fact that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence, it is easily seen from (4.12) that $\{y_i\}_{i=1}^{\infty}$ is also a Cauchy sequence in $\mathbb{R}^m$. Hence $\lim_{i\to\infty} y_i = y^*$ exists and $y^* \in B(y_0, r) \cap D$ since $B(y_0, r) \cap D$ is closed. We define $y(x^*) = y^*$. By (4.9), we have

$$\begin{aligned}\|y(x) - y_i\| \le& \frac{1}{\Gamma(\alpha)}\Big\| \int_{x_i}^{x} (x - t)^{\alpha-1} h(t)dt\Big\| + \Big\|(x - x_i)p_i\Big\| \\ &+ \frac{1}{\Gamma(\alpha)}\Big\| \int_{a}^{x_i} \big[(x - t)^{\alpha-1} - (x_i - t)^{\alpha-1}\big] h(t)dt\Big\| \\ \le& \frac{M_r}{\Gamma(\alpha+1)}\Big[2(x - x_i)^{\alpha} + (x - a)^{\alpha} - (x_i - a)^{\alpha}\Big] + \frac{x - x_i}{n}.\end{aligned} \tag{4.15}$$

This, alone with the fact that $\lim_{i\to\infty} y_i = y^*$ implies that $\lim_{x\uparrow x^*} y(x) = y^*$. Accordingly, y is continuous on $[a, x^*]$.

We assert that $x^* \notin L_n$ for sufficiently large n. Indeed, if $x^* \in L_n$, then there are only finitely many $x_i \notin L_n$ since $[x_0, x^*]\backslash L_n$ is closed. Hence there is a positive integer i_0 such that $x_i \in L_n$ for all $i \ge i_0$. But then $[x_{i_0}, x^*] \subset L_n$ by (4.5), which contradicts the fact that $m(L_n) < \frac{1}{n}$ for sufficiently large n.

Now we assume by negation that $x^* < x_0 + T$. Then we can choose $h^* \in (0, \frac{1}{n}]$ such that

$$d(y^* + \frac{h^{*\alpha}}{\Gamma(\alpha+1)} f(x^*, y^*); D) \le \frac{h^*}{8n}. \tag{4.16}$$

Since $d_i > \frac{1}{2}\delta_i$ and $d_i = x_{i+1} - x_i \to 0$ as $i \to \infty$, there is a positive integer i_0 such that $\delta_i < h^*$ for all $i > i_0$. On the basis of (4.8), we have

$$d(y_i + \frac{h^{*\alpha}}{\Gamma(\alpha+1)} f(x_i, y_i); D) > \frac{h^*}{6n} \tag{4.17}$$

for all $i > i_0$ and $x_i \notin L_n$. Letting $i \to \infty$ in (4.17), one gets an inequality which contradicts (4.16). Therefore $x^* = x_0 + T$, which completes the proof. □

Proof of Theorem 4.1. Let $\{L_n\}$ be the sequence of open subsets of $\mathbb{R}$ such that $Z \subset L_n$ and $m(L_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Take $L = \cap_{n\ge1} L_n$ and a sequence of n-approximate solutions $\{y^n\}$ and $\{x_i^n\}$ obtained in Lemma 4.2. Define

$$g^n(x) = \sum_{k=0}^{i-1} (x_{k+1} - x_k)p_k + (x - x_i^n)p_i^n$$

for $x \in [x_i, x_{i+1})$. Then $\|g^n(x)\| \leq \frac{T}{n}$ for all $x \in [x_0, x_0+T]$ and y^n can be written in the form

$$\begin{aligned} y^n(x) =& y_0 + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} h^n(t)dt + g^n(x) + \frac{1}{\Gamma(\alpha)} \int_a^{x_0} \left[(x-t)^{\alpha-1} - (x_0-t)^{\alpha-1}\right] h^n(t)dt \\ =& y_0 + \frac{1}{\Gamma(\alpha)} \int_a^{x} (x-t)^{\alpha-1} h^n(t)dt - \frac{1}{\Gamma(\alpha)} \int_a^{x_0} (x_0-t)^{\alpha-1} h^n(t)dt + g^n(x). \end{aligned} \tag{4.18}$$

From the construction of y^n we know that $\{y^n\}$ is uniformly bounded. We now prove that $\{y^n\}$ is equicontinuous. Take $x', x'' \in [x_0, x_0+T]$ with $x' < x''$. Then for any $n \in \mathbb{N}$, by (4.18) we have

$$\begin{aligned} \|y(x'') - y(x')\| \leq& \frac{1}{\Gamma(\alpha)} \Big\| \int_a^{x'} \left[(x''-t)^{\alpha-1} - (x'-t)^{\alpha-1}\right] h^n(t)dt \\ &+ \int_{x'}^{x''} (x''-t)^{\alpha-1} h^n(t)dt \Big\| + \Big\| g^n(x'') - g^n(x') \Big\| \\ \leq& \frac{M_r}{\Gamma(\alpha+1)} \left[(x''-a)^{\alpha} - (x'-a)^{\alpha} + 2(x''-x')^{\alpha}\right] + \frac{x''-x'}{n}, \end{aligned} \tag{4.19}$$

which converges to 0 as $x'' - x' \to 0$ and the convergence is independent of n. Hence $\{y^n\}$ is equicontinuous. By Arzela-Ascoli's theorem, $\{y^n\}$ is relatively compact in $C([a, x_0+T]; \mathbb{R}^m)$ and hence has a convergent subsequence. Without loss of generality, we may assume that $\{y^n\}$ itself is convergent and

$$\lim_{n\to\infty} y^n(x) = y(x)$$

uniformly on $[x_0, x_0+T]$ (recall that $y^n(x) \equiv \bar{y}(x)$ for $x \in [a.x_0]$). Notice that $\lim_{n\to\infty} g^n(x) = 0$ uniformly for $x \in [x_0, x_0+T]$. Let us investigate $h^n(t) = f(\alpha^n(t), y^n(\beta^n(t)))$. If $t \notin L$, then $t \notin L_n$ for sufficiently large n and then we have $\alpha^n(t) \to t$ as $n \to \infty$. Also we have $\beta^n(t) \to t$ as $n \to \infty$ for all $t \in [x_0, x_0+T]$. Therefore $h^n(t) \to f(t, y(t))$ as $n \to \infty$ for a.e $t \in [x_0, x_0+T]$. Moreover, $y^n(\beta^n(t)) \in B(y_0, r) \cap D$ implies $y(t) \in B(y_0, r) \cap D$ (which is closed). Finally, passing to the limit in (4.18), one obtains that

$$y(x) = y_0 + \frac{1}{\Gamma(\alpha)} \int_a^{x} (x-t)^{\alpha-1} f(t, y(t))dt - \frac{1}{\Gamma(\alpha)} \int_a^{x_0} (x_0-t)^{\alpha-1} f(t, y(t))dt$$

and $y(x) \in B(y_0, r) \cap D$ for $x \in [x_0, x_0+T]$, which completes the proof. □

Concerning the continuation of the solution to (1.1) satisfying (1.2). Recall that a solution $\tilde{y} : [x_0, x_0 + T_1] \to \mathbb{R}^m$ to (1.1) with $T_1 > T$ is said to be a continuation to the right of the solution $y : [x_0, x_0+T] \to \mathbb{R}^m$ to (1.1), if $\tilde{y}(x) = y(x)$ for all $x \in [x_0, x_0+T]$. A solution $\tilde{y}$ is said to be non-continuable if it has no proper continuation. Using a standard argument based on Zorn's lemma, one can easily verify that, if the hypotheses of Theorem 4.1 hold and $\tilde{y} : [x_0, b_0) \to \mathbb{R}^m$ is a non-continuable solution to (1.1) satisfying (1.2), then either $b_0 = b$ or $\lim_{x\uparrow b_0} \|\tilde{y}(x)\| = +\infty$. Precisely, we have

Theorem 4.3. *Under the hypotheses of* Theorem 4.1*, a sufficient condition for every $x_0 \in (a, b)$, $y_0 \in D$ to have a non-continuable solution $y(x) \in D$ to* (1.1) *satisfying* (1.2) *is the tangency condition* (2.1).

Consider $D = \mathbb{R}^m_+ = \{y = (y_1, y_2, \cdots, y_m) \in \mathbb{R}^m : y_i > 0, i = 1, 2, \cdots, m\}$. If $y_0 \in \mathbb{R}^m_+$ and f is continuous, then the tangency condition (2.1) is satisfied automatically since $\mathbb{R}^m_+$ is open. In this case, we can get the existence of a positive solution for IVP (1.1)-(1.2) from Theorem 4.1.

Corollary 4.4. *Suppose that $D = \mathbb{R}^m_+$ and f is continuous. If $y_0 \in \mathbb{R}^m_+$, then there exists a $T > 0$ with $x_0 + T \leq b$ such that the IVP* (1.1)-(1.2) *has a positive solution on* $[x_0, x_0+T]$.

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